Perturbation theory of selfadjoint matrices and sign characteristics under generic structured rank one perturbations*

Christian Mehl [†] Volker Mehrmann[†] André C. M. Ran[‡]
Leiba Rodman [§]

Dedicated to Heinrich Voss on the occasion of his 65th birthday

Abstract

For selfadjoint matrices in an indefinite inner product, possible canonical forms are identified that arise when the matrix is subjected to a selfadjoint generic rank one perturbation. Genericity is understood in the sense of algebraic geometry. Special attention is paid to the perturbation behavior of the sign characteristic. Typically, under such a perturbation, for every given eigenvalue, the largest Jordan block of the eigenvalue is destroyed and (in case the eigenvalue is real) all other Jordan blocks keep their sign characteristic. The new eigenvalues, i.e. those eigenvalues of the perturbed matrix that are not eigenvalues of the original matrix, are typically simple, and in some cases information is provided about their sign characteristic (if the new eigenvalue is real). The main results are proved by using the well known canonical forms of selfadjoint matrices in an indefinite inner product, a version of the Brunovsky canonical form and on general results concerning rank one perturbations obtained.

Key Words: indefinite inner product, selfadjoint matrices, perturbation analysis, generic perturbation, rank one perturbation.

Mathematics Subject Classification 2010: 15A63, 15B57, 47A55, 47B50.

*This research was supported by *Deutsche Forschungsgemeinschaft*, through the DFG Research Center MATHEON *Mathematics for key technologies* in Berlin.

[†]Technische Universität Berlin, Institut für Mathematik, MA 4-5, Straße des 17. Juni 136, 10623 Berlin, Germany. Email: {mehl,mehrmann}@math.tu-berlin.de.

[‡]Afdeling Wiskunde, Faculteit der Exacte Wetenschappen, Vrije Universiteit Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands. E-mail: ran@few.vu.nl.

[§]College of William and Mary, Department of Mathematics, P.O.Box 8795, Williamsburg, VA 23187-8795, USA. E-mail: lxrodm@math.wm.edu. *The research of this author was supported by Plumeri Award and Faculty Research Assignment at the College of William and Mary. Large part of this work was done while this author visited at Vrije Universiteit Amsterdam and TU Berlin, whose hospitality are gratefully acknowledged.*

1 Introduction

We consider matrices which are selfadjoint with respect to an indefinite inner product structure given by a Hermitian invertible matrix.

Definition 1.1 *Let $H = H^*$ be an invertible Hermitian $n \times n$ complex matrix. An $n \times n$ complex matrix A is called H-*selfadjoint *if $HA = A^*H$. Here H^* denotes the conjugate transpose of the matrix H.*

In this paper we study the perturbation theory of the canonical forms, including the Jordan forms, of such H-selfadjoint matrices. We focus on generic rank one perturbations which in turn are also H-selfadjoint. Our main results derive the possible Jordan forms of the perturbed H-selfadjoint matrix, depending on the canonical form associated with the original selfadjoint matrix and the indefinite inner product. As the sign characteristic is an essential part of the canonical form, we also identify the sign characteristic of the perturbed matrix.

The general perturbation analysis of eigenvalues of general square matrices under generic low rank perturbations, in particular, for rank one perturbations, has been studied in [2, 10, 13, 20, 23, 24]. Motivated by numerous applications, see e.g. [16, 17, 25], the eigenvalue perturbation analysis of generic structured rank one perturbations of matrices with various structures has been studied in [16]; the sense in which "generic" is used is carefully presented in [16]. Here, we continue this line of investigation, and focus on H-selfadjoint matrices. In contrast to [16], where general eigenvalue perturbation results were obtained and several classes of structured complex matrices were investigated, in this paper the sign characteristic of H-selfadjoint matrices and its behavior under H-selfadjoint generic rank one perturbation plays a key role. The analysis of the behavior of the sign characteristic under perturbations is of particular importance in the context of perturbations that perturb a passive system to a nearby non-passive system, because in this application eigenvalues have to be perturbed off the imaginary axis by small norm perturbations, and whether this is possible or not strongly depends on the sign characteristic, see [7, 9, 18].

Our main results are stated in Section 3; the rather long proof of Theorem 3.3 is relegated to Section 4. In Section 5 we investigate the sign characteristic attached to new real eigenvalues of the perturbed matrix, namely those real eigenvalues that are not eigenvalues of the original matrix. Finally, our conclusions are presented in the last section.

The following notation is used throughout the paper. $\mathbb{C}$ and $\mathbb{R}$ stand for the complex and real field, respectively, and we use $\mathbb{F}$ to denote either $\mathbb{C}$ or $\mathbb{R}$. The real, imaginary parts of a complex number λ will be denoted by $\mathrm{Re}(\lambda) = \frac{\lambda+\bar{\lambda}}{2}$, $\mathrm{Im}(\lambda) = \frac{\lambda-\bar{\lambda}}{2i}$, respectively.

The set of positive integers is denoted by $\mathbb{N}$. $\mathcal{J}_m(\lambda)$ denotes an upper triangular $m \times m$ Jordan block with eigenvalue λ and R_m stands for the $m \times m$ matrix with 1 on the leftbottom - topright diagonal and zeros elsewhere, i.e.

$$\mathcal{J}_m(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad R_m = \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & & \iddots & 0 \\ 0 & \iddots & & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}.$$

The k-th standard basis vector of length n will be denoted by $e_{k,n}$ or in short e_k if the length is clear from the context. The spectrum of a matrix $A \in \mathbb{F}^{n\times n}$, i.e. the set of eigenvalues including possibly nonreal eigenvalues of real matrices, is denoted by $\sigma(A)$. An eigenvalue $\lambda \in \sigma(A)$ is said to be *simple* if the corresponding algebraic multiplicity is one, i.e. λ is a simple zero of the characteristic polynomial of A.

A block diagonal matrix with diagonal blocks $X_1, \dots, X_q$ (in that order) is denoted by $X_1 \oplus X_2 \oplus \cdots \oplus X_q$. We will also use the notation $X^{\oplus k}$ for $X \oplus X \oplus \cdots \oplus X$ (k times).

If $v^T = [v_1, \dots, v_n]^T \in \mathbb{C}^n$ then $\mathrm{Toep}\,(v)$ denotes the $n \times n$ upper triangular Toeplitz matrix

$$\mathrm{Toep}\,(v) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \\ 0 & v_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_2 \\ 0 & \dots & 0 & v_1 \end{bmatrix}.$$

If $\mathcal{M} \subseteq \mathbb{F}^m$ is a subspace, we denote by $\mathcal{M}^\perp$ the orthogonal complement of $\mathcal{M}$ with respect to the standard Euclidean metric in $\mathbb{F}^m$.

We say that a set $W \subseteq \mathbb{R}^n$ is *algebraic* if there exists a finite set of polynomials $f_1(x_1, \dots, x_n), \dots, f_k(x_1, \dots, x_n)$ with real coefficients such that a vector $[a_1, \dots, a_n]^T \in \mathbb{R}^n$ belongs to W if and only if

$$f_j(a_1, \dots, a_n) = 0, \quad j = 1, 2, \dots, k.$$

In particular, the empty set is algebraic and $\mathbb{R}^n$ is algebraic. We say that a set $W \subseteq \mathbb{R}^n$ is *generic* if W is not empty and the complement $\mathbb{R}^n \setminus W$ is contained in the union of finitely many algebraic sets which is not $\mathbb{R}^n$.

2 Canonical form, partial Brunovsky form

In this section we recall two known key theorems needed for the proofs of our main results. The first is the well-known canonical form for H-selfadjoint matrices, where H is Hermitian and invertible; see e.g. [7, 9, 14] for details.

Theorem 2.1 *Let $H \in \mathbb{C}^{n\times n}$ be Hermitian and invertible, and let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Then there exists an invertible matrix $P \in \mathbb{C}^{n\times n}$ such that $P^{-1}AP$ and P^*HP are block diagonal matrices*

$$P^{-1}AP = A_1 \oplus A_2, \quad P^*HP = H_1 \oplus H_2, \tag{2.1}$$

where

(i) $$A_1 = A_{1,1} \oplus \cdots \oplus A_{1,\mu}, \quad H_1 = H_{1,1} \oplus \cdots \oplus H_{1,\mu},$$

and

$$A_{1,j} = \mathcal{J}_{n_{j,1}}(\lambda_j) \oplus \cdots \oplus \mathcal{J}_{n_{j,p_j}}(\lambda_j), \quad H_{1,j} = \sigma_{j,1}R_{n_{j,1}} \oplus \cdots \oplus \sigma_{j,p_j}R_{n_{j,p_j}},$$

with $n_{j,1}, \ldots, n_{j,p_j} \in \mathbb{N}$, $n_{j,1} \geq \cdots \geq n_{j,p_j}$, and $\sigma_{j,1}, \ldots, \sigma_{j,p_j} \in \{+1, -1\}$, for $j = 1, \ldots, \mu$ and $\lambda_1, \ldots, \lambda_\mu \in \mathbb{R}$ being pairwise distinct;

(ii) $$A_2 = A_{2,1} \oplus \cdots \oplus A_{2,\nu}, \quad H_2 = H_{2,1} \oplus \cdots \oplus H_{2,\nu}\ ,$$

and

$$A_{2,j} = \begin{bmatrix} \mathcal{J}_{m_{j,1}}(\tau_j) & 0 \\ 0 & \mathcal{J}_{m_{j,1}}(\tau_j)^* \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{m_{j,q_j}}(\tau_j) & 0 \\ 0 & \mathcal{J}_{m_{j,q_j}}(\tau_j)^* \end{bmatrix},$$

$$H_{2,j} = \begin{bmatrix} 0 & I_{m_{j,1}} \\ I_{m_{j,1}} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{m_{j,q_j}} \\ I_{m_{j,q_j}} & 0 \end{bmatrix},$$

with $m_{j,1}, \ldots, m_{j,q_j} \in \mathbb{N}$, $m_{j,1} \geq \cdots \geq m_{j,q_j}$, and $\tau_j \in \mathbb{C}$ with $\mathrm{Im}(\tau_j) > 0$ for $j = 1, \ldots, \nu$. Moreover, $\tau_1, \ldots, \tau_\nu$ are pairwise distinct.

The form (2.1) is uniquely determined by the pair (A, H), up to a simultaneous permutation of diagonal blocks in the right hand sides of (2.1).

The signs $\sigma_{j,1}, \ldots, \sigma_{j,p_j}$, $j = 1, 2, \ldots, \mu$, form the *sign characteristic* of the pair (A, H). Thus, the sign characteristic attaches a sign to every block associated with a real eigenvalue in the canonical form.

The most important tool for obtaining the main results of this paper is the so-called *partial Brunovsky form* developed in [16].

Theorem 2.2 (Partial Brunovsky form, [16, Theorem 2.10]) *Let*

$$A = \left(\mathcal{J}_{n_1}(\widehat{\lambda})^{\oplus \ell_1}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(\widehat{\lambda})^{\oplus \ell_m}\right) \oplus \widetilde{A} \in \mathbb{C}^{n\times n}, \tag{2.2}$$

where $n_1 > \cdots > n_m$ and $\sigma(\widetilde{A}) \subseteq \mathbb{C} \setminus \{\widehat{\lambda}\}$. Moreover, let $a = \ell_1 n_1 + \cdots + \ell_m n_m$ denote the algebraic multiplicity of $\widehat{\lambda}$ and let $B = uv^T$, where $u \in \mathbb{C}^n$ and

$$v = \begin{bmatrix} v^{(1)} \\ \vdots \\ v^{(m)} \\ \widetilde{v} \end{bmatrix}, \quad v^{(i)} = \begin{bmatrix} v^{(i,1)} \\ \vdots \\ v^{(i,\ell_i)} \end{bmatrix}, \quad v^{(i,j)} \in \mathbb{C}^{n_i}, \quad j = 1, \ldots, \ell_i, \quad i = 1, \ldots, m.$$

Assume that the first component of each vector $v^{(i,j)}$, $j = 1, \ldots, \ell_i$, $i = 1, \ldots, m$ is nonzero. Then the following statements hold:

(1) *The inverse of the matrix*

$$S := \left(\bigoplus_{j=1}^{\ell_1} \mathrm{Toep}(v^{(1,j)}) \oplus \cdots \oplus \bigoplus_{j=1}^{\ell_m} \mathrm{Toep}(v^{(m,j)}) \right) \oplus I_{n-a}$$

exists and

$$SAS^{-1} = A, \quad SBS^{-1} = w \left[\underbrace{e^T_{1,n_1}, \ldots, e^T_{1,n_1}}_{\ell_1 \text{ times}}, \ldots, \underbrace{e^T_{1,n_m}, \ldots, e^T_{1,n_m}}_{\ell_m \text{ times}}, z^T \right] \tag{2.3}$$

where $w = Su$, and for some appropriate vector $z \in \mathbb{C}^{n-a}$.

(2) *The matrix $S(A + B)S^{-1}$ has at least $\ell_1 + \cdots + \ell_m - 1$ Jordan chains associated with $\widehat{\lambda}$ given as follows, starting with eigenvectors:*

a) *$\ell_1 - 1$ Jordan chains of length at least n_1:*

$$\begin{array}{lll} e_1 - e_{n_1+1}, & \ldots, & e_{n_1} - e_{2n_1}; \\ \vdots & \ddots & \vdots \\ e_1 - e_{(\ell_1-1)n_1+1}, & \ldots, & e_{n_1} - e_{\ell_1 n_1}; \end{array} \tag{2.4}$$

b) *ℓ_i Jordan chains of length at least n_i for $i = 2, \ldots, m$:*

$$\begin{array}{lll} e_1 - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + 1}, & \ldots, & e_{n_i} - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + n_i}; \\ e_1 - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + n_i + 1}, & \ldots, & e_{n_i} - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + 2n_i}; \\ \vdots & \ddots & \vdots \\ e_1 - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + (\ell_i - 1) n_i + 1}, & \ldots, & e_{n_i} - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + \ell_i n_i}; \end{array} \tag{2.5}$$

The vectors in (2.4), (2.5) are in their totality linearly independent. But generally speaking we do not claim that the vectors in (2.4), (2.5), when multiplied on the left by S^{-1}, form a basis for the root subspace of $A + B$ associated with $\widehat{\lambda}$.

To illustrate Theorem 2.2, let $m = 2$, $\ell_1 = \ell_2 = 2$, $n_1 = 3$, $n_2 = 2$, $\widehat{\lambda} = 0$ and $\widetilde{A}$ empty, in other words,

$$A = \mathcal{J}_3(0) \oplus \mathcal{J}_3(0) \oplus \mathcal{J}_2(0) \oplus \mathcal{J}_2(0) \in \mathbb{C}^{10 \times 10}.$$

Then $S(A+uv^T)S^{-1} = S(A+B)S^{-1}$ has the form

$$\begin{bmatrix} w_1 & 1 & 0 & w_1 & 0 & 0 & w_1 & 0 & w_1 & 0 \\ w_2 & 0 & 1 & w_2 & 0 & 0 & w_2 & 0 & w_2 & 0 \\ w_3 & 0 & 0 & w_3 & 0 & 0 & w_3 & 0 & w_3 & 0 \\ w_4 & 0 & 0 & w_4 & 1 & 0 & w_4 & 0 & w_4 & 0 \\ w_5 & 0 & 0 & w_5 & 0 & 1 & w_5 & 0 & w_5 & 0 \\ w_6 & 0 & 0 & w_6 & 0 & 0 & w_6 & 0 & w_6 & 0 \\ w_7 & 0 & 0 & w_7 & 0 & 0 & w_7 & 1 & w_7 & 0 \\ w_8 & 0 & 0 & w_8 & 0 & 0 & w_8 & 0 & w_8 & 0 \\ w_0 & 0 & 0 & w_9 & 0 & 0 & w_9 & 0 & w_9 & 1 \\ w_{10} & 0 & 0 & w_{10} & 0 & 0 & w_{10} & 0 & w_{10} & 0 \end{bmatrix},$$

where the w_j's are the components of $w = Su$.

3 Main results

In this section we let $H \in \mathbb{C}^{n\times n}$ be Hermitian and invertible, and consider the perturbations of eigenvalues as well as the sign characteristic under generic H-selfadjoint rank one perturbations. We will restrict ourselves to perturbations of the form $B = uu^*H$. Note that rank one perturbations of the form $-uu^*H$ can be treated in a similar fashion, or alternatively consider $-H$ in place of H.

Applying the general results from [16] to this particular situation, we obtain the following result on the effect of generic H-selfadjoint rank one perturbations of H-selfadjoint matrices.

Theorem 3.1 *Let $H \in \mathbb{C}^{n\times n}$ be Hermitian and invertible, let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint, and let $\lambda \in \mathbb{C}$. If A has the Jordan canonical form*

$$\left(\mathcal{J}_{n_1}(\lambda)^{\oplus \ell_1}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(\lambda)^{\oplus \ell_m}\right) \oplus \widetilde{A}, \tag{3.1}$$

where $n_1 > \cdots > n_m$ and where $\sigma(\widetilde{A}) \subseteq \mathbb{C} \setminus \{\lambda\}$ and if $B \in \mathbb{C}^{n\times n}$ is a rank one perturbation of the form $B = uu^*H$, then generically (with respect to $2n$ independent real variables that represent the real and imaginary components of u) the matrix $A+B$ has the Jordan canonical form

$$\left(\mathcal{J}_{n_1}(\lambda)^{\oplus \ell_1 - 1}\right) \oplus \left(\mathcal{J}_{n_2}(\lambda)^{\oplus \ell_2}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(\lambda)^{\oplus \ell_m}\right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}}$ contains all the Jordan blocks of $A+B$ associated with eigenvalues different from λ.

Proof. This follows immediately from Theorem 2.1, and [16, Theorems 3.1, 3.2]. $\square$

Observe that Theorem 3.1 describes the Jordan structure after generic structured rank one perturbations, but does not discuss the canonical form of the pair $(A +$

$uu^*H, H)$ (cf. Theorem 2.1). More precisely, Theorem 3.1 gives no information concerning the relation between the signs in the sign characteristic of (A, H) corresponding to an eigenvalue λ, and the signs in the sign characteristic of the pair $(A + uu^*H, H)$ corresponding to the same eigenvalue λ.

The following example is illustrative.

Example 3.2 Consider the matrices

$$A = 0_{n\times n}, \quad H = \begin{bmatrix} I_{\kappa_+} & 0 \\ 0 & -I_{\kappa_-} \end{bmatrix},$$

where $\kappa_+ + \kappa_- = n$. Then $A + uu^*H = uu^*H$. Assume that $u^*Hu \neq 0$, which is a generic condition. Then u is an eigenvector of $A + uu^*H$ corresponding to the nonzero eigenvalue u^*Hu. Let $v_1, \ldots, v_{n-1}$ be an H-orthogonal basis for $(\mathrm{Span}\{Hu\})^\perp$ (which exists because of Theorem 2.1). The signs in the sign characteristic of (A, H) corresponding to the zero eigenvalue of $A + uu^*H$ are then given by the signs of the numbers $v_i^*Hv_i$, $i = 1, \ldots, n-1$. Considering the basis $u, v_1, \ldots, v_{n-1}$ of $\mathbb{C}^n$ and computing the sign characteristic of H using this basis, we see the following:

	sign characteristic of the eigenvalue zero		sign of the eigenvalue u^*Hu
	# of signs $+1$	# of signs -1	
$u^*Hu > 0$	$\kappa_+ - 1$	κ_-	$+1$
$u^*Hu < 0$	κ_+	$\kappa_- - 1$	-1

It is easy to see that the sets

$$\Omega_+ := \{u \in \mathbb{C}^n \,:\, u^*Hu > 0\}, \quad \Omega_- := \{u \in \mathbb{C}^n \,:\, u^*Hu < 0\}$$

are the two connected components of the set of vectors u for which $u^*Hu \neq 0$. Observe that on each of the components Ω_+ and Ω_-, the sign characteristic of the eigenvalue 0 (of algebraic multiplicity $n-1$) of $A + uu^*H$ is constant (as a function of u), but it is different for the different connected components. □

This situation turns out to be typical, as the following theorem shows. In the theorem, "generically" means "generically with respect to the real and imaginary components of u".

Theorem 3.3 *Let $H \in \mathbb{C}^{n\times n}$ be Hermitian and invertible and let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Assume that the pair (A, H) has the canonical form $(\widehat{A}, \widehat{H})$ with*

$$\begin{aligned} \widehat{A} \;=\; & \bigoplus_{j=1}^{\mu} \Big(\big(\mathcal{J}_{n_{1,j}}(\lambda_j)^{\oplus \ell_{1,j}}\big) \oplus \big(\mathcal{J}_{n_{2,j}}(\lambda_j)^{\oplus \ell_{2,j}}\big) \oplus \cdots \oplus \Big(\mathcal{J}_{n_{m_j,j}}(\lambda_j)^{\oplus \ell_{m_j,j}}\Big)\Big) \\ & \oplus \bigoplus_{j=1}^{\nu} \left(\bigoplus_{s=1}^{q_j} \begin{bmatrix} \mathcal{J}_{k_{s,j}}(\tau_j) & 0 \\ 0 & \mathcal{J}_{k_{s,j}}(\tau_j)^* \end{bmatrix} \right), \end{aligned} \tag{3.2}$$

where $\lambda_j \in \mathbb{R}$, $n_{1,j} > \cdots > n_{m_j,j}$, $j = 1, \ldots, \mu$, *and* $\tau_j \in \mathbb{C} \setminus \mathbb{R}$, $k_{1,j} \geq \cdots \geq k_{q_j,j}$, $j = 1, \ldots, \nu$ *(note that we group together Jordan blocks of the same size for real eigenvalues* λ_j*, but not so for nonreal eigenvalues), and with*

$$\widehat{H} = \bigoplus_{j=1}^{\mu} \left(\left(\bigoplus_{s=1}^{\ell_{1,j}} \sigma_{1,s,j} R_{n_{1,j}} \right) \oplus \left(\bigoplus_{s=1}^{\ell_{2,j}} \sigma_{2,s,j} R_{n_{2,j}} \right) \oplus \cdots \oplus \left(\bigoplus_{s=1}^{\ell_{m_j,j}} \sigma_{m_j,s,j} R_{n_{m_j,j}} \right) \right) \oplus \bigoplus_{j=1}^{\nu} \left(\bigoplus_{s=1}^{q_j} \begin{bmatrix} 0 & I_{k_{s,j}} \\ I_{k_{s,j}} & 0 \end{bmatrix} \right),$$

where $\sigma_{i,s,j} \in \{+1, -1\}$, $s = 1, \ldots, \ell_{i,j}$, $i = 1, \ldots, m_j$, $j = 1, \ldots, \mu$. *If* $B \in \mathbb{C}^{n \times n}$ *is a rank one perturbation of the form* $B = uu^*H$*, then:*

(a) *generically the pair* $(A + B, H)$ *has the canonical form* (A', H')*, given by*

$$A' = \bigoplus_{j=1}^{\mu} \left(\left(\mathcal{J}_{n_{1,j}}(\lambda_j)^{\oplus \ell_{1,j}-1} \right) \oplus \left(\mathcal{J}_{n_{2,j}}(\lambda_j)^{\oplus \ell_{2,j}} \right) \oplus \cdots \oplus \left(\mathcal{J}_{n_{m_j,j}}(\lambda_j)^{\oplus \ell_{m_j,j}} \right) \right) \oplus \bigoplus_{j=1}^{\nu} \left(\bigoplus_{s=2}^{q_j} \begin{bmatrix} \mathcal{J}_{k_{s,j}}(\tau_j) & 0 \\ 0 & \mathcal{J}_{k_{s,j}}(\tau_j)^* \end{bmatrix} \right) \oplus A_3',$$

$$H' = \bigoplus_{j=1}^{\mu} \left(\left(\bigoplus_{s=1}^{\ell_{1,j}-1} \sigma'_{1,s,j} R_{n_{1,j}} \right) \oplus \left(\bigoplus_{s=1}^{\ell_{2,j}} \sigma_{2,s,j} R_{n_{2,j}} \right) \oplus \cdots \oplus \left(\bigoplus_{s=1}^{\ell_{m_j,j}} \sigma_{m_j,s,j} R_{n_{m_j,j}} \right) \right) \oplus \bigoplus_{j=1}^{\nu} \left(\bigoplus_{s=2}^{q_j} \begin{bmatrix} 0 & I_{k_{s,j}} \\ I_{k_{s,j}} & 0 \end{bmatrix} \right) \oplus H_3',$$

where A_3' *consists of Jordan blocks with eigenvalues different from the eigenvalues of* A*, and where the list* $(\sigma'_{1,1,j}, \ldots, \sigma'_{1,\ell_{1,j}-1,j})$ *is obtained from* $(\sigma_{1,1,j}, \ldots, \sigma_{1,\ell_{1,j},j})$ *by removing either exactly one sign* $+1$ *or exactly one sign* -1*;*

(b) *generically all eigenvalues of* $A + uu^*H$ *which are not eigenvalues of* A *are simple;*

(c) *let* $\Omega \subseteq \mathbb{C}^n$ *be the generic (with respect to the real and imaginary parts of vectors) set such that for every* $u \in \Omega$ *properties* (a) *and* (b) *hold. Then, within each connected component* Ω_0 *of* Ω*, the sign characteristic of the pair* $(A + uu^*H, H)$*,* $u \in \Omega_0$*, corresponding to those among the* λ_j*'s that are eigenvalues of* $A + uu^*H$*, is constant, and the sign characteristic of any simple real eigenvalue* $\gamma = \gamma(u)$ *of* $A + uu^*H$ *which is different from the* λ_j*'s is also constant, assuming* $\gamma(u)$ *is chosen to be continuous function of* $u \in \Omega_0$*.*

We see in Theorem 3.3 that the sign characteristic of the pair $(A + B, H)$ for the eigenvalue λ_j is the same as this for (A, H), except that, for the set of Jordan blocks with eigenvalue λ_j and maximal size, one sign is dropped.

The rather long proof of Theorem 3.3 will be given in Section 4.

4 Proof of Theorem 3.3

Proof of parts (a) and (c). First note that the Jordan canonical form of $A+B$ in part (a) follows by applying Theorem 3.1 to each eigenvalue of A and taking advantage of the fact that the intersection of finitely many generic sets is again generic. We next show the part of the assertion concerning the sign characteristic. To this end, pick a fixed eigenvalue $\lambda_j = \widehat{\lambda}$ and assume without loss of generality that the pair (A, H) is in canonical form, where the diagonal blocks have been permuted in such a way that the blocks associated with $\widehat{\lambda}$ come first.

For simplicity, let $n_i := n_{i,j}$, $\ell_i := \ell_{i,j}$, $m := m_j$, and $\sigma_{i,s} := \sigma_{i,s,j}$, i.e. A and H have the forms

$$\begin{aligned} A &= \left(\mathcal{J}_{n_1}(\widehat{\lambda})^{\oplus \ell_1}\right) \oplus \left(\mathcal{J}_{n_2}(\widehat{\lambda})^{\oplus \ell_2}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(\widehat{\lambda})^{\oplus \ell_m}\right) \oplus \breve{A}, \\ H &= \left(\bigoplus_{i=1}^{\ell_1} \sigma_{1,i} R_{n_1}\right) \oplus \left(\bigoplus_{i=1}^{\ell_2} \sigma_{2,i} R_{n_2}\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{\ell_m} \sigma_{m,i} R_{n_m}\right) \oplus \breve{H}, \end{aligned}$$

where $\breve{A}$ contains all the blocks associated with eigenvalues different from $\widehat{\lambda}$. Let

$$u = \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(m)} \\ \widetilde{u} \end{bmatrix}, \quad u^{(i)} = \begin{bmatrix} u^{(i,1)} \\ \vdots \\ u^{(i,\ell_i)} \end{bmatrix}, \quad u^{(i,k)} = \begin{bmatrix} u_1^{(i,k)} \\ \vdots \\ u_{n_i}^{(i,k)} \end{bmatrix} \in \mathbb{C}^{n_i}, \quad \widetilde{u} \in \mathbb{C}^{n-a},$$

where $a = \sum_{i=1}^{m} \ell_i n_i$ denotes the algebraic multiplicity of the eigenvalue $\widehat{\lambda}$. By Theorem 2.2, the transformation matrix S that brings $A+B$ into partial Brunovsky form takes the form $S = \widehat{S} \oplus I_{n-a}$, where

$$\widehat{S} = \left(\bigoplus_{i=1}^{\ell_1} \operatorname{Toep}\left(\sigma_{1,i} R_{n_1} \overline{u^{(1,i)}}\right)\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{\ell_m} \operatorname{Toep}\left(\sigma_{m,i} R_{n_m} \overline{u^{(m,i)}}\right)\right).$$

Note that the inverse of the matrix S exists if $u_{n_i}^{(k,i)} \neq 0$ for $k = 1, \ldots, \ell_i$, $i = 1, \ldots, m$ which is generically (in the sense of the theorem) the case. Now $S(A+B)S^{-1}$ is in partial Brunovsky form (2.3) and $S^{-*}HS^{-1}$-selfadjoint, where

$$S^{-*}HS^{-1} = \widehat{H} \oplus \widetilde{H}, \quad \widehat{H} = \left(\bigoplus_{i=1}^{l_1} H^{(1,i)}\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{l_m} H^{(m,i)}\right)$$

and where each $H^{(k,i)} \in \mathbb{C}^{n_i \times n_i}$ takes the form

$$H^{(k,i)} = \begin{bmatrix} 0 & \ldots & 0 & \sigma_{k,i} |u_{n_i}^{(k,i)}|^{-2} \\ \vdots & \iddots & \iddots & * \\ 0 & \iddots & \iddots & \vdots \\ \sigma_{k,i} |u_{n_i}^{(k,i)}|^{-2} & * & \ldots & * \end{bmatrix}. \tag{4.1}$$

Note that by Theorem 3.1 the algebraic multiplicity of the eigenvalue $\widehat{\lambda}$ of $A+B$ is $a-n_1$, thus the Jordan chains (2.4) and (2.5) form a basis of the corresponding root space.

We are now going to compute the sign characteristic of the eigenvalue $\widehat{\lambda}$ of $(A+B)$. We do this by using the description of the sign characteristic given in [9, Section 5.8] (see also alternative "second description" in [7]). Thus, let $\Psi_1 = \operatorname{Ker}(A-\widehat{\lambda}I_n)$ and let $\nu(x)$ be the maximal length of a Jordan chain of A beginning with the eigenvector $x \in \Psi_1 \setminus \{0\}$, and let Ψ_i denote the subspace of Ψ_1 spanned by all $x \in \Psi_1$ with $\nu(x) \geq i$, $i = 1, \ldots, n_1$. Observe that

$$\Psi_1 \supseteq \Psi_2 \supseteq \cdots \supseteq \Psi_{n_1}$$

and

$$\dim \Psi_{n_1} = \ell_1 - 1,\ \dim \Psi_{n_2} = \ell_1 - 1 + \ell_2,\ \ldots,\ \dim \Psi_{n_i} = \ell_1 - 1 + \ell_2 + \cdots + \ell_i.$$

Finally let

$$f_i(x,y) := x^* H y^{(i)}, \quad x \in \Psi_i, \quad y \in \Psi_i \setminus \{0\},$$

where $y = y^{(1)}, y^{(2)}, \ldots, y^{(i)}$ is a Jordan chain of A associated with $\widehat{\lambda}$ with the eigenvector y, and let $f_i(x,0) = 0$. Then by [9, Theorem 5.8.1] the value $f_i(x,y)$ does not depend on the choice of $y^{(2)}, \ldots, y^{(i)}$. Furthermore, there exists a selfadjoint linear transformation $G_i : \Psi_i \to \Psi_i$ such that

$$f_i(x,y) = x^* G_i y \quad \text{for all } x, y \in \Psi_i$$

and the number of positive (negative) eigenvalues of G_i, counting multiplicities, coincides with the number of positive (negative) signs in the sign characteristic of (A,H) corresponding to the Jordan blocks of size i associated with the eigenvalue $\widehat{\lambda}$. Thus, it remains to calculate the signature of a matrix representation M_{n_i} of G_{n_i} for $i = 1, \ldots, m$ in order to compute the sign characteristic of $\widehat{\lambda}$. Note that there are $\ell_1 - 1 + \ell_2 + \cdots + \ell_i$ eigenvectors in the chains (2.4) and (2.5) which are in Ψ_{n_i}, so these eigenvectors form a basis of Ψ_{n_i} and we will compute the matrix representation M_{n_i} with respect to this basis. First let $i > 1$. Note that by [9, Theorem 5.8.1 (iii)] we have $\operatorname{Ker} G_{n_i} = \Psi_{n_i+1}$, so it is sufficient to consider those basis vectors that are in Ψ_{n_i}, but not in Ψ_{n_i+1}, i.e. the vectors

$$e_1 - e_{\eta_{i,1}+1},\ e_1 - e_{\eta_{i,2}+1}, \ldots,\ e_1 - e_{\eta_{i,\ell_i}+1},$$

where $\eta_{i,k} := \ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + (k-1) n_i$, $k = 1, \ldots, \ell_i$. Then, the (κ, π)-entry of M_{n_i} is given by

$$\begin{aligned} f_{n_i}(e_1 - e_{\eta_{i,\kappa}+1}, e_1 - e_{\eta_{i,\pi}+1}) &= (e_1 - e_{\eta_{i,\kappa}+1})^* S^{-*} H S^{-1} (e_{n_i} - e_{\eta_{i,\pi}+n_i}) \\ &= \begin{cases} 0 & \text{if } \kappa \neq \pi, \\ e^*_{\eta_{i,\kappa}+1} H^{(i,\kappa)} e_{\eta_{i,\kappa}+n_i} = \sigma_{i,\kappa} |u^{(i,\kappa)}_{n_i}|^{-2} & \text{if } \kappa = \pi, \end{cases} \end{aligned}$$

because $S^{-*}HS^{-1}$ is block diagonal and, since $e_1^* H^{(1,1)} e_{n_i} = 0$, because $H^{(1,1)} \in \mathbb{C}^{n_1 \times n_1}$ has the special form (4.1) and $n_i < n_1$. Thus, M_{n_i} is diagonal and the number of positive (negative) eigenvalues of M_{n_i} equals the number of positive (negative) signs among $\sigma_{i,1}, \ldots, \sigma_{i,\ell_i}$. This means that the sign characteristic of $(A+B, H)$ corresponding to the blocks of size n_i associated with the eigenvalue $\widehat{\lambda}$ is the same as that for (A, H).

For $i = 1$, setting $\eta_{1,k} := \ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + k n_i$, $k = 1, \ldots, \ell_1 - 1$ we similarly obtain that the (κ, π)-entry of the $(\ell_1 - 1) \times (\ell_1 - 1)$ matrix M_{n_1} takes the form

$$\begin{aligned} f_{n_1}(e_1 - e_{\eta_{i,\kappa}+1}, e_1 - e_{\eta_{1,\pi}+1}) &= (e_1 - e_{\eta_{1,\kappa}+1})^* S^{-*} H S^{-1} (e_{n_1} - e_{\eta_{1,\pi}+n_1}) \\ &= \begin{cases} \sigma_{1,1} |u_{n_1}^{(1,1)}|^{-2} & \text{if } \kappa \neq \pi, \\ \sigma_{1,1} |u_{n_1}^{(1,1)}|^{-2} + \sigma_{1,\kappa} |u_{n_1}^{(1,\kappa)}|^{-2} & \text{if } \kappa = \pi. \end{cases} \end{aligned}$$

Thus, we have

$$M_{n_1} = \begin{bmatrix} \sigma_{1,2} |u_{n_1}^{(1,2)}|^{-2} & & 0 \\ & \ddots & \\ 0 & & \sigma_{1,\ell_1} |u_{n_1}^{(1,\ell_1)}|^{-2} \end{bmatrix} + \sigma_{1,1} |u_{n_1}^{(1,1)}|^{-2} \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix},$$

i.e. M_{n_1} is a Hermitian rank one perturbation of a Hermitian diagonal matrix. The result then follows using an interlacing theorem which is a special case of Weyl's Theorem on eigenvalues. Indeed, assuming that M_{n_1} is invertible (which is a generic condition with respect to the real and imaginary parts of the components of u), let there be π signs $+1$ among $\sigma_{1,1}, \ldots, \sigma_{1,\ell_1}$. Then by [11, Corollary 4.3.3 and Theorem 4.3.4] it is guaranteed that M_{n_1} has at least $\pi - 1$ and at most π positive eigenvalues. Thus, the sign characteristic of $(A + B, H)$ corresponding to the Jordan blocks of size n_1 associated with the eigenvalue $\widehat{\lambda}$ is the same as that for (A, H), except that exactly one sign is dropped.

Part (c) follows from results on perturbation of sign characteristic [22, Theorem 3.6], [3]. □

It remains to prove part (b) of the theorem. In the proof, the following two examples of matrices Z and their characteristic polynomials $\chi(Z) = \det(xI - Z)$ will be used. The first example is well known.

Example 4.1 Let

$$Z^{(1)}(\lambda, \alpha) = \begin{bmatrix} \lambda & 1 & \ldots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ \alpha & \ldots & 0 & \lambda \end{bmatrix} = \mathcal{J}_m(\lambda) + \alpha e_m e_m^T R_m \in \mathbb{C}^{m \times m}, \quad \lambda \in \mathbb{C},\ \alpha \in \mathbb{C} \setminus \{0\}.$$

Then $\chi\big(Z^{(1)}(\lambda, \alpha)\big) = (x - \lambda)^m - \alpha$; in particular, $Z^{(1)}(\lambda, \alpha)$ has m distinct eigenvalues. □

Example 4.2 Let

$$\begin{aligned} Z^{(2)}(\tau,\alpha) &= \begin{bmatrix} \mathcal{J}_m(\tau) & 0 \\ 0 & \mathcal{J}_m(\tau)^* \end{bmatrix} + \begin{bmatrix} \alpha e_m \\ \alpha e_1 \end{bmatrix} \begin{bmatrix} \alpha e_m \\ \alpha e_1 \end{bmatrix}^* \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \in \mathbb{C}^{2m\times 2m}, \\ & \tau \in \mathbb{C}, \quad \operatorname{Im}\tau > 0, \quad \alpha \in \mathbb{C}\setminus\{0\}. \end{aligned}$$

Using the Laplace expansion theorem for determinants with respect to the first m rows of $\det(xI - Z^{(2)}(\tau,\alpha))$, and omitting terms that are obviously vanishing, we find

$$\begin{aligned} & \chi\big(Z^{(2)}(\tau,\alpha)\big) = \chi\big(Z^{(1)}(\tau,|\alpha|^2)\big)\chi\big(Z^{(1)}(\overline{\tau},|\alpha|^2)\big) - |\alpha|^4 \\ = \ & \big((x-\tau)^m - |\alpha|^2\big)\big((x-\overline{\tau})^m - |\alpha|^2\big) - |\alpha|^4. \end{aligned}$$

Elementary calculations show that $Z^{(2)}(\tau,\alpha)$ is guaranteed to have $2m$ distinct simple eigenvalues if α is chosen so that

$$|\alpha|^2 < \frac{|\overline{\tau}-\tau|^m}{2}.$$

Indeed, assuming that x_0 is a common zero of $\chi\big(Z^{(2)}(\tau,\alpha)\big)$ and of $\frac{\partial}{\partial x}\chi\big(Z^{(2)}(\tau,\alpha)\big)$, we have (with $\beta = |\alpha|^2$):

$$(x_0-\tau)^m(x_0-\overline{\tau})^m - \beta(x_0-\tau)^m - \beta(x_0-\overline{\tau})^m = 0, \tag{4.2}$$

$$(x_0-\tau)^{m-1}(x_0-\overline{\tau})^m + (x_0-\tau)^m(x_0-\overline{\tau})^{m-1} - \beta(x_0-\tau)^{m-1} - \beta(x_0-\overline{\tau})^{m-1} = 0. \tag{4.3}$$

Multiplying (4.3) by $x_0-\tau$ and using (4.2), after simple algebraic manipulations, we get

$$(x_0-\tau)^{m+1} = \beta(\overline{\tau}-\tau).$$

Analogously,

$$(x_0-\overline{\tau})^{m+1} = \beta(\tau-\overline{\tau}).$$

These two identities are contradictory if β is sufficiently small, namely if $\beta < \frac{|\overline{\tau}-\tau|^m}{2}$. □

We denote by Ω' the generic set of vectors $u \in \mathbb{C}^n$ for which Theorem 3.3 (a) holds. We may assume Ω' is open.

Lemma 4.3 *Let Ω' be the generic set of vectors $u \in \mathbb{C}^n$ for which Theorem 3.3 (a) holds. Then there exists $\epsilon > 0$ and an open dense (in $\{u \in \mathbb{C}^n : \|u\| < \epsilon\}$) set $\Omega'' \subseteq \Omega'$ such that for every $u \in \Omega''$, $\|u\| < \epsilon$, the Jordan form of $A+uu^*H$ is as in Theorem 3.3, where A_3 consists of simple eigenvalues different from any of the λ_j's and from any of the τ_k's and $\overline{\tau_k}$'s.*

Proof. The proof follows the same approach as that of [16, Lemma 2.5]. However, additional considerations are needed here, due to the presence of paired nonreal eigenvalues τ_j, $\overline{\tau_j}$.

Denote by $D(z, \epsilon)$ the closed disc of radius ϵ centered at $z \in \mathbb{C}$. Let $\epsilon > 0$ be so small that for every $u \in \mathbb{C}^n$ with $\|u\| < \epsilon$, all eigenvalues of $A + B$ lie within the union of the closed pairwise nonintersecting discs of radius ϵ centered at each of the points $\lambda_1, \ldots, \lambda_\mu$, $\tau_1, \overline{\tau_1}, \ldots, \tau_\nu, \overline{\tau_\nu}$. We also suppose that ϵ is sufficiently small, namely that

$$\left(\frac{1}{2}\epsilon^n\right)^2 < \frac{1}{2} \min_{k=1,2,\ldots,\nu} \{|\tau_k - \overline{\tau_k}|^s, \quad s = 1, 2, \ldots, n\}.$$

(This is to make sure that in a subsequent application of Example 4.2 the values of the parameter α in that example are such that the simplicity of the relevant eigenvalues is guaranteed.) It will be assumed from now on in the proof that $\|u\| < \epsilon$.

Let $\chi(\lambda_j, u)$ for $j = 1, 2, \ldots, \mu$, and $\chi(\tau_k, u)$, $\chi(\overline{\tau_k}, u)$ for $k = 1, 2, \ldots, \nu$, be the characteristic polynomials of the independent variable x for the restrictions of $A + B$ to its spectral invariant subspaces corresponding to the eigenvalues of $A+B$ within the disc $D(\lambda_j, \epsilon)$ (or the discs $D(\tau_k, \epsilon)$, $D(\overline{\tau_k}, \epsilon)$, respectively). Notice that the coefficients of $\chi(\lambda_j, u)$, $\chi(\tau_k, u)$, $\chi(\overline{\tau_k}, u)$ are real analytic functions of the real and imaginary parts of u. Indeed, this follows from the formula for the projection onto the spectral invariant subspace

$$\frac{1}{2\pi i} \int_\Gamma (zI - (A + B))^{-1} dz,$$

for a suitable closed simple contour Γ.

Let $q(\lambda_j, u)$, resp., $q(\tau_k, u)$, be the number of distinct eigenvalues of $A + B$ in the disc $D(\lambda_j, \epsilon)$, resp., $D(\tau_k, \epsilon)$. (We need not consider separately the number of distinct eigenvalues of $A + B$ in the disc $D(\overline{\tau_k}, \epsilon)$, since it is equal to $q(\tau_k, u)$ in view of the H-selfadjointness of $A + B$.) Let

$$q_{\max}(\lambda_j) = \max_{u \in \mathbb{C}^n, \quad \|u\| < \epsilon} \{q(\lambda_j, u)\}, \quad q_{\max}(\tau_k) = \max_{u \in \mathbb{C}^n, \quad \|u\| < \epsilon} \{q(\tau_k, u)\}.$$

Next, we fix λ_j, and denote by $\mathcal{S}(p_1, p_2)$ the Sylvester resultant matrix of the two polynomials $p_1(x)$, $p_2(x)$ (see e.g. [1, 5]); note that $\mathcal{S}(p_1, p_2)$ is of square size degree (p_1) + degree (p_2) and recall the well known fact (see [15] for example) that the rank deficiency of $p_1(x)$, $p_2(x)$ coincides with the degree of the greatest common divisor of the polynomials $p_1(x)$ and $p_2(x)$. We have

$$q(\lambda_j, u) = \operatorname{rank} \mathcal{S}\left(\chi(\lambda_j, u), \frac{\partial \chi(\lambda_j, u)}{\partial x}\right) - (n_{1,j} + \cdots + n_{m_j,j}) + 1.$$

The entries of $\mathcal{S}(\chi(\lambda_j, u), \frac{\partial \chi(\lambda_j, u)}{\partial x})$ are scalar (independent of u) multiples of the coefficients of $\chi(\lambda_j, u)$, and therefore the set $Q(\lambda_j)$ of all vectors $u \in \mathbb{C}^n$, $\|u\| < \epsilon$, for which $q(\lambda_j, u) = q_{\max}(\lambda_j)$ is the complement of the set of common zeros of finitely many real analytic functions of the real and imaginary parts of u. In particular, $Q(\lambda_j)$ is open and dense in $\{u \in \mathbb{C}^n : \|u\| < \epsilon\}$.

On the other hand, still for a fixed λ_j, consider

$$u_0 := \frac{1}{2}\epsilon^n \begin{bmatrix} u_{1,1} \\ \vdots \\ u_{1,\mu} \\ u_{2,1} \\ \vdots \\ u_{2,\nu} \end{bmatrix},$$

partitioned conformably with the partitioning in (3.2), where all the entries $u_{1,i}$ and $u_{2,k}$ are zero, except for $u_{1,j}$ which has 1 in the $n_{1,j}$th position and zeros elsewhere. One checks easily (cf. Example 4.1) that in the disc $D(\lambda_j, \epsilon)$ the matrix $A + u_0 u_0^* H$ has:

(1) $n_{1,j}$ simple eigenvalues different from λ_j; and

(2) the eigenvalue λ_j with partial multiplicities $\ell_{1,j} - 1$ times $n_{1,j}$ and $\ell_{i,j}$ times $n_{i,j}$, $i = 2, \ldots, m_j$.

If by chance u_0 is not in Ω', then we slightly perturb u_0 to obtain a new vector $u_0' \in \Omega'$ such that (1) and (2) are still valid for the matrix $A + u_0'(u_0')^* H$. (This is possible because Ω' is generic, the property of eigenvalues being simple persists under small perturbations, and the total number of eigenvalues of $A + uu^*H$ within $D(\lambda_j, \epsilon)$, counted with multiplicities, is equal to $n_{1,j} + \cdots + n_{m_j,j}$, for every $u \in \mathbb{C}^n$, $\|u\| < \epsilon$.) Since Ω' is open, clearly there exists $\delta > 0$ such that (1) and (2) are valid for every $A + uu^*H$, where $u \in \mathbb{C}^n$ and $\|u - u_0\| < \delta$. Since the set of all such u's is open in $\mathbb{C}^n$, it follows from the properties of the set $Q(\lambda_j)$ established in the preceding paragraph that in fact we have

$$q(\lambda_j, u) = q_{\max}(\lambda_j), \quad \text{for all } \ u \in \mathbb{C}^n, \quad \|u - u_0\| < \delta.$$

So for the following open dense (in $\{u \in \mathbb{C}^n : \|u\| < \epsilon\}$) set

$$\Omega_j^{(1)} := Q(\lambda_j) \cap \Omega'$$

the following property holds: For every $u \in \Omega_j^{(1)}$, the part of the Jordan form of $A + uu^*H$ corresponding to the eigenvalues within $D(\lambda_j, \epsilon)$ consists of

$$\left(\mathcal{J}_{n_{1,j}}(\lambda_j)^{\oplus \ell_{1,j}-1}\right) \oplus \left(\mathcal{J}_{n_{2,j}}(\lambda_j)^{\oplus \ell_{2,j}}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_{m_j,j}}(\lambda_j)^{\oplus \ell_{m_j,j}}\right)$$

and $n_{1,j}$ simple eigenvalues different from λ_j.

Apply now a similar argument to the blocks associated with nonreal eigenvalues (τ_j, τ_j^*) for a fixed j ($j = 1, 2, \ldots, \nu$), using instead of u_0 the vector

$$u_0' := \frac{1}{2}\epsilon^n \begin{bmatrix} u_{1,1}' \\ \vdots \\ u_{1,\mu}' \\ u_{2,1}' \\ \vdots \\ u_{2,\nu}' \end{bmatrix},$$

partitioned conformably with the partitioning in (3.2), where all the entries $u_{1,i}'$ and $u_{2,\ell}'$ are zeros except for $u_{2,j}'$ which has 1 in the $k_{1,j}$th and $k_{1,j}+1$th positions and zeros elsewhere. Note that the $2k_{1,j} \times 2k_{1,j}$ matrix

$$\Phi(\alpha) := \begin{bmatrix} \mathcal{J}_{k_{1,j}}(\tau_j) & 0 \\ 0 & \mathcal{J}_{k_{1,j}}(\tau_j)^* \end{bmatrix} + \begin{bmatrix} \alpha e_{k_{1,j}} \\ \alpha e_1 \end{bmatrix} \cdot \begin{bmatrix} \alpha e_{k_{1,j}} \\ \alpha e_1 \end{bmatrix}^* \begin{bmatrix} 0 & I_{k_{1,j}} \\ I_{k_{1,j}} & 0 \end{bmatrix}$$

has $2k_{1,j}$ (necessarily simple) distinct eigenvalues none of which is equal to τ_j, $\overline{\tau_j}$, for every complex $\alpha \neq 0$ with $|\alpha| \neq 1$. (See Example 4.2.) Consequently, in the union of the discs $D(\tau_j, \epsilon) \cup D(\overline{\tau_j}, \epsilon)$ the matrix $A + u_0'(u_0')^* H$ has:

(1) $k_{1,j}$ simple eigenvalues different from $\tau_j, \overline{\tau_j}$; and

(2) the eigenvalues $\tau_j, \overline{\tau_j}$ each with partial multiplicities $k_{2,j}, \ldots, k_{q_j,j}$.

As a consequence we obtain an open dense (in $\{u \in \mathbb{C}^n : \|u\| < \epsilon\}$) set $\Omega_j^{(2)}$ such that the part of Jordan form of $A + uu^*H$, where $u \in \Omega_j^{(2)}$, corresponding to the eigenvalues within $D(\tau_j, \epsilon) \cup D(\overline{\tau_j}, \epsilon)$ consists of (more precisely, is similar to)

$$\begin{bmatrix} \mathcal{J}_{k_{2,j}}(\tau_j) & 0 \\ 0 & \mathcal{J}_{k_{2,j}}(\tau_j)^* \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{k_{q_j,j}}(\tau_j) & 0 \\ 0 & \mathcal{J}_{k_{q_j,j}}(\tau_j)^* \end{bmatrix}$$

and $2k_{1,j}$ simple eigenvalues different from $\tau_j, \overline{\tau_j}$.

Now let

$$\Omega'' = \left(\cap_{j=1}^{\mu} \Omega_j^{(1)}\right) \cap \left(\cap_{j=1}^{\nu} \Omega_j^{(2)}\right) \cap \Omega'$$

to satisfy Lemma 4.3. $\square$

Proof of part (b). Let $\chi_u(x)$ be the characteristic polynomial (in the independent variable x) of $A + B = A + uu^*H$. Then the number of distinct roots of $\chi_u(x)$ is given by the rank of the Sylvester resultant matrix $\mathcal{S}(\chi_u(x), \frac{\partial}{\partial x}\chi_u(x))$ minus $n-1$ (cf. the proof of Lemma 4.3). Therefore, the set Ω_0 of all vectors u on which the number of distinct roots of $\chi_u(x)$ is maximal, is a generic set. By Lemma 4.3, the maximal number of distinct roots of $\chi_u(x)$ is equal to

$$n_{1,1} + \cdots + n_{p,1} + \sum_{j=1}^{p} \min\{g_j - 1, 1\}.$$

Thus, for the generic set $U = \Omega_0 \cap \Omega'$ the Jordan structure of $A + uu^*H$ is described by (a) and (b), as required. $\square$

5 Local behavior of the sign characteristic: new real eigenvalues

We continue our study of the local behavior of the sign characteristic of H-selfadjoint matrices under generic H-selfadjoint rank one perturbations. In Section 3 we have considered the real eigenvalues of the perturbed matrices that are also the eigenvalues of the original matrix. Here, we consider "new" real eigenvalues of the perturbed matrix - those that are not eigenvalues of the original matrix - under *small* generic rank one perturbations. To this end we use the description of the sign characteristic in terms of "analytic eigenvalues". This technique was used in [6, 7], and in more general contexts in [8, 21]. We provide the necessary background in the next subsection.

5.1 Analytic eigenvalues and sign characteristics

Let $H \in \mathbb{C}^{n\times n}$ be Hermitian and invertible, and let A be H-selfadjoint. The function $xH - HA$ of the real variable x clearly takes Hermitian matrix values. It is well known (Rellich's theorem, see e.g. [12], a proof can be also found in [6, Chapter S.6]) that the eigenvalues $\mu_1(x), \ldots, \mu_n(x)$ of $xH - HA$ can be enumerated so that they become real analytic functions of x. So let $\mu_1^A(x), \ldots, \mu_n^A(x)$ be the eigenvalues of $xH - HA$, for every $x \in \mathbb{R}$, and assume that they are analytic functions of x. Clearly, $\lambda_0 \in \mathbb{R}$ is an eigenvalue of A if and only if λ_0 is a zero of one of the functions $\mu_j^A(x)$. The following lemma was proved in [7].

Lemma 5.1 *Let λ_0 be a real eigenvalue of A, and let*

$$\mu_{j_1}^A(x), \ldots, \mu_{j_s}^A(x)$$

be all the functions among the $\mu_j^A(x)$'s that have a zero at λ_0. Suppose that λ_0 is a zero of $\mu_{j_w}^A(x)$ of multiplicity κ_w, $w = 1, 2, \ldots, s$. Then the partial multiplicities of λ_0 as an eigenvalue of A are $\kappa_1, \ldots, \kappa_s$, and the sign in the sign characteristic of (A, H) associated with the multiplicity κ_w coincides with the sign of the nonzero real number $(\mu_{j_w}^A)^{(\kappa_w)}(\lambda_0)$ (the κ_wth derivative of $\mu_{j_w}^A(x)$ evaluated at λ_0).

Now fix a nonzero vector $u \in \mathbb{C}^n$, and let $B = \pm uu^* H$. For the subsequent analysis we choose the sign $-$; if the sign is $+$, then just replace H with $-H$ to reduce the consideration to the case of the sign $-$. Analogously we have the analytic eigenvalues $\mu_1^{A+B}(x), \ldots, \mu_n^{A+B}(x)$ of $xH - H(A+B)$. Note that

$$xH - H(A+B) - (xH - HA) = Huu^*H$$

is positive semidefinite. Thus, by the well known monotonicity property of eigenvalues of Hermitian matrices [11, Corollary 4.3.3], we have

$$\#\{j \,:\, \mu_j^{A+B}(x) \geq q\} \geq \#\{j \,:\, \mu_j^{A}(x) \geq q\} \tag{5.1}$$

for every $x \in \mathbb{R}$ and every real number q. (Here, $\#L$ denotes the cardinality of a finite set L.)

We also note the following fact:

Lemma 5.2 *For a fixed real x, suppose that there are s eigenvalues (counted with multiplicities) of $xH - HA$ in the real interval $[\alpha, \beta]$. Then there are at least s eigenvalues of $xH - H(A - uu^*H)$ in the interval $[\alpha - \|uu^*H\|, \beta + \|uu^*H\|]$.*

For the proof, observe that Lemma 5.2 follows easily from Mirsky's inequality for eigenvalues of two Hermitian matrices [4, 19].

5.2 The sign characteristic of new real eigenvalues: main result

Let $H \in \mathbb{C}^{n\times n}$ be Hermitian and invertible, and let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Fix a real eigenvalue λ_0 of A. Let $n_1 > \cdots > n_p$ be the distinct partial multiplicities of A corresponding to λ_0, and let there be ℓ_j blocks in the Jordan form of A having size n_j and eigenvalue λ_0, for $j = 1, 2, \ldots, p$, with the signs $\xi_{j,k} = \pm 1$, $k = 1, 2, \ldots, \ell_j$ attached to the partial multiplicities $n_j, \ldots, n_j$ (repeated ℓ_j times) in the sign characteristic of (A, H) associated with the eigenvalue λ_0. Recall (Theorem 2.1) that the signs $\xi_{j,k}$, for every fixed j, are uniquely determined up to a permutation. For the purpose of our analysis, it will be convenient to distinguish $\xi_{1,1}$ and classify the various possibilities according to the value $\xi_{1,1} = 1$ or $\xi_{1,1} = -1$.

We distinguish two cases: (e) n_1 is even; (o) n_1 is odd. According to Theorem 3.3, for a generic set (with respect to the real and imaginary parts of the components) of vectors $u \in \mathbb{C}^n$, we have one of the following four (not necessarily mutually exclusive) situations:

(e+) n_1 is even, $\xi_{1,1} = 1$, and at the eigenvalue λ_0 the H-selfadjoint matrix $A - uu^*H$ has distinct partial multiplicities $n_1 > \cdots > n_p$ repeated $\ell_1 - 1, \ell_2, \ldots, \ell_p$ times respectively (if $\ell_1 = 1$, then n_1 is omitted), with signs in the sign characteristic $\xi_{1,k}$, $k = 2, \ldots, \ell_1$ corresponding to the partial multiplicities n_1 (repeated $\ell_1 - 1$ times) and

$\xi_{j,k}$, $k = 1, 2, \ldots, \ell_j$ corresponding to the partial multiplicities n_j (repeated ℓ_j times) for $j = 2, 3, \ldots, p$.

(e–) n_1 is even, $\xi_{1,1} = -1$, and all other properties as described in (e+).

(o+) n_1 is odd, $\xi_{1,1} = 1$, and all other properties as described in (e+).

(o–) n_1 is odd, $\xi_{1,1} = -1$, and all other properties as described in (e+).

In addition, we shall assume $\|u\|$ is sufficiently small, so that $A - uu^*H$ has generically n_1 eigenvalues $\nu_1, \ldots, \nu_{n_1}$ (which may be real or complex) different from λ_0 that are clustered around λ_0. By Theorem 3.3, we may assume that generically the eigenvalues $\nu_1, \ldots \nu_{n_1}$ are all simple. Renumbering the eigenvalues so that $\nu_1, \ldots, \nu_m$ are real and the rest are nonreal, we let (generically) $\nu_1 < \cdots < \nu_m$. (Note that m may depend on u, but this dependence is not reflected in the notation.) Thus, there is a sign η_q associated with ν_q, $q = 1, 2, \ldots, m$, in the sign characteristic of $(A - uu^*H, H)$. Obviously, $m \leq n_1$.

We now state our main result on the "new" eigenvalues ν_q and their sign characteristic. Denote by Ω the open generic (with respect to the real and imaginary parts of the components of u) set of vectors $u \in \mathbb{C}^n$ for which one of (e+), (e–), (o+), (o–) holds and the eigenvalues $\nu_1, \ldots, \nu_{n_1}$ are all distinct, simple, and none of them is equal to λ_0.

Theorem 5.3 (a) *Under the above notation, and assuming that $u \in \Omega$ and $\|u\|$ is sufficiently small (the sufficiency of the smallness of $\|u\|$ is determined by the pair (A, H) only), m is even and $\eta_1 + \cdots + \eta_m = 0$ in cases (e+) and (e–), and m is odd and $\eta_1 + \cdots + \eta_m = \pm 1$ in cases (o±).*

(b) *Assuming in addition that the geometric multiplicity of λ_0 as the eigenvalue of A is equal to one, then:*

(b1) *if (e+) holds, then the ν_q are all nonreal, i.e. $m = 0$;*

(b2) *if (e–) holds, then for some odd k, $k < m$, we have*

$$\nu_1 < \nu_2 < \cdots < \nu_k < \lambda_0 < \nu_{k+1} < \cdots < \nu_m,$$

with $\eta_q = (-1)^{q-1}$, for $q = 1, 2, \ldots, m$.

(b3) *if (o+) holds, then $\nu_1 < \nu_2 < \cdots < \nu_m < \lambda_0$, with $\eta_q = (-1)^{q-1}$, for $q = 1, 2, \ldots, m$.*

(b4) *if (o–) holds, then $\lambda_0 < \nu_1 < \nu_2 < \cdots < \nu_m$, with $\eta_q = (-1)^q$, for $q = 1, 2, \ldots, m$.*

We emphasize that the number m in Theorem 5.3 may depend on $u \in \Omega$ (although this is not reflected in the notation).

Proof. Fix a disc $\{z \in \mathbb{C} : |z - \lambda_0| < \delta\}$, where δ is chosen so that λ_0 is the only eigenvalue of A in the disc. Part (a) concerning the number m follows easily from the fact that the number of nonreal eigenvalues of $A + B$ in a disc $\{z \in \mathbb{C} : |z - \lambda_0| < \delta\}$ is even and the total number of eigenvalues of $A + B$ in the disc is equal to $n_1\ell_1 + \cdots + n_p\ell_p$

(for sufficiently small $\|u\|$). The statements about η_j's then follow from the general perturbation theory for H-selfadjoint matrices, see, for example, [9, Chapter 9].

We prove (b). We give a detailed proof for the cases (b1) and (b2) only, the proof in the other cases is obtained by analogous considerations. Thus, let n_1 be the even algebraic multiplicity of λ_0, with the sign -1. Following the analysis and notation of Subsection 5.1, let $\mu^A(x)$ be the analytic (as function of the real variable x) eigenvalue of $xH - HA$ so that $\mu^A(x)$ has a zero at λ_0 of multiplicity n_1 and $(\mu^A)^{(n_1)}(\lambda_0) < 0$. Clearly, there exists $\delta > 0$ such that λ_0 is the only zero of any $\mu^A(x)$ in the interval $[\lambda_0 - \delta, \lambda_0 + \delta]$ and that the graphs of all other analytic eigenvalues of $xH - HA$ do not intersect the closed rectangle

$$\{(\lambda_0 + w, y) \in \mathbb{R}^2 \, : \, |w| \leq \delta, \quad |y| \leq \delta\}. \tag{5.2}$$

In view of Lemma 5.2, there exists $\epsilon > 0$ such that for every $u \in \Omega$, $\|u\| < \epsilon$, there is exactly one analytic eigenvalue $\mu^{A+B}(x)$ of $xH - H(A - uu^*H) = xH - H(A + B)$ that intersects the rectangle (5.2). Moreover, by taking ϵ smaller if necessary, we may assume also that $\mu^{A+B}(\lambda_0 \pm \delta) \neq 0$ and that $\mu^A(\lambda_0 \pm \delta)$ and $\mu^{A+B}(\lambda_0 \pm \delta)$ have the same sign. Because of these conditions, and taking into account that $\mu^A(\lambda_0 \pm \delta) < 0$ (since λ_0 is the only zero of $\mu^A(x)$ on the interval $[\lambda_0 - \delta, \lambda_0 + \delta]$), and we are in the case (b2)), we have

$$\mu^{A+B}(\lambda_0 \pm \delta) < 0. \tag{5.3}$$

On the other hand, property (5.1) (applied with $x = \lambda_0$ and $q = 0$) yields

$$\mu^{A+B}(\lambda_0) > 0. \tag{5.4}$$

In view of Lemma 5.1, inequalities (5.3) and (5.4) now easily lead to the desired conclusion in the case (b2).

Suppose now that n_1 is even with the sign $+1$. Let $\mu^A(x)$ and $\mu^{A+B}(x)$ be the analytic eigenvalues of $xH - HA$ and of $xH - H(A + B)$ respectively, having the properties as in the case (b2), for $u \in \Omega$ with $\|u\|$ sufficiently small. By property (5.1), we have $\mu^{A+B}(x) \geq \mu^A(x)$ for every $x \in [\lambda_0 - \delta, \lambda_0 + \delta]$. Since $\mu^{A+B}(\lambda_0) \neq 0$, we must have $\mu^{A+B}(x) > 0$ for all $x \in [\lambda_0 - \delta, \lambda_0 + \delta]$, and the result follows. □

Example 5.4 To illustrate Theorem 5.3, we consider the matrices

$$A = \mathcal{J}_4(0), \quad H = -R_4.$$

Thus, we are in the case (b2) of Theorem 5.3, so for a given sufficiently small vector u, the following situations are possible for the eigenvalues of the matrix $A - uu^*H$:

i) two real eigenvalues, one positive, one negative;

ii) four real eigenvalues, one negative, three positive;

iii) four real eigenvalues, three negative, one positive.

Indeed, it seems that all three possibilities can be realized by arbitrarily small perturbations. For an example realizing iii), one can take the vector

$$u := \varepsilon \begin{bmatrix} 1 \\ 2 \\ 1 \\ \frac{1}{10}\varepsilon \end{bmatrix}.$$

Then MATLAB examples show that for $\varepsilon = 10^{-1}, 10^{-2}, \dots, 10^{-16}$ the matrix $A - uu^*H$ has one positive and three negative eigenvalues. For example, by taking $\varepsilon = 10^{-3}$, the eigenvalues of $A - uu^*H$ are -0.000885, -0.000113, -0.000092, and 0.001093. However, it should be noted that if ε is kept fixed, but the vector u is scaled down in norm by $\widetilde{u} = \tau u$, then the situation changes from iii) to i). E.g., taking in the above example $\varepsilon = 10^{-3}$ and $\tau = 1/10$, the eigenvalues of $A - \widetilde{u}\widetilde{u}^*H$ become -0.000062, 0.000162, and $-0.000050 \pm 0.000087i$. Numerical experiments suggest that this is true in general: for a fixed vector u that realizes situations iii), scaling down the norm of u has the effect that at some point the situation changes from iii) to i) and continues to be i) when the norm is scaled further down. □

6 Conclusions

We have discussed the perturbation theory for selfadjoint matrices in an indefinite inner product under generic selfadjoint rank one perturbations. We have derived the Jordan structures of the perturbed matrices and also characterized the behavior of the sign characteristic associated with the real eigenvalues under these perturbations.

References

[1] S. Barnett. Greatest common divisor of several polynomials. *Proc. Cambridge Philos. Soc.*, 70:263–268, 1971.

[2] M.A. Beitia, I. de Hoyos, and I. Zaballa. The change of the Jordan structure under one row perturbations. *Linear Algebra Appl.*, 401:119–134, 2005.

[3] T. Bella, V. Olshevsky, and U. Prasad. Lipschitz stability of canonical Jordan bases of H-selfadjoint matrices under structure-preserving perturbations. *Linear Algebra Appl.*, 428:2130–2176, 2008.

[4] R. Bhatia. Analysis of spectral variation and some inequalities. *Trans. Amer. Math. Soc.*, 272:323–331, 1982.

[5] I. Gohberg and G. Heinig. The resultant matrix and its generalizations. I. The resultant operator of matrix polynomials. *Acta Sci. Math. (Szeged)*, 37:41–61, 1975. (Russian).

[6] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials.* Academic Press, New York, 1982.

[7] I. Gohberg, P. Lancaster, and L. Rodman. *Matrices and Indefinite Scalar Products.* Birkhäuser, Basel, 1983.

[8] I. Gohberg, P. Lancaster, and L. Rodman. A sign characteristic for selfadjoint meromorphic matrix functions. *Applicable Anal.*, 16:165–185, 1983.

[9] I. Gohberg, P. Lancaster, and L. Rodman. *Indefinite Linear Algebra and Applications.* Birkhäuser, Basel, 2005.

[10] L. Hörmander and A. Melin. A remark on perturbations of compact operators. *Math. Scand.*, 75:255–262, 1994.

[11] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, UK, 1985.

[12] T. Kato. *Perturbation Theory for Linear Operators.* Springer-Verlag, New York, NY, 1966.

[13] M. Krupnik. Changing the spectrum of an operator by perturbation. *Linear Algebra Appl.*, 167:113–118, 1992.

[14] P. Lancaster and L. Rodman. Canonical forms for Hermitian matrix pairs under strict equivalence and congruence. *SIAM Review*, 47:407–443, 2005.

[15] P. Lancaster and M. Tismenetsky. *The Theory of Matrices.* Academic Press, Orlando, FL, 2nd edition, 1985.

[16] C. Mehl, V. Mehrmann, A. C. M. Ran, and L. Rodman. Eigenvalue perturbation theory of structured matrices under generic structured rank one perturbations: General results and complex matrices. Technical Report 673, Matheon - DFG Research Center "Mathematics for key technologies", Berlin, Germany, 2009.

[17] V. Mehrmann and H. Voss. Nonlinear eigenvalue problems: A challenge for modern eigenvalue methods. *Mitt. d. Ges. f. Angewandte Mathematik und Mechanik*, 27:121–151, 2005.

[18] V. Mehrmann and H. Xu. Perturbation of purely imaginary eigenvalues of Hamiltonian matrices under structured perturbations. *Electr. J. Lin. Alg.*, 17:234–257, 2008.

[19] L. Mirsky. Symmetric gauge functions and unitarily invariant norms. *Quart. J. Math. Oxford (2)*, 11:50–59, 1960.

[20] J. Moro and F. Dopico. Low rank perturbation of Jordan structure. *SIAM J. Matrix Anal. Appl.*, 25:495–506, 2003.

[21] A. C. M. Ran and L. Rodman. Semidefinite perturbations of analytic Hermitian matrix functions. *Integral Equations Operator Theory*, 12:739–745, 1989.

[22] L. Rodman. Similarity vs unitary similarity: Complex and real indefinite inner products. *Linear Algebra Appl.*, 416:945–1009, 2006.

[23] S.V. Savchenko. Typical changes in spectral properties under perturbations by a rank-one operator. *Mat. Zametki*, 74:590–602, 2003. (Russian). Translation in Mathematical Notes. 74:557–568, 2003.

[24] S.V. Savchenko. On the change in the spectral properties of a matrix under a perturbation of a sufficiently low rank. *Funktsional. Anal. i Prilozhen*, 38:85–88, 2004. (Russian). Translation in Funct. Anal. Appl. 38:69–71, 2004.

[25] M. Stammberger and H. Voss. Automated multi-level substructuring for a fluid-solid vibration problem. In K. Kunisch, G. Of, and O. Steinbach, editors, *Numerical Mathematics and Advanced Applications. ENUMATH 2007*, pages 563–570, Berlin, 2008. Springer Verlag.