Jordan forms of real and complex matrices under rank one perturbations*

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Abstract

New perturbation results for the behavior of eigenvalues and Jordan forms of real and complex matrices under generic rank one perturbations are discussed. Several results that are available in the complex case are proved as well for the real case and the assumptions on the genericity are weakened. Rank one perturbations that lead to maximal algebraic multiplicities of the "new" eigenvalues are also discussed.

Key Words: Eigenvalues, generic perturbation, rank one perturbation, Jordan canonical form.

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1 Introduction

The goal of this paper is to present new results on eigenvalues and Jordan forms of real and complex matrices under generic rank one perturbations, where the genericity is understood in various ways. This topic has been studied intensively in recent years. Basic results were obtained in [10, 11, 18, 20, 21] but mostly for the complex case; we extend these results to the real case.

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Our interest in the topic is motivated by the perturbation analysis of structured matrices, in particular Hamiltonian matrices. Such perturbation problems arise in the analysis of linear quadratic optimal control problems [13, 17] and in particular in the passivation of linear systems [3, 4, 8, 19, 22]. It has been recently shown how to construct minimum norm perturbations that move eigenvalues of Hamiltonian matrices from the imaginary axis [2] and these perturbations typically turn out to be of low rank. This has motivated the question of what happens to the eigenvalues of Hamiltonian matrices (and in particular to the purely imaginary eigenvalues) under generic, Hamiltonian, low rank perturbations, see [15, 16].

In this paper, we study the general unstructured case. We combine the results for real and complex matrices under weakened genericity assumptions in Section 2. In Section 3 we extend these results to matrix polynomials and in Section 4 we study generic low rank perturbations that lead to maximal algebraic multiplicities in the perturbed eigenvalues.

2 Main Theorems

In order to state our results on the influence of generic rank one perturbations we first recall from [15] the general concept of genericity. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We say that a set $W \subseteq \mathbb{F}^n$ (abbreviation for $\mathbb{F}^{n\times 1}$) is *algebraic* if there exists a finite set of polynomials $f_1(x_1,\dots,x_n),\dots,f_k(x_1,\dots,x_n)$ with coefficients in $\mathbb{F}$ such that a vector $[a_1,\dots,a_n]^T \in \mathbb{F}^n$ belongs to W if and only if

$$f_j(a_1,\dots,a_n) = 0, \quad j = 1,2,\dots,k.$$

In particular, the empty set is algebraic and $\mathbb{F}^n$ is algebraic. We say that a set $W \subseteq \mathbb{F}^n$ is *generic* if the complement $\mathbb{F}^n \setminus W$ is contained in an algebraic set which is not $\mathbb{F}^n$. Note that the union of finitely many algebraic sets is again algebraic.

We then have our main Theorems, in which $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and the used Jordan forms are understood over $\mathbb{C}$, where in the following we denote by $\mathcal{J}_m(\lambda_0)$ an $m\times m$ upper triangular Jordan block with eigenvalue λ_0.

Theorem 2.1 *Let $A \in \mathbb{F}^{n\times n}$ be a matrix having pairwise distinct eigenvalues $\lambda_1,\dots,\lambda_p$ with geometric multiplicities $g_1,\dots,g_p$ and a Jordan canonical form*

$$\bigoplus_{k=1}^{g_1} \mathcal{J}_{n_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{g_p} \mathcal{J}_{n_{p,k}}(\lambda_p), \tag{2.1}$$

where $n_{j,1} \geq \cdots \geq n_{j,g_j}$, $j = 1,\dots,p$. Consider the rank one matrix $B = uv^T$, with $u, v \in \mathbb{F}^n$. Then generically (with respect to the entries of u and v) the Jordan blocks of $A+B$ with eigenvalue λ_j are just the $g_j - 1$ smallest Jordan blocks of A with eigenvalue λ_j, and all other eigenvalues of $A+B$ are simple, i.e., of algebraic multiplicity one; if $g_j = 1$, then generically λ_j is not an eigenvalue of $A+B$.

More precisely, the set $\Omega \subseteq \mathbb{F}^n \times \mathbb{F}^n$, consisting of all $(u, v) \in \mathbb{F}^n \times \mathbb{F}^n$ for which the Jordan structure of $A + uv^T$ is as described in the statements (a) *and* (b) *below, is generic.*

(a) *the Jordan structure of $A + uv^T$ for the eigenvalues $\lambda_1, \ldots, \lambda_p$ is given by*

$$\bigoplus_{k=2}^{g_1} \mathcal{J}_{n_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=2}^{g_p} \mathcal{J}_{n_{p,k}}(\lambda_p);$$

(b) *the eigenvalues of $A + uv^T$ that are different from any of $\lambda_1, \ldots, \lambda_p$ are all simple.*

Note that in the case $\mathbb{F} = \mathbb{R}$, the distinct eigenvalues $\lambda_1, \ldots, \lambda_p$ include the non-real ones (if there are any).

In the complex case, different proofs of Theorem 2.1 were given in [10, 18, 15, 20] (part (a)) and in [15, 20] (part (b)). To be more precise, in these references the existence of a generic set $\widetilde{\Omega}$ was proved such that for every $(u, v) \in \widetilde{\Omega}$, the Jordan structure of $A + uv^T$ is as described in the statements (a) and (b) of Theorem 2.1. However, the genericity of the set Ω in Theorem 2.1 then immediately follows from $\widetilde{\Omega} \subseteq \Omega$. On the other hand, the proof of the real case in Theorem 2.1 follows immediately from the following lemma.

Lemma 2.2 *Let $W \in \mathbb{C}^n$ be a* proper *(i.e., different from $\mathbb{C}^n$) algebraic set. Then $W_r = W \cap \mathbb{R}^n$ is a proper algebraic set in $\mathbb{R}^n$.*

Proof. Clearly W_r is an algebraic set. Let

$$W = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \; : \; f_1(z_1, \ldots, z_n) = \cdots = f_k(z_1, \ldots, z_n) = 0\}, \tag{2.2}$$

where the $f_j(z_1, \ldots, z_n)$ are non-identically zero polynomials of $z_1, \ldots, z_n$ (with complex coefficients). Write

$$f_j(z_1, \ldots, z_n) = \sum_{(i_1, \ldots, i_n) \in \mathbb{N}^n} c^{(j)}_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n}, \quad c^{(j)}_{i_1, \ldots, i_n} \in \mathbb{C}, \quad j = 1, 2, \ldots, k, \tag{2.3}$$

(where we use the convention that the natural numbers include zero, that is, $\mathbb{N} = \{0, 1, 2, \ldots\}$) and the sums in (2.3) are finite. Then

$$\begin{aligned} W_r = \{(z_1, \ldots, z_n) \in \mathbb{R}^n \quad : \quad & \sum_{(i_1, \ldots, i_n) \in \mathbb{N}^n} \left(\operatorname{Re} c^{(j)}_{i_1, \ldots, i_n}\right) z_1^{i_1} \cdots z_n^{i_n} \\ = \; & \sum_{(i_1, \ldots, i_n) \in \mathbb{N}^n} \left(\operatorname{Im} c^{(j)}_{i_1, \ldots, i_n}\right) z_1^{i_1} \cdots z_n^{i_n} \; = \; 0, \quad j = 1, \ldots, k\}. \end{aligned}$$

To show that W_r is proper, we use induction on n. For $n = 1$, this is trivial because a subset of $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) is a proper algebraic set if and only if it is finite.

Assume that the assertion has been proved for proper algebraic sets in $\mathbb{C}^{n-1}$, and let W be given by (2.2). If all the $f_j(z_1, \ldots, z_n)$ are independent of z_n, then we are done by the induction hypothesis. So, let us assume that at least one of $f_j(z_1, \ldots, z_n)$ depends on z_n, say, without loss of generality,

$$f_1(z_1, \ldots, z_n) = \sum_{\ell=0}^{\ell_0} d_\ell(z_1, \ldots, z_{n-1}) z_n^\ell,$$

where $\ell_0 \geq 1$ and the $d_0, \ldots, d_{\ell_0}$ are polynomials in $z_1, \ldots, z_{n-1}$, with d_{ℓ_0} not identically zero. By the induction hypothesis, there exist $(z_1', \ldots, z_{n-1}') \in \mathbb{R}^{n-1}$ such that $d_{\ell_0}(z_1', \ldots, z_{n-1}') \neq 0$. Let z_n' be any real number which is not a zero of the polynomial

$$\sum_{\ell=0}^{\ell_0} d_\ell(z_1', \ldots, z_{n-1}') z_n^\ell.$$

considered as a polynomial of z_n. Then $(z_1', \ldots, z_{n-1}', z_n')$ does not belong to W_r. □

Motivated from the situation arising in structured perturbations, in the following we refine Theorem 2.1 by using a more restrictive notion of genericity.

Theorem 2.3 *Let $A \in \mathbb{F}^{n \times n}$ be as in Theorem 2.1. Suppose that a matrix $X \in \mathbb{F}^{n \times n}$ is such that for some vector $u_0 \in \mathbb{F}^n$ the Jordan structure of $A + u_0(Xu_0)^T$ is given by (a) and (b) of Theorem 2.1, where $u = u_0$, $v = Xu_0$.*

Then the set $\Omega' \subseteq \mathbb{F}^n$ consisting of all $u \in \mathbb{F}^n$, for which the Jordan structure of $A + u(Xu)^T$ is as described in the statements (a) and (b) below, is generic (with respect to the entries of u).

(a) *the Jordan structure of $A + u(Xu)^T$ for the eigenvalues $\lambda_1, \ldots, \lambda_p$ is given by*

$$\bigoplus_{k=2}^{g_1} \mathcal{J}_{n_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=2}^{g_p} \mathcal{J}_{n_{p,k}}(\lambda_p);$$

(b) *the eigenvalues of $A + u(Xu)^T$ that are different from any of $\lambda_1, \ldots, \lambda_p$, are all simple.*

Note that the genericity condition in Theorem 2.3 is not equivalent to the genericity condition in Theorem 2.1.

We also have the following dual version of Theorem 2.3.

Theorem 2.4 *Let $A \in \mathbb{F}^{n \times n}$ be as in Theorem 2.1. Suppose a matrix $Y \in \mathbb{F}^{n \times n}$ is such that for some vector $v_0 \in \mathbb{F}^n$ the Jordan structure of $A + Yv_0v_0^T$ is given by (a) and (b) of Theorem 2.1, where $u = Yv_0$, $v = v_0$.*

Then the set $\Omega'' \subseteq \mathbb{F}^n$ consisting of all $v \in \mathbb{F}^n$, for which the Jordan structure of $A + Yvv^T$ is as described in the statements (a) and (b) below, is generic (with respect to the entries of v).

(a) *the Jordan structure of* $A + Yvv^T$ *for the eigenvalues* $\lambda_1, \ldots, \lambda_p$ *is given by*

$$\bigoplus_{k=2}^{g_1} \mathcal{J}_{n_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=2}^{g_p} \mathcal{J}_{n_{p,k}}(\lambda_p);$$

(b) *the eigenvalues of* $A + Yvv^T$ *that are different from any of* $\lambda_1, \ldots, \lambda_p$*, are all simple.*

For the proofs of Theorems 2.3 and 2.4 in the complex case we make use of the following lemma.

Lemma 2.5 *Let* Ω *be the generic set as in Theorem* 2.1*. Then the complement of* Ω *in* $\mathbb{C}^n \times \mathbb{C}^n$ *is a proper algebraic set.*

Proof. We have $\Omega = \Omega_1 \cap \Omega_2$, where Ω_1 and Ω_2 are defined as follows: Ω_1 consists of all pairs $(u, v) \in \mathbb{C}^n \times \mathbb{C}^n$ such that the matrix $A + uv^T$ has the following eigenvalue structure:

(a) λ_j is an eigenvalue of $A + uv^T$ of algebraic multiplicity $\sum_{k=2}^{g_j} n_{j,k}$, for $j = 1, 2, \ldots, p$;

(b) $A + uv^T$ has exactly
$$n - \sum_{j=1}^{p} \sum_{k=2}^{g_j} n_{j,k}$$
simple eigenvalues different from any of $\lambda_1, \ldots, \lambda_p$.

The set Ω_2 consists of all pairs $(u, v) \in \mathbb{C}^n \times \mathbb{C}^n$ such that the partial multiplicities of λ_j as an eigenvalue of $A+uv^T$ are exactly $n_{j,2} \geq \cdots \geq n_{j,g_j}$, respectively, for $j = 1, 2, \ldots, p$. Evidently, it suffices to prove that the complement of each of Ω_1 and Ω_2 is an algebraic set (the properness of this set is immediate because Ω is generic by Theorem 2.1).

We start with Ω_1. For $u, v \in \mathbb{C}^n$, we denote by $S(u, v)$ the Sylvester resultant matrix of the two polynomials in the independent variable x given by

$$\det (xI - (A + uv^T)), \qquad \frac{\partial \det (xI - (A + uv^T))}{\partial x}.$$

Recall (see, e.g., [1, 6, 9, 12, 14]) that $S(u, v)$ is a $(2n - 1) \times (2n - 1)$ matrix, and its *rank defect*, i.e., the difference $2n - 1 - \operatorname{rank} S(u, v)$, is equal to

$$\sum_{j=1}^{k} ((\text{algebraic multiplicity of } \mu_j) - 1),$$

where $\mu_1, \ldots, \mu_k$ are all the distinct eigenvalues of $A + uv^T$ (we omit in this notation the dependence of these algebraic multiplicities as well as of the number k on u and v).

Denote by $\operatorname{rd} S(u,v)$ the rank defect of the matrix $S(u,v)$. By Theorem 2.1 we have:

$$\operatorname{rd} S(u,v) = \sum_{j=1,\dots,p;\ g_j\geq 2} \left(\left(\sum_{k=2}^{g_j} n_{j,k} \right) - 1 \right)$$

for all (u,v) in a generic set in $\mathbb{C}^n \times \mathbb{C}^n$. Since the rank defect of $S(u,v)$ can only increase when passing to the limit of a convergent sequence of vectors (u,v), it follows that

$$\operatorname{rd} S(u,v) \geq \sum_{j=1,\dots,p;\ g_j\geq 2} \left(\left(\sum_{k=2}^{g_j} n_{j,k} \right) - 1 \right) \tag{2.4}$$

for all $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$. On the other hand, since Ω is generic, there is (u_0,v_0) such that

$$\operatorname{rd} S(u_0,v_0) = \sum_{j=1,\dots,p;\ g_j\geq 2} \left(\left(\sum_{k=2}^{g_j} n_{j,k} \right) - 1 \right).$$

Since $S(u,v)$ depends continuously on u, the rank defect of $S(u,v)$ may only decrease for (u,v) in a sufficiently small neighborhood of (u_0,v_0). Thus, we obtain that

$$\operatorname{rd} S(u,v) \leq \sum_{j=1,\dots,p;\ g_j\geq 2} \left(\left(\sum_{k=2}^{g_j} n_{j,k} \right) - 1 \right)$$

for all (u,v) sufficiently close to (u_0,v_0). Comparing with (2.4), we see that

$$\operatorname{rd} S(u,v) = \sum_{j=1,\dots,p;\ g_j\geq 2} \left(\left(\sum_{k=2}^{g_j} n_{j,k} \right) - 1 \right) := \Upsilon$$

for all (u,v) sufficiently close to (u_0,v_0). Let W be the algebraic set which consists of all common zeros of the determinants of the $(2n-1-\Upsilon)\times(2n-1-\Upsilon)$ submatrices of $S(u,v)$. Since there is a least one $(2n-1-\Upsilon)\times(2n-1-\Upsilon)$ submatrix of $S(u,v)$ whose determinant is nonzero at $(u=u_0, v=v_0)$, it follows that W is proper. Then it follows that the complement of W coincides with Ω_1.

As second step we prove that the complement of Ω_2 is an algebraic set. For this we shall use the Segre and Weyr characteristics of the matrix $A+uv^T$. Recall that the *Segre characteristic* of $A+uv^T$ corresponding to one of its eigenvalues λ_j is the non-increasing list of sizes of Jordan blocks of $A+uv^T$ with eigenvalue λ_j. The Weyr characteristic, on the other hand, is the non-increasing list of dimensions of the null spaces of the powers of $A+uv^T-\lambda_j$. Note that both these lists of numbers are partitions of n, in fact, they are dual partitions, as can be seen most easily by representing both the Weyr and the Segre characteristic in a so-called Ferrer diagram. For a nice introduction to the theory of the Weyr and Segre characteristics and their relation see [23].

Consider the set T of all $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$ that satisfy the following inequalities for $j = 1, 2, \ldots, p$:

$$\mathrm{rd}\,(A + uv^T - \lambda_j I) \le g_j - 1;$$

$$\begin{aligned} \mathrm{rd}\,((A + uv^T - \lambda_j I)^2) &\le \#\{\alpha \in \{2,3,\ldots,g_j\} \,:\, n_{j,\alpha} \ge 2\} \\ &+ \#\{\alpha \in \{2,3,\ldots,g_j\} \,:\, n_{j,\alpha} \ge 1\}; \end{aligned}$$

and so on, and finally

$$\mathrm{rd}\,((A + uv^T - \lambda_j I)^{n_{j,2}}) \le \sum_{k=1}^{n_{j,2}} \#\{\alpha \in \{2,3,\ldots,g_j\} \,:\, n_{j,\alpha} \ge k\}.$$

Note that because of duality between the Weyr and Segre characteristic

$$\sum_{k=1}^{n_{j,2}} \#\{\alpha \in \{2,3,\ldots,g_j\} \,:\, n_{j,\alpha} \ge k\} = n_{j,2} + \ldots + n_{j,g_j}.$$

We have

$$T = \cap_{j=1}^{p} \cap_{k=1}^{n_{j,2}} T_{j,k},$$

where $T_{j,k}$ consists of all $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$ such that

$$\mathrm{rd}\,((A + uv^T - \lambda_j I)^k) \le \sum_{\ell=1}^{k} \#\{\alpha \in \{2,3,\ldots,g_j\} \,:\, n_{j,\alpha} \ge \ell\}.$$

As in the preceding paragraph, we now show that the complement of $T_{j,k}$ is an algebraic set, and therefore the complement of T is an algebraic set. The set

$$T'_{j,k} := \{(u,v) \in \mathbb{C}^n \times \mathbb{C}^n \,:\, \mathrm{rd}\,((A + uv^T - \lambda_j I)^k) \ge \sum_{\ell=1}^{k} \#\{\alpha \in \{2,3,\ldots,g_j\} \,:\, n_{j,\alpha} \ge \ell\}$$

is algebraic, and Theorem 2.1 shows that $T'_{j,k}$ contains the generic set Ω. It follows that $T'_{j,k} = \mathbb{C}^n \times \mathbb{C}^n$ for all $k = 1, 2, \ldots, n_{j,2}$ and all $j = 1, 2, \ldots, p$. Thus, in fact, T consists of all such pairs $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$ for which the identities

$$\mathrm{rd}\,((A + uv^T - \lambda_j I)^k) = \sum_{\ell=1}^{k} \#\{\alpha \in \{2,3,\ldots,g_j\} \,:\, n_{j,\alpha} \ge \ell\}.$$

hold for all $k = 1, 2, \ldots, n_{j,2}$ and all $j = 1, 2, \ldots, p$.

The set Ω_2 is defined in terms of the Segre characteristic of $A + uv^T$, whereas T is defined in terms of the Weyr characteristic of the same matrix. Since these two characteristics are each other's duals, and the Weyr characteristic is expressed in the rank defects as above, it is easy to see that $T = \Omega_2$, and so the complement of Ω_2 is an algebraic set as required. $\square$

Proof of Theorem 2.3. In the complex case, by Lemma 2.5 the complement of Ω is an algebraic set W; thus

$$W = \{(u, v) \in \mathbb{C}^n \times \mathbb{C}^n \, : \, f_k(u_1, \dots, u_n, v_1, \dots, v_n) = 0, \quad k = 1, 2, \dots, q\},$$

where $f_1, \dots, f_q$ are polynomials in the components $(u_1, \dots, u_n)$ of u and $(v_1, \dots, v_n)$ of v. Then clearly Ω' coincides with the complement of the algebraic set

$$W' := \{u \in \mathbb{C}^n \, : \, f_k(u_1, \dots, u_n, (Xu)_1, \dots, (Xu)_n) = 0, \quad k = 1, 2, \dots, q\}.$$

Since $u_0 \notin W'$, the set W' is proper.

The real case again follows from the complex case by applying Lemma 2.2. $\square$

Proof of Theorem 2.4. The proof is a consequence of the fact that the Jordan canonical form of a matrix A and that of A^T are the same. Applying Theorem 2.3 to A^T in place of A, Y in place of X and v in place of u, we arrive at the desired conclusion. $\square$

We continue with several remarks.

Remark 2.6 The form of the rank one perturbation $u(Xu)^T$ in Theorem 2.3 appears in studies of rank one perturbations of structured matrices, for particular choices of X, see e.g. [15, 16] for applications to Hamiltonian matrices and to selfadjoint matrices in an indefinite inner product space.

Remark 2.7 The following example shows that the hypothesis on X in Theorem 2.3, namely, that there exists $u_0 \in \mathbb{F}^n$ such that the Jordan structure of $A + u_0(Xu_0)^T$ is given by Theorem 2.1 (a), (b), is essential. For

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$$

this hypothesis does not hold. If $u_0 \neq 0$, then $A + u_0(Xu_0)^T = u_0(Xu_0)^T$ has one nilpotent Jordan block of size 2, and it is immediate that the conclusion of Theorem 2.3 does not hold either.

Remark 2.8 One can prove that, for a given A, the set $X(A)$ of all matrices $X \in \mathbb{F}^{n\times n}$ for which the hypothesis in Theorem 2.3 holds is generic (with respect to the entries of X). Note that this set depends on A. Indeed, let Ω be the generic set of Theorem 2.1. Fix $(u', v') \in \Omega$, where both u' and v' are nonzero. Then for any matrix X' such that $v' = X'u'$ we have that the hypothesis on X in Theorem 2.3 holds. On the other hand, we have that $X \in X(A)$ provided that $(u, Xu) \in \Omega$ for some u. Let W be the proper algebraic set such that the complement of Ω is contained in W, and let

$$W = \{(x, y) \in \mathbb{F}^n \times \mathbb{F}^n \, : \, f_j(x_1, \dots, x_n, y_1, \dots, y_n) = 0 \quad \text{for} \quad j = 1, 2, \dots, k\}.$$

Here $f_1, \ldots, f_k$ are certain polynomials with coefficients in $\mathbb{F}$, and by $z_1, \ldots, z_n$ we denote the components of $z \in \mathbb{F}^n$. Thus, if the matrix X is such that at least one of the following inequalities holds

$$f_j(v'_1, \ldots, v'_n, (Xv')_1, \ldots, (Xv')_n) \neq 0, \quad j = 1, 2, \ldots, k,$$

then $X \in X(A)$. So, the complement of $X(A)$ is contained in the union of two algebraic sets (with respect to the entries of X), one defined by $\det X = 0$, and the other algebraic set is defined by the identities

$$f_j(v'_1, \ldots, v'_n, (Xv')_1, \ldots, (Xv')_n) = 0, \quad j = 1, 2, \ldots, k.$$

The latter algebraic set is proper in view of the existence of an X' with $(u', X'u') \in \Omega$ (we select $(u', v' = X'u')$ in the complement of W). It follows that $X(A)$ is generic.

For example, if $A = 0 \in \mathbb{C}^{2\times 2}$, then $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in X(A)$ if and only if $x_{11}x_{22} \neq x_{12}x_{21}$ and at least one of the numbers $x_{11}, x_{12} + x_{21}, x_{22}$ is nonzero.

We now illustrate our main results by an example.

Example 2.9 Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Assume that every eigenvalue of $A \in \mathbb{F}^{n\times n}$ has geometric multiplicity one (including the nonreal eigenvalues of a real matrix A). Then A is similar (with the similarity matrix in $\mathbb{F}^{n\times n}$) to a matrix in the companion form

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \ldots & -c_{n-2} & -c_{n-1} \end{bmatrix} \in \mathbb{F}^{n\times n}, \quad c_j \in \mathbb{F}. \tag{2.5}$$

So we may assume without loss of generality that A is given by (2.5). Consider a rank one perturbation of A given by

$$A + B = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -c_0 - \epsilon & -c_1 & -c_2 & \ldots & -c_{n-2} & -c_{n-1} \end{bmatrix}, \quad \epsilon \in \mathbb{F}.$$

Since the characteristic polynomials of A and $A + B$ are given by

$$\det(xI - A) = x^n + \sum_{j=0}^{n-1} x^j c_j, \qquad \det(xI - (A+B)) = x^n + (c_0 + \epsilon) + \sum_{j=1}^{n-1} x^j c_j,$$

respectively, it follows that for all values of ϵ, with the possible exception of at most $n-1$ values, the polynomial $\det(xI-(A+B))$ has all roots (in the complex plane) simple. Indeed, the Sylvester resultant $(2n-1)\times(2n-1)$ matrix of $\det(xI-(A+B))$ and that of

$$\frac{\partial(\det(xI-(A+B)))}{\partial x}$$

has the form

$$R := \begin{bmatrix} 1 & c_{n-1} & c_{n-2} & \dots & c_0+\epsilon & \dots & 0 \\ 0 & 1 & c_{n-1} & \dots & c_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & c_1 & c_0+\epsilon \\ n & (n-1)c_{n-1} & (n-2)c_{n-2} & \dots & c_1 & \dots & 0 \\ 0 & n & (n-1)c_{n-1} & \dots & 2c_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & 2c_2 & c_1 \end{bmatrix},$$

and it is easy to see that the determinant $p(\epsilon)$ of R is a polynomial of ϵ of degree $n-1$ whose leading term is $\pm n^n\epsilon^{n-1}$. Thus, for every $\epsilon\in\mathbb{F}$ that is not a root of $p(\epsilon)$, the polynomial $\det(xI-(A+B))$ has all roots simple, and hence for those values of ϵ the matrix $A+B$ has all eigenvalues (including the nonreal ones in case $A+B$ is real) distinct.

3 A generalization and an application to matrix polynomials

The setup of Theorem 2.3 can be extended (with essentially the same proof) to the following situation. Let $\mathcal{K}$ be any $\mathbb{F}$-subspace of $\mathbb{F}^n\times\mathbb{F}^n$. Then for a given matrix A as in Theorem 2.1, generically the Jordan structure of matrices $A+uv^T$, where $(u,v)\in\mathcal{K}$, is described by (a) and (b) of Theorem 2.3, provided that for at least one element $(u_0,v_0)\in\mathcal{K}$ the Jordan structure of $A+u_0v_0^T$ is given by (a) and (b) of Theorem 2.1. The genericity is understood with respect to the coefficients of (u,v) in some basis for $\mathcal{K}$. Thus, Theorems 2.3 and 2.4 are stated for

$$\begin{aligned} \mathcal{K} &= \mathcal{K}(X)=\{(u,v)\,:\,v=Xu,\quad u\in\mathbb{F}^n\}, \\ \mathcal{K} &= \mathcal{K}(Y)=\{(u,v)\,:\,u=Yv,\quad v\in\mathbb{F}^n\}, \end{aligned}$$

respectively.

As an application we consider rank one perturbations for matrix polynomials as they were studied in [24]. We restrict our attention here to monic matrix polynomials, i.e., having leading coefficient I. Let

$$L(\lambda)=\lambda^m I_n+\sum_{j=0}^{m-1}\lambda^j A_j,\qquad A_j\in\mathbb{F}^{n\times n} \tag{3.1}$$

be a matrix polynomial, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We say that $\lambda_0 \in \mathbb{C}$ is an *eigenvalue* of $L(\lambda)$ if $\det L(\lambda_0) = 0$. A *Jordan chain of length* m *with eigenvector* v_0 of $L(\lambda)$ associated with the eigenvalue λ is, by definition, see e.g., [14], a chain of vectors $v_0, \ldots, v_{m-1} \in \mathbb{C}^n$ such that

$$\sum_{j=0}^{k} \frac{1}{j!} L^{(j)}(\lambda_0) v_{k-j} = 0, \quad k = 0, 1, \ldots, m-1, \qquad v_0 \neq 0,$$

where $L^{(j)}(\lambda_0)$ stands for the j-th derivative of $L(\lambda)$ with respect to λ. *Partial multiplicities* of $L(\lambda)$ at the eigenvalue λ_0 are the lengths of Jordan chains in any collection of Jordan chains associated with λ_0 of maximal total length, subject to the restriction that the eigenvectors in the collection of Jordan chains are linearly independent. One can prove that the partial multiplicities of $L(\lambda)$ at λ_0 do not depend on the choice of the collection of Jordan chains with the above properties, see [7] for more details.

Theorem 3.1 *Let $L(\lambda)$ be a matrix polynomial* (3.1)*, with pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, and partial multiplicities $n_{j,1} \geq \cdots \geq n_{j,g_j}$ corresponding to eigenvalues λ_j, $j = 1, 2, \ldots, p$, respectively. Fix $X_0, \ldots, X_{m-1} \in \mathbb{F}^{n\times n}$, and consider matrix polynomials of the form*

$$K_u(\lambda) := L(\lambda) + \sum_{j=0}^{m-1} \lambda^j u (X_j u)^T, \qquad u \in \mathbb{F}^n. \tag{3.2}$$

Assume that for some $u_0 \in \mathbb{F}^n$, we have the following properties.

(a) *the partial multiplicities of $K_{u_0}(\lambda)$ at λ_j are $n_{j,2} \geq \cdots \geq n_{j,g_j}$, for $j = 1, \ldots, p$; if $g_j = 1$, then λ_j is not an eigenvalue of $K(\lambda)$;*

(b) *all eigenvalues of $K_{u_0}(\lambda)$ that are different from $\lambda_1, \ldots, \lambda_p$ are simple, i.e., simple roots of* $\det K_{u_0}(\lambda)$.

Then (a) *and* (b) *hold for a generic (with respect to the entries of u) set of vectors u.*

Proof. The proof follows by combining several facts. First, it is clear that the Jordan form of the companion matrix C_L of $L(\lambda)$ is given by (2.1), and similarly for the companion matrices of the matrix polynomials $K_u(\lambda)$ (see, e.g., [7] for details). Recall that

$$C_L = \begin{bmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & I \\ -A_0 & -A_1 & -A_2 & \ldots & -A_{m-2} & -A_{m-1} \end{bmatrix} \in \mathbb{F}^{mn\times mn}.$$

We can then employ the remarks made at the beginning of this section concerning the subspaces $\mathcal{K}$ and apply Theorem 2.3, with

$$\mathcal{K} = \left\{ \left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ u \end{bmatrix}, v \right) \in \mathbb{F}^{mn} \times \mathbb{F}^{mn} \, : \, v = \begin{bmatrix} X_0 u \\ X_1 u \\ \vdots \\ X_{m-1} u \end{bmatrix} \right\}. \quad \square$$

Theorem 2.4 can be also extended to matrix polynomials in a fashion analogous to Theorem 3.1, with a proof similar to that of Theorem 3.1. We omit the statement and proof of this extension.

4 Perturbations with maximal total algebraic multiplicity at new eigenvalues

Let $A \in \mathbb{F}^{n \times n}$ have the Jordan canonical form (2.1). Theorems 2.1, 2.3, and 2.4 assert that generically (understood in various senses) matrices $A + B$, where B is of rank one, have the sum of the algebraic multiplicities at distinct eigenvalues different from $\lambda_1, \ldots, \lambda_p$, equal to $n_{1,1} + \cdots + n_{p,1}$. This is in fact the maximal total algebraic multiplicity at "new" eigenvalues under rank one perturbations, see Proposition 4.1 below. It is of interest therefore to find out the Jordan forms of *all* (in contrast with a generic set) matrices of the form $A + B$, where B is of rank one and have the above mentioned property of the total algebraic multiplicity at eigenvalues other than $\lambda_1, \ldots, \lambda_p$. In this section a complete answer to this problem is given.

Recall that the *partial multiplicities* of an eigenvalue λ of $A \in \mathbb{F}^{n \times n}$ are just the sizes of Jordan blocks with eigenvalue λ in the Jordan form of A, each multiplicity m repeated as many times as the number of Jordan blocks $\mathcal{J}_m(\lambda)$ indicates.

We start with some information about algebraic and geometric multiplicities at the "new" eigenvalues:

Proposition 4.1 *Let $A \in \mathbb{F}^{n \times n}$ be as in Theorem* 2.1. *Assume that $B \in \mathbb{F}^{n \times n}$ is a rank one matrix such that the total algebraic multiplicity of $A + B$ at eigenvalues different from $\lambda_1, \ldots, \lambda_p$ is at least $n_{1,1} + \ldots + n_{p,1}$. Then:*

(1) *the sum of the algebraic multiplicities of $A+B$ at its pairwise distinct eigenvalues different from $\lambda_1, \ldots, \lambda_p$ is equal to $n_{1,1} + \cdots + n_{p,1}$;*

(2) *each eigenvalue of $A+B$ different from $\lambda_1, \ldots, \lambda_p$ has geometric multiplicity one.*

Proof. Assume that the total algebraic multiplicity of $A+B$ at the eigenvalues different from $\lambda_1, \ldots, \lambda_p$ is actually larger than $n_{1,1} + \ldots + n_{p,1}$. Approximate B with a sequence $\{B_m = u_m v_m^T\}_{m=1}^{\infty}$ so that for each $A + u_m v_m^T$ the statements (a) and (b) of Theorem 2.1 hold. By continuity of the eigenvalues, for the limit $A + B = \lim_{m \to \infty}(A + B_m)$ we have that the total algebraic multiplicity of $A+B$ at eigenvalues different from $\lambda_1,\dots,\lambda_p$ cannot exceed that of $A+B_m$, namely $n_{1,1}+\ldots+n_{p,1}$, which is a contradiction.

If for some rank one B the matrix $A+B$ would have geometric multiplicity at least two at some eigenvalue $\mu\notin\{\lambda_1,\dots,\lambda_p\}$, then $A=(A+B)-B$, being a rank one perturbation of $A+B$, must have μ as an eigenvalue, which contradicts the original hypothesis that $\sigma(A)=\{\lambda_1,\dots,\lambda_p\}$. This proves (2). □

The following simple example shows that in Proposition 4.1 (2) the geometric multiplicity cannot be replaced by algebraic multiplicity.

Example 4.2 Let

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix},\qquad B=\begin{bmatrix}0&0\\-b^2&-2b\end{bmatrix},$$

where $b\in\mathbb{F}\setminus\{0\}$. Then $A+B$ has an eigenvalue $-b$ of geometric multiplicity one and algebraic multiplicity two.

More detailed information in terms of partial multiplicities at "old" eigenvalues is given in the following theorem. We state the result for the complex field (the case of the real field is subsumed as a particular situation).

Theorem 4.3 *Let $A\in\mathbb{C}^{n\times n}$ be as in Theorem* 2.1. *Then for every $B\in\mathbb{C}^{n\times n}$ with the properties* (a) *and* (b) *below:*

(a) *the rank of B is one;*

(b) *the sum of the algebraic multiplicities of the matrix $A+B$ at its distinct eigenvalues different from $\lambda_1,\dots,\lambda_p$, is at least $n_{1,1}+\cdots+n_{p,1}$;*

the following holds: the matrix $A+B$ has the partial multiplicities $n_{j,2}\ge\cdots\ge n_{j,g_j}$ for the eigenvalue λ_j, for $j=1,2,\dots,p$.

The proof is based on the following result.

Theorem 4.4 *Let*

$$A=\left(\bigoplus_{j=1}^{\ell_1}\mathcal{J}_{n_1}(\lambda)\right)\oplus\cdots\oplus\left(\bigoplus_{j=1}^{\ell_m}\mathcal{J}_{n_m}(\lambda)\right)\oplus\widetilde{A}\in\mathbb{C}^{n\times n},\tag{4.1}$$

where $n_1>\cdots>n_m$, $\widetilde{A}\in\mathbb{C}^{\widetilde{n}\times\widetilde{n}}$, and $\sigma(\widetilde{A})\subseteq\mathbb{C}\setminus\{\lambda\}$. Let $B\in\mathbb{C}^{n\times n}$ be an arbitrary rank one matrix. Then the matrix $A+B$ has at least ℓ_1-1 Jordan chains of lengths at least n_1 and ℓ_i Jordan chains of lengths at least n_i for $i=2,\dots,m$, associated with the eigenvalue λ, and the set of vectors obtained from all those chains is linearly independent.

For the proof of Theorem 4.4, we introduce the following notation. If

$$w = [w_1, \dots, w_n]^T \in \mathbb{C}^n,$$

then we denote by Toep (w, k, p), $p \le n$, the upper triangular $p \times p$ Toeplitz matrix with

$$\begin{bmatrix} w_{k+1} & w_{k+2} & \dots & w_{k+p} \end{bmatrix}.$$

as its first row, where we define $w_j = 0$ for $j > n$. Note that Toep (w, k, p) is invertible if and only if $w_{k+1} \neq 0$. E.g., for $w = [w_1, w_2, w_3, w_4, w_5]^T$, then

$$\mathrm{Toep}\,(w,0,4) = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ 0 & w_1 & w_2 & w_3 \\ 0 & 0 & w_1 & w_2 \\ 0 & 0 & 0 & w_1 \end{bmatrix} \quad \text{and} \quad \mathrm{Toep}\,(w,2,4) = \begin{bmatrix} w_3 & w_4 & w_5 & 0 \\ 0 & w_3 & w_4 & w_5 \\ 0 & 0 & w_3 & w_4 \\ 0 & 0 & 0 & w_3 \end{bmatrix}.$$

We also let $B = uv^T$, and partition v as follows:

$$v = \begin{bmatrix} v^{(1)} \\ \vdots \\ v^{(m)} \\ \widetilde{v} \end{bmatrix}, \; v^{(i)} = \begin{bmatrix} v^{(i,1)} \\ \vdots \\ v^{(i,\ell_i)} \end{bmatrix}, \; v^{(i,j)} = \begin{bmatrix} v_1^{(i,j)} \\ \vdots \\ v_{n_i}^{(i,j)} \end{bmatrix} \in \mathbb{C}^{n_i}, \; j = 1, \dots, \ell_i, \; i = 1, \dots, m. \tag{4.2}$$

Lemma 4.5 *Suppose that Theorem* 4.4 *holds for all matrices of the form* (4.1) *with smaller values of $\ell_1 + \cdots + \ell_m$ and all rank one matrices B' of suitable size. Let the vector $v \in \mathbb{C}^n$ be given, and let it be partitioned as in* (4.2)*. If one of the $v^{(i,j)}$ is the zero vector, then Theorem* 4.4 *also holds for A given by* (4.1) *and for any rank one matrix $B \in \mathbb{C}^{n \times n}$ of the form $B = uv^T$ (where $u \in \mathbb{C}^n$ is non-zero but otherwise arbitrary).*

Proof. As above, let $B = uv^T$, and let v be partitioned as in (4.2). Assume that $v^{(i,k)} = 0$ for some index k, $k \in \{1, 2, \dots, \ell_i\}$ and for some i, $i = 1, 2, \dots, m$. Then setting $p = \ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + (k-1)n_i$, it follows that the columns $p+1, \dots, p+n_i$ of the matrix A coincide with the columns $p+1, \dots, p+n_i$ of the matrix $A+B$. But then a Jordan chain of length at least n_i for the eigenvalue λ of $A+B$ is given by the standard basis vectors $e_{p+1}, \dots, e_{p+n_i}$. We then arrive at the assertion of Lemma 4.5 by applying Theorem 4.4 to the matrix A' obtained from A by deleting the kth Jordan block in $\bigoplus_{j=1}^{\ell_i} \mathcal{J}_{n_i}(\lambda)$, and to the matrix $B' = u'(v')^T$, where v', resp. u', is obtained from v, resp. u, by deleting the vector $v^{(i,k)}$, resp. by deleting the corresponding part of u. $\square$

Proof of Theorem 4.4. In the proof, we will use the fact that in Theorem 4.4 one can apply simultaneous similarity

$$A \;\mapsto\; S^{-1}AS, \quad B \;\mapsto S^{-1}BS = S^{-1}u(S^Tv)^T, \tag{4.3}$$

where the invertible matrix $S \in \mathbb{C}^{n\times n}$ commutes with A, so that $S^{-1}AS = A$, without affecting the hypotheses or the conclusions of the theorem. Note that under transformation (4.3), the vector v is replaced by $S^T v$.

Step 1. In view of Lemma 4.5, and using induction on $\ell_1 + \cdots + \ell_m$, we may (and do) assume that

$$v^{(i,j)} \neq 0, \qquad j = 1, \ldots, \ell_i; \quad i = 1, 2, \ldots, m.$$

(Note that the base of induction, i.e., the case when $m = 1$ and $\ell_1 = 1$ in Theorem 4.4, is trivial.) Let

$$k_i := \min_{j=1,2,\ldots,\ell_i} \left\{ \max \left\{ k \,\middle|\, v_1^{(i,j)} = v_2^{(i,j)} = \cdots = v_k^{(i,j)} = 0 \right\} \right\}, \quad i = 1, \ldots, m.$$

Note that $k_i = 0$ is possible. Then we have $k_i < n_i$, so $v_{k_i+1}^{(i,j)} \neq 0$ for some $j \in \{1, \ldots, \ell_i\}$. Without loss of generality, we may assume that $v_{k_i+1}^{(i,1)} \neq 0$, otherwise we can apply a suitable permutation (as in (4.3)). Let

$$S = S_1 \oplus \cdots \oplus S_m \oplus I_{\widetilde{n}},$$

where

$$S_i := \mathrm{Toep}\,(v^{(i,1)}, k_i, n_i)^{-1} \oplus I_{(\ell_i-1)n_i}, \quad i = 1, \ldots, m.$$

Note that $\mathrm{Toep}\,(v^{(i,1)}, k_i, n_i)$ is invertible and $e_{k_i+1,n_i}^T \mathrm{Toep}\,(v^{(i,1)}, k_i, n_i) = (v^{(i,1)})^T$, where e_{k_i+1,n_i} denotes the (k_i+1)st standard basis column vector of length n_i. Therefore,

$$(v^{(i,1)})^T \mathrm{Toep}\,(v^{(i,1)}, k_i, n_i)^{-1} = e_{k_i+1,n_i}^T.$$

Thus, S is well defined, invertible, $S^{-1}AS = A$ and

$$S^T v = \begin{bmatrix} \check{v}^{(1)} \\ \vdots \\ \check{v}^{(m)} \\ \widetilde{v} \end{bmatrix}, \quad \check{v}^{(i)} = \begin{bmatrix} e_{k_i+1,n_i} \\ v^{(i,2)} \\ \vdots \\ v^{(i,\ell_i)} \end{bmatrix}, \quad i = 1, \ldots, m.$$

Furthermore, note that

$$e_{k_i+1,n_i}^T \mathrm{Toep}\,(v^{(i,j)}, k_i, n_i) = [\,\underbrace{0, \ldots, 0}_{k_i \text{ times}}, v_{k_i+1}^{(i,j)}, \ldots, v_{n_i}^{(i,j)}] = (v^{(i,j)})^T,$$

because by the definition of k_i, all nonzero entries of $v^{(i,j)}$ are among the $v_{k_i+1}^{(i,j)}, \ldots, v_{n_i}^{(i,j)}$. Thus, setting

$$\check{S} = \check{S}_1 \oplus \cdots \oplus \check{S}_m \oplus I_{\widetilde{n}},$$

where

$$\check{S}_i = \begin{bmatrix} I_{n_i} & -\mathrm{Toep}\,(v^{(i,2)}, k_i, n_i) & \dots & -\mathrm{Toep}\,(v^{(i,\ell_i)}, k_i, n_i) \\ 0 & I_{n_i} & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I_{n_i} \end{bmatrix},$$

we obtain that $\check{S}^{-1}A\check{S} = A$, $(S\check{S})^{-1}A(S\check{S}) = A$, and

$$(S\check{S})^T v = \check{S}^T S^T v = \begin{bmatrix} \check{v}^{(1)} \\ \vdots \\ \check{v}^{(m)} \\ \widetilde{v} \end{bmatrix}, \quad \check{v}^{(i)} = \begin{bmatrix} e_{k_i+1,n_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{C}^{\ell_i n_i}, \quad i = 1, \dots, m.$$

Using a suitable transformation (4.3), Lemma 4.5, and induction on $\ell_1 + \cdots + \ell_m$, we may assume therefore that $\ell_1 = \cdots = \ell_m = 1$, and

$$v = \begin{bmatrix} e_{k_1+1,n_1} \\ \vdots \\ e_{k_m+1,n_m} \\ \widetilde{v} \end{bmatrix}. \tag{4.4}$$

Step 2. Assume that there exist indices i_1, i_2 such that $i_1 < i_2$ and $k_{i_1} \leq k_{i_2}$. Then setting T_{i_1,i_2} the $(n_{i_2} + (n_{i_1} - n_{i_2})) \times n_{i_2}$ matrix with as first n_{i_2} rows the matrix $\mathrm{Toep}\,(-e_{k_{i_2}-k_{i_1}+1,n_{i_2}}, 0, n_{i_2})$, and as last $n_{i_1} - n_{i_2}$ rows zeros, that is,

$$T_{i_1,i_2} := \begin{bmatrix} \mathrm{Toep}\,(-e_{k_{i_2}-k_{i_1}+1,n_{i_2}}, 0, n_{i_2}) \\ 0 \end{bmatrix},$$

we obtain that $e^T_{k_{i_1}+1,n_{i_1}} T_{i_1,i_2} = -e^T_{k_{i_2}+1,n_{i_2}}$. Thus, letting

$$\widetilde{T} = T \oplus I_{\widetilde{n}},$$

where T is the $m \times m$ block matrix with $I_{n_1}, \dots, I_{n_m}$ as diagonal blocks, T_{i_1,i_2} as the block in the (i_1, i_2)-block position and zero blocks elsewhere, we obtain from [5, Chapter VIII] that A and $\widetilde{T}$ commute, so $\widetilde{T}^{-1}A\widetilde{T} = A$, and

$$\widetilde{T}^T v = \left[e^T_{k_1+1,n_1}, \dots, e^T_{k_{i_2-1}+1,n_{i_2-1}}, 0, e^T_{k_{i_2+1}+1,n_{i_2+1}}, \dots, e^T_{k_m+1,n_m}\right].$$

Using again Lemma 4.5 and induction on $\ell_1 + \cdots + \ell_m$ as many times as necessary, we reduce the proof to the situation where A and v have the forms

$$A = \mathcal{J}_{n_1}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widetilde{A} \in \mathbb{C}^{n \times n}, \qquad v = \begin{bmatrix} e_{k_1+1,n_1} \\ \vdots \\ e_{k_m+1,n_m} \\ \widetilde{v} \end{bmatrix}, \tag{4.5}$$

where $n_1 > \cdots > n_m$, $\widetilde{A} \in \mathbb{C}^{\widetilde{n}\times\widetilde{n}}$, $\sigma(\widetilde{A}) \subseteq \mathbb{C}\setminus\{\lambda\}$, $\widetilde{v} \in \mathbb{C}^{\widetilde{n}}$, and where *in addition we have* $k_1 > \cdots > k_m$.

Step 3. Finally, we are able to construct the necessary Jordan chains working with A and $B = uv^T$ as in (4.5). Observe that indeed the following are Jordan chains associated with λ of $A+B$:

$$e_1, \ldots, e_{k_1-k_2}, e_{k_1-k_2+1} - e_{n_1+1}, \ldots, e_{k_1+n_2-k_2} - e_{n_1+n_2}; \tag{4.6}$$

$$e_{n_1+1}, \ldots, e_{n_1+k_2-k_3}, e_{n_1+k_2-k_3+1} - e_{n_1+n_2+1}, \ldots, e_{n_1+k_2-k_3+n_3} - e_{n_1+n_2+n_3}; \tag{4.7}$$

and so on, the last Jordan chain being

$$e_{y+1}, \ldots, e_{y+k_{m-1}-k_m}, e_{y+k_{m-1}-k_m+1} - e_{y+n_{m-1}+1}, \ldots, e_{y+k_{m-1}-k_m+n_m} - e_{y+n_{m-1}+n_m}, \tag{4.8}$$

where we have set $y = n_1 + \cdots + n_{m-2}$. So we have constructed a total of $m-1$ Jordan chains, of lengths $k_1 - k_2 + n_2 > n_2$, $k_2 - k_3 + n_3 > n_3$, and so on, $k_{m-1} - k_m + n_m > n_m$, respectively. Notice also that the vectors in the union of chains (4.6) throughout (4.8) are clearly linearly independent. We have satisfied all the requirements of Theorem 4.3 and the proof is complete $\square$

Example 4.6 Let

$$A = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}, \quad v = \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ \hline 1 \\ 0 \end{array}\right].$$

Then $n_1 = 3$, $n_2 = 2$, $k_1 = 1$, $k_2 = 0$. Then the Jordan chain of the matrix

$$A + uv^T = \left[\begin{array}{ccc|cc} 0 & 1+u_1 & 0 & u_1 & 0 \\ 0 & u_2 & 1 & u_2 & 0 \\ 0 & u_3 & 0 & u_3 & 0 \\ \hline 0 & u_4 & 0 & u_4 & 1 \\ 0 & u_5 & 0 & u_5 & 0 \end{array}\right]$$

mentioned in Step 3 of the proof is $e_1, e_2 - e_4, e_3 - e_5$ and it has length $k_1 - k_2 + n_2 = 3$.

Note that the result of Theorem 4.4 is valid for any matrix similar to (4.1). Indeed, use the transformation analogous to (4.3) but with the invertible matrix S not necessarily commuting with A.

Proof of Theorem 4.3. It will be convenient to use the notation am$\,(X, \lambda)$ for the algebraic multiplicity of the eigenvalue λ of a complex matrix X. Under the hypotheses of Theorem 4.3, and using Proposition 4.1, we have

$$\sum \mathrm{am}\,(A+B, \mu) = n_{1,1} + \cdots + n_{p,1}, \tag{4.9}$$

where the sum is taken over all distinct eigenvalues μ of $A+B$ which are different from any of the λ_j's. On the other hand,

$$\text{am}\,(A+B,\lambda_j) \geq n_{j,2} + \cdots + n_{j,g_j}, \qquad j = 1,2,\ldots,p. \tag{4.10}$$

Indeed, (4.10) holds for a generic set U of rank one matrices by Theorem 2.1, and if B does not belong to U, then (4.10) follows by using approximations of B by elements of U (cf. the proof of Proposition 4.1). Combining (4.9) and (4.10) we see that in fact the equalities hold in (4.10), for $j = 1,2,\ldots,p$. It follows that the Jordan chains constructed in Theorem 4.4 (with λ replaced by λ_j) form a basis for the eigenspace of $A+B$ associated with the eigenvalue λ_j. The result of Theorem 4.3 follows. $\square$

5 Conclusion

We have studied the perturbation analysis for real and complex matrices as well as monic matrix polynomials under generic rank one perturbations. We have shown that previous results for the complex case also hold in the real case and we have also proved analogous results for a different form of generic rank one perturbations that is extremely useful in structured perturbations. Furthermore, we have analyzed in detail rank one perturbations for which the total algebraic multiplicity becomes maximal for the perturbed eigenvalues. The resulting Jordan structure of perturbed matrices is fully described.

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