Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations *

Christian Mehl † Volker Mehrmann† André C. M. Ran‡
Leiba Rodman §

Dedicated to G.W. (Pete) Stewart on the occasion of his 70th birthday

Abstract

We study the perturbation theory of structured matrices under structured rank one perturbations, and then focus on several classes of complex matrices. Generic Jordan structures of perturbed matrices are identified. It is shown that the perturbation behavior of the Jordan structures in the case of singular J-Hamiltonian matrices is substantially different from the corresponding theory for unstructured generic rank one perturbation as it has been studied in [18, 28, 30, 31]. Thus a generic structured perturbation would not be generic if considered as an unstructured perturbation. In other settings of structured matrices, the generic perturbation behavior of the Jordan structures, within the confines imposed by the structure, follows the pattern of that of unstructured perturbations.

Key Words: structured matrices, Brunovsky form, complex Hamiltonian Jordan form, perturbation analysis, generic perturbation, rank one perturbation.

Mathematics Subject Classification: 15A63, 15A21, 15A57, 47A55, 93B10, 93B35, 93C73.

---

*This research was supported by *Deutsche Forschungsgemeinschaft*, through the DFG Research Center MATHEON *Mathematics for key technologies* in Berlin.

†Technische Universität Berlin, Institut für Mathematik, MA 4-5, Straße des 17. Juni 136, 10623 Berlin, Germany. Email: {mehl,mehrmann}@math.tu-berlin.de.

‡Afdeling Wiskunde, Faculteit der Exacte Wetenschappen, Vrije Universiteit Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands. E-mail: ran@few.vu.nl.

§College of William and Mary, Department of Mathematics, P.O.Box 8795, Williamsburg, VA 23187-8795, USA. E-mail: lxrodm@math.wm.edu. *The research of this author was supported by NSF Grant DMS-0456625 and by Plumeri Award and Faculty Research Assignment at the College of William and Mary. Large part of this work was done while this author visited at Vrije Universiteit Amsterdam and TU Berlin, whose hospitality is gratefully acknowledged.*

1 Introduction

In this paper, we consider the perturbation theory for Jordan structures associated with complex matrices in several classes of structured matrices under generic perturbations that have rank one and are structure preserving. We also present results on the behavior of Jordan structures under rank one structured perturbations for rather general classes of structured matrices, both real and complex, that cover many particular cases and support the perturbation theory developed in this paper, and will be used in subsequent publications as well. The classes that we consider are defined as follows.

Let $\mathbb{F}$ denote either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$ and let I_n denote the $n \times n$ identity matrix. The superscript $(\cdot)^T$ denotes the transpose and $(\cdot)^*$ denotes the conjugate transpose of a matrix or vector; thus $X^* = X^T$ for $X \in \mathbb{R}^{m\times n}$.

Definition 1.1 *Let $J \in \mathbb{F}^{2n\times 2n}$ be an invertible skew-symmetric matrix. A matrix $A \in \mathbb{F}^{2n\times 2n}$ is called J-*Hamiltonian *if $JA = (JA)^T$.*

The classical and most important example in applications, see Section 1.3, are the classes obtained with the matrix

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{1.1}$$

Other types of symmetries are introduced using an invertible symmetric matrix instead of a skew-symmetric J in Definition 1.1:

Definition 1.2 *Let $H \in \mathbb{F}^{n\times n}$ be an invertible symmetric matrix. A matrix $A \in \mathbb{F}^{n\times n}$ is called H-*symmetric *if $HA = (HA)^T$.*

If J is skew-symmetric invertible, and N is such that $JN = -(JN)^T$, then N is called *J-skew-Hamiltonian*. Note that the rank of any J-skew-Hamiltonian matrix is even, and since we are concerned only with rank one perturbations in this paper, J-skew-Hamiltonian matrices will not be considered here. For a similar reason, we do not consider here matrices N such that $HN = -(HN)^T$, where H is symmetric and invertible.

In this paper we consider the complex case in the above definitions. The real case, as well as rank one perturbation analysis of *J-symplectic* matrices (defined by the equality $S^TJS = J$ with invertible skew symmetric matrix J) and *H-orthogonal* matrices (defined by the equality $S^THS = H$ with invertible symmetric matrix H) will be studied in subsequent papers. An analogous but different perturbation theory for rank one structured perturbations can be also developed for the case when H is taken to be Hermitian and the transpose is replaced by the conjugate transpose (in the complex case) in Definition 1.2. This will be addressed elsewhere as well.

1.1 Notation

In the following the set of positive integers is denoted by $\mathbb{N}$. $\mathcal{J}_m(\lambda)$ denotes an upper triangular $m \times m$ Jordan block with eigenvalue λ and R_m stands for the $m \times m$ matrix with 1 on the leftbottom - topright diagonal and zeros elsewhere, i.e.,

$$\mathcal{J}_m(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad R_m = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}.$$

The k-th standard basis vector of length n will be denoted by $e_{k,n}$ or in short e_k if the length is clear from the context. The spectrum of a matrix $A \in \mathbb{F}^{n \times n}$, i.e., the set of eigenvalues including possibly nonreal eigenvalues of real matrices, is denoted by $\sigma(A)$. An eigenvalue $\lambda \in \sigma(A)$ is said to be *simple* if the corresponding algebraic multiplicity is one, i.e., λ is a simple root of the characteristic polynomial of A.

$\chi(Z) = \det(Z - xI)$ is the characteristic polynomial of a square size matrix Z.

Throughout the paper we will use a fixed matrix norm $\|\cdot\|$ which denotes the spectral norm $\|\cdot\|_2$, and a fixed vector norm $\|\cdot\|$, namely the Euclidean norm.

A block diagonal matrix with diagonal blocks $X_1, \ldots, X_q$ (in that order) is denoted by $X_1 \oplus X_2 \oplus \cdots \oplus X_q$. We also use the notation $X^{\oplus k}$ for $X \oplus X \oplus \cdots \oplus X$ (k times).

If $v^T = [v_1, \ldots, v_n]^T \in \mathbb{C}^n$ then $\mathrm{Toep}\,(v)$ denotes the $n \times n$ upper triangular Toeplitz matrix

$$\mathrm{Toep}\,(v) = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \\ 0 & v_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_2 \\ 0 & \ldots & 0 & v_1 \end{bmatrix}.$$

We also introduce the anti-diagonal matrices

$$\Sigma_k = \begin{bmatrix} 0 & \cdots & 0 & (-1)^0 \\ \vdots & \iddots & (-1)^1 & 0 \\ 0 & \iddots & & \vdots \\ (-1)^{k-1} & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \iddots & & & & 0 \end{bmatrix} = (-1)^{k-1}\Sigma_k^T, \qquad (1.2)$$

i.e., Σ_k is symmetric if k is odd, and skew-symmetric if k is even.

1.2 Motivation

The perturbation theory for eigenvalues of matrices is well established [34]. This is also the case if the perturbations are generic low rank matrices, see [4, 18, 20, 28, 30, 31]. But when the perturbations are restricted to be structure preserving then surprisingly different effects may occur.

Example 1.3 Let

$$A = \begin{bmatrix} \mathcal{J}_3(0) & 0 \\ 0 & -\mathcal{J}_3(0)^T \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix}.$$

Then A is J-Hamiltonian and has two Jordan blocks of size 3 associated with the eigenvalue zero. The perturbation analysis under unstructured generic rank 1 perturbations, Theorem 3.1 in [28] (a particular case of which is part of Theorem 2.3 below), yields that the perturbed matrix still has one block $J_3(0)$, while the other block has vanished and split into three (generically different) nonzero eigenvalues.

In contrast to this (as we will show below) a generic J-Hamiltonian rank one perturbation will lead to a Jordan structure with a 4×4 block $\mathcal{J}_4(0)$ plus two (generically) nonzero simple eigenvalues. Thus the size of the largest block even increases. □

This example demonstrates that the classical understanding of perturbation theory has to be changed for classes of structured matrices. The perturbation theory for structured generic low rank perturbations is dominated by two conflicting effects, the generic structured perturbation trying to destroy the most sensitive part in the Jordan structure (which is the largest Jordan block) and the structure which requires certain Jordan structures.

1.3 Applications

The perturbation theory that we present in this paper has several important applications in control.

Let us first discuss the problem of passivity of systems. Consider a linear time-invariant control system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = 0, \\ y &= Cx + Du, \end{aligned} \tag{1.3}$$

with matrices $A \in \mathbb{F}^{n,n}$, $B \in \mathbb{F}^{n,m}$, $C \in \mathbb{F}^{p,n}$, $D \in \mathbb{F}^{p,m}$. Here u is the input, x the state, and y the output. Let us assume that all eigenvalues of A are in the open left half complex plane and that D is square and invertible. The system is called *passive*, see e.g. [2], if there exists a nonnegative scalar valued function Θ such that the *dissipation inequality*

$$\Theta(x(t_1)) - \Theta(x(t_0)) \le \int_{t_0}^{t_1} (u^* y + y^* u)\, dt$$

holds for all $t_1 \ge t_0$, i.e., the system absorbs supply energy. It is well known, [2, 16], that one can check whether the system is passive by checking whether the *Hamiltonian matrix*

$$\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^* \end{bmatrix} := \begin{bmatrix} A - BR^{-1}C & -BR^{-1}B^* \\ -C^*R^{-1}C & -(A - BR^{-1}C)^* \end{bmatrix} \tag{1.4}$$

has no purely imaginary eigenvalues, where $R = D + D^*$.

In many real world applications the system model (1.3) is only an approximation arising from a discretization of an infinite dimensional problem, a linearization of a nonlinear system, a realization or a reduced order approximate model, see e.g. [12, 13, 16, 17, 29, 33], and often in this approximation process the passivity is lost, and one tries to modify the non-passive approximate system by a small norm (typically also small rank) perturbation to a nearby passive system. Our perturbation theory will be important in understanding and computing minimal perturbations.

Another important application arises in robust control. Consider a control system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \qquad x(t_0) = x^0, \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t). \end{aligned} \tag{1.5}$$

In this system, x is again the state, u the input, and w is an exogenous input that may include noise, linearization errors, and un-modeled dynamics. The vector y contains measured outputs, while z is a regulated output or an estimation error.

The optimal $\mathcal{H}_\infty$ control problem is the task of designing a dynamic controller that minimizes (or at least approximately minimizes) the influence of the disturbances w on the output z in the $\mathcal{H}_\infty$-norm, see [36]. The computation of this controller is usually achieved by first solving two Hamiltonian eigenvalue problems that both are low rank perturbations (rank one in the single input case) of other Hamiltonian matrices where the perturbation matrices depend on the same parameter γ that gives an upper bound for the $\mathcal{H}_\infty$-norm to be minimized. Minimizing the value of γ under certain constraints then allows to find the optimal controller. Very often the optimal solution is obtained when an eigenvalue of the Hamiltonian matrix (as a function of γ) hits the imaginary axis, and thus becomes a multiple eigenvalue. The structured perturbation analysis of the eigenvalues as functions of this low rank perturbation allows the analysis and computation of the optimal controller, see [5, 27].

There are many further applications of the perturbation theory for structured matrices, such as the analysis of numerical methods for the Hamiltonian eigenvalue problem or its generalizations, see e.g. [1, 6, 7, 11, 26], or the solution of algebraic Riccati equations [19, 21]. Although in most applications the system matrices are real, in this paper we first study the complex case to lay down the basis for the structured perturbation theory. The real case will be discussed in a subsequent paper.

1.4 Review of contents

Besides the introduction and the conclusion, the paper consists of four sections. In Section 2 we focus on unstructured generic rank one perturbations, collect some known canonical forms, as well as describe ranks of perturbations of nilpotent matrices (Theorem 2.2). We refine the known results on generic rank one perturbations by showing that the "disappearing" Jordan block splits into simple eigenvalues (Theorem 2.3). The partial Brunovsky form leads to formulas for the characteristic polynomial of the perturbed matrices and for some of its coefficients (Theorem 2.10). The main results of Section 3 (Theorems 3.1 and 3.2) provide descriptions of Jordan canonical forms under generic rank one perturbations, in general settings of structured matrices that encompass many particular structures, including complex J-Hamiltonian and H-symmetric, as well complex H-selfadjoint and real structures (to be studied elsewhere). In Sections 4 and 5 we state and prove our main results on generic rank one perturbations within the classes of complex J-Hamiltonian and of complex H-symmetric matrices, respectively (Theorems 4.2 and 5.1).

2 General results

In this section we recall and/or derive some mathematical results on generic rank one perturbations, with emphasis on the unstructured setting, that will become important in the further analysis, in this and subsequent papers.

2.1 Perturbations of nilpotent matrices

We say that a set $W \subseteq \mathbb{F}^n$ (abbreviation for $\mathbb{F}^{n\times 1}$) is *algebraic* if there exists a finite set of polynomials $f_1(x_1,\ldots,x_n),\ldots,f_k(x_1,\ldots,x_n)$ with coefficients in $\mathbb{F}$ such that a vector $[a_1,\ldots,a_n]^T \in \mathbb{F}^n$ belongs to W if and only if

$$f_j(a_1,\ldots,a_n)=0, \quad j=1,2,\ldots,k.$$

In particular, the empty set is algebraic and $\mathbb{F}^n$ is algebraic. We say that a set $W \subseteq \mathbb{F}^n$ is *generic* if W is not empty and the complement $\mathbb{F}^n \setminus W$ is contained in an algebraic set which is not $\mathbb{F}^n$. Note here that the union of finitely many algebraic sets is again algebraic. Clearly, if the set $W \subseteq \mathbb{F}^n$ is generic and if $S \in \mathbb{F}^{n\times n}$ is invertible then SW is also generic. In the following, we say that a set $W \subseteq \mathbb{F}^n \times \mathbb{F}^n$ is generic if W, canonically identified with a subset of $\mathbb{F}^{2n}$ is generic as a subset of $\mathbb{F}^{2n}$.

The following lemma is almost obvious, but useful.

Lemma 2.1 *Let $Y(x_1,\ldots,x_r) \in \mathbb{F}^{m\times n}[x_1,\ldots,x_r]$ be a matrix whose entries are polynomials in the variables $x_1,\ldots,x_r$. If*

$$\operatorname{rank} Y(a_1,\ldots,a_r)=k$$

for some $[a_1,\ldots,a_r]^T \in \mathbb{F}^r$, then the set

$$\{[b_1,\ldots,b_r]^T \in \mathbb{F}^r \,:\, \operatorname{rank} Y(b_1,\ldots,b_r) \geq k\} \tag{2.1}$$

is generic.

Proof. Let

$$f_j(x_1, \ldots, x_r) = \det Y_j(x_1, \ldots, x_r), \quad j = 1, 2, \ldots, s,$$

where $Y_1(x_1, \ldots, x_r), \ldots, Y_s(x_1, \ldots, x_r)$ are the $k \times k$ submatrices of $Y(x_1, \ldots, x_r)$. Then the complement of the set (2.1) consists of the common zeros of the polynomials $f_1, \ldots, f_s$, i.e., it is an algebraic set, and the set (2.1) is nonempty by hypothesis. This shows that (2.1) is generic. □

In the following result we discuss ranks of powers of generic rank one perturbations of nilpotent matrices.

Theorem 2.2 *Consider a matrix $A \in \mathbb{F}^{n\times n}$ satisfying $A^m = 0$ for some $m \in \mathbb{N}$.*

(1) *If $X \in \mathbb{F}^{n\times n}$ is any rank one matrix, then*

$$\operatorname{rank}((A + X)^m) \leq m.$$

(2) *If in addition $A^{m-1} \neq 0$, then*

$$\operatorname{rank}((A + uv^T)^m) = m$$

for a generic set of vectors $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{F}^{2n}$.

(3) *If in addition $A^{m-1} \neq 0$, then for every invertible $B, C \in \mathbb{F}^{n\times n}$ we have*

$$\operatorname{rank}((A + Cuu^T B)^m) = m$$

for a generic set of vectors $u \in \mathbb{F}^n$.

Proof. Multiplying out $(A + X)^m$ we obtain

$$\begin{aligned}(A+X)^m &= A^m + A^{m-1}X + A^{m-2}X(A+X) + A^{m-3}X(A+X)^2 \\ &+ \cdots + AX(A+X)^{m-2} + X(A+X)^{m-1}. \end{aligned} \tag{2.2}$$

Since $A^m = 0$ and all other summands in the right hand side of (2.2) have ranks at most one, part (1) follows.

For part (2) let us assume without loss of generality that A is in Jordan canonical form, i.e.,

$$A = J_{k_1}(0) \oplus J_{k_2}(0) \oplus \cdots \oplus J_{k_t}(0), \tag{2.3}$$

where $k_j \leq k_1 = m$, $j = 2, 3, \ldots, t$.

We obviously have that

$$\operatorname{rank}((A + e_m e_1^T)^m) = m,$$

so by Lemma 2.1, part (2) follows (note that we cannot have $\operatorname{rank}((A + uv^T)^m) > m$ by part (1)).

Finally, consider part (3). From

$$A + Cuu^T B = C(C^{-1}AC + uu^T BC)C^{-1},$$

we see that without loss of generality we may assume that $C = I$. Furthermore,

$$A + uu^T B = S(S^{-1}AS + (S^{-1}u)(u^T(S^{-1})^T)(S^T BS))S^{-1},$$

and choosing the invertible matrix S so that $S^{-1}AS$ is in Jordan canonical form, we may also assume without loss of generality that A is given by (2.3). Denote by $\Delta(u)$, $u = [u_1, \ldots, u_n]^T \in \mathbb{F}^n$, the determinant of the $m \times m$ upper left corner of the matrix $(A + uu^T B)^m$. In view of Lemma 2.1, we only need to show that

$$\Delta(w_1, \ldots, w_n) \neq 0 \quad \text{for some} \;\; w_1, \ldots, w_n \in \mathbb{F}. \tag{2.4}$$

By (2.2) we have that

$$\begin{aligned} & (A + uu^T B)^m \\ = \; & A^{m-1}uu^T B + A^{m-2}uu^T B(A + uu^T B) + A^{m-3}uu^T B(A + uu^T B)^2 + \\ & \cdots + \; Auu^T B(A + uu^T B)^{m-2} + uu^T B(A + uu^T B)^{m-1}. \end{aligned} \tag{2.5}$$

This formula shows that $\Delta(u)$ is a polynomial in $u_1, \ldots, u_n$ of the form

$$\Delta(u) = \Delta_{2m}(u) + \Delta_{2m+2}(u) + \cdots + \Delta_{2m^2}(u),$$

where $\Delta_p(u)$ is a homogeneous polynomial of degree p. Clearly, to prove (2.4), we only need to find $w_1, \ldots, w_n \in \mathbb{F}$ such that

$$\Delta_{2m}(w_1, \ldots, w_n) \neq 0 \tag{2.6}$$

(here we use the easily proved fact, that if one homogeneous component of a polynomial in several variables takes a nonzero value, then the whole polynomial takes a nonzero value).

Note that $\Delta_{2m}(u)$ is the determinant of the upper left $m \times m$ corner of the matrix

$$A^{m-1}uu^T B + A^{m-2}uu^T BA + A^{m-3}uu^T BA^2 + \cdots + \; Auu^T BA^{m-2} + uu^T BA^{m-1} \tag{2.7}$$

(cf. formula (2.5)). Let $[b_1, \ldots, b_n]^T$ be the first column of B. Since the upper left $m \times m$ corner of A is the nilpotent Jordan block $J_m(0)$, it follows that the upper left $m \times m$ corner of the matrix (2.7) is an upper triangular $m \times m$ matrix with

$$u_m(u_1 b_1 + \ldots + u_n b_n)$$

on the main diagonal. Clearly, one can choose $w_1, \ldots, w_n \in \mathbb{F}$ so that

$$w_m(w_1 b_1 + \ldots + w_n b_n) \neq 0,$$

(here we use the hypothesis that B is invertible, and therefore at least one of $b_1, \ldots, b_n$ is nonzero), and (2.6) follows. $\square$

2.2 Unstructured generic rank one perturbation theory

The general perturbation analysis for generic low rank perturbations has been studied in [18, 20, 28, 30, 31, 32]. For the case of rank one perturbations - which is of interest in this paper - we have the following result.

Theorem 2.3 *Let $A \in \mathbb{C}^{n\times n}$ be a matrix having the pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_p$ with geometric multiplicities $g_1, \ldots, g_p$ and having the Jordan canonical form*

$$\bigoplus_{k=1}^{g_1} \mathcal{J}_{n_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{g_p} \mathcal{J}_{n_{p,k}}(\lambda_p),$$

where $n_{j,1} \geq \cdots \geq n_{j,g_j}$, $j = 1, \ldots, p$. Consider the rank one matrix $B = uv^T$, with $u, v \in \mathbb{C}^n$. Then generically (with respect to the entries of u and v) the Jordan blocks of $A+B$ with eigenvalue λ_j are just the $g_j - 1$ smallest Jordan blocks of A with eigenvalue λ_j, and all other eigenvalues of $A+B$ are simple; if $g_j = 1$, then generically λ_j is not an eigenvalue of $A + B$.

More precisely, there is a generic set $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^n$ such that for every $(u, v) \in \Omega$, the Jordan structure of $A + uv^T$ is described in (a) *and* (b) *below:*

(a) *the Jordan structure of $A + uv^T$ for the eigenvalues $\lambda_1, \ldots, \lambda_p$ is given by*

$$\bigoplus_{k=2}^{g_1} \mathcal{J}_{n_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=2}^{g_p} \mathcal{J}_{n_{p,k}}(\lambda_p);$$

(b) *the eigenvalues of $A+uv^T$ that are different from any of $\lambda_1, \ldots, \lambda_p$, are all simple.*

Part (a) of Theorem 2.3 is the main theorem of [28] specialized to the case of rank one perturbations; a result similar to that of [28] has been obtained in [30]. For the proof of part (b) we need some preparations. We start with the following well known example:

Example 2.4 Let

$$Z^{(1)}(\lambda, \alpha) = \mathcal{J}_m(\lambda) + \alpha e_m e_1^T = \begin{bmatrix} \lambda & 1 & \ldots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ \alpha & \ldots & 0 & \lambda \end{bmatrix} \in \mathbb{C}^{m\times m}, \quad \lambda \in \mathbb{C},\ \alpha \in \mathbb{C} \setminus \{0\}.$$

Then $\chi(Z^{(1)}(\lambda, \alpha)) = (-1)^m((x - \lambda)^m - \alpha)$; in particular, $Z^{(1)}(\lambda, \alpha)$ has m distinct eigenvalues. □

Next, we note that by [28], it follows that there exists a generic set Ω' of vectors $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$ for which Theorem 2.3 (a) holds. Clearly, we may assume Ω' is open; indeed, if the complement of Ω' is contained in an algebraic set $\Xi \neq \mathbb{C}^n \times \mathbb{C}^n$, we may replace Ω' with a smaller open set whose complement is Ξ. We then obtain the following lemma.

Lemma 2.5 *Let A be as in Theorem 2.3. Then there exists $\epsilon > 0$ and an open dense (in $\{(u,v) \in \mathbb{C}^n \times \mathbb{C}^n : \|u\|, \|v\| < \epsilon\}$) set*

$$\Omega'' \subseteq \Omega' \cap \{(u,v) \in \mathbb{C}^n \times \mathbb{C}^n : \|u\|, \|v\| < \epsilon\}$$

such that for every $(u,v) \in \Omega''$, the Jordan form of $A + uv^T$ is as in Theorem 2.3.

Proof. Denote by $D(z,\epsilon)$ the closed disc of radius ϵ centered at $z \in \mathbb{C}$. Fix $\epsilon > 0$ so small that for every $u, v \in \mathbb{C}^n$ with $\|u\|, \|v\| < \epsilon$, all eigenvalues of $A + B = A + uv^T$ are within the union of the closed pairwise nonintersecting discs of radius $\epsilon^{2/n}$ centered at each of the points $\lambda_1, \ldots, \lambda_p$. It will be assumed from now on that $\|u\|, \|v\| < \epsilon$.

Let $\chi(\lambda_j, u, v)$ for $j = 1, 2, \ldots, p$ be the characteristic polynomials in the independent variable x for the restrictions of $A + B$ to its spectral invariant subspaces corresponding to the eigenvalues of $A + B$ within the disc $D(\lambda_j, \epsilon^{2/n})$. Notice that the coefficients of $\chi(\lambda_j, u, v)$ are analytic functions of the components of u and v. Indeed, this follows from the formula for the projection onto the spectral invariant subspace

$$\frac{1}{2\pi i} \int_\Gamma (zI - (A+B))^{-1} dz,$$

for a suitable closed simple contour Γ. The integral is analytic as function of u and v; to prove that, use approximation of the integral by Riemann sums, and within every summand of the Riemann sum use the formula

$$(z_0 I - (A+B))^{-1} = (\operatorname{adj}(z_0 I - (A+B)))/(\det(z_0 I - (A+B))),$$

where $\operatorname{adj} Z$ stands for the algebraic adjoint of a matrix Z.

Let $q(\lambda_j, u, v)$ be the number of distinct eigenvalues of $A+B$ in the disc $D(\lambda_j, \epsilon^{2/n})$. Let

$$q_{\max}(\lambda_j) = \max_{u,v \in \mathbb{C}^n, \quad \|u\|,\|v\| < \epsilon} \{q(\lambda_j, u, v)\}.$$

Next, we fix λ_j. Denote by $S(p_1, p_2)$ the Sylvester resultant matrix of the two polynomials $p_1(x)$, $p_2(x)$ (see e.g. [3, 15]); note that $S(p_1, p_2)$ is a square matrix of size $\operatorname{degree}(p_1) + \operatorname{degree}(p_2)$ and recall the well known fact (see [23] for example) that the rank deficiency of $S(p_1, p_2)$ coincides with the degree of the greatest common divisor of the polynomials $p_1(x)$ and $p_2(x)$. We have

$$q(\lambda_j, u, v) = \operatorname{rank} S\left(\chi(\lambda_j, u, v), \frac{\partial \chi(\lambda_j, u, v)}{\partial x}\right) - (n_{j,1} + \cdots + n_{j,g_j}) + 1.$$

The entries of $S(\chi(\lambda_j,u,v),\frac{\partial\chi(\lambda_j,u,v)}{\partial x})$ are scalar (independent of u,v) multiples of the coefficients of $\chi(\lambda_j,u,v)$, and therefore the set $Q(\lambda_j)$ of all vectors $(u,v)\in\mathbb{C}^n\times\mathbb{C}^n$, $\|u\|,\|v\|<\epsilon$, for which $q(\lambda_j,u,v)=q_{\max}(\lambda_j)$ is the complement of the set of common zeros of finitely many analytic functions of the components of u and v. In particular, $Q(\lambda_j)$ is open and dense in

$$\{(u,v)\in\mathbb{C}^n\times\mathbb{C}^n\,:\,\|u\|,\|v\|<\epsilon\}.$$

On the other hand, still for a fixed λ_j, consider

$$u_0:=\frac{1}{2}\epsilon\begin{bmatrix}u_1\\ \vdots\\ u_p\end{bmatrix},\tag{2.8}$$

where the vectors $u_k\in\mathbb{C}^{n_{k,1}+\cdots+n_{k,g_k}}$ are such that all u_k's are zeros except for u_j which has 1 in the $n_{j,1}$-th position and zeros elsewhere. Also let

$$v_0:=\frac{1}{2}\epsilon\begin{bmatrix}v_1\\ \vdots\\ v_p\end{bmatrix},\tag{2.9}$$

partitioned conformably with (2.8), where all all v_k's are zeros except for v_j which has 1 in the first position and zeros elsewhere. (The coefficient $(1/2)\epsilon$ in (2.8) and in (2.9) is chosen so that the properties (1) and (2) below can be guaranteed.) One checks easily (cf. Example 2.4) that in the disc $D(\lambda_j,\epsilon^{2/n})$ the matrix $A+u_0v_0^T$ has:

(1) $n_{j,1}$ simple eigenvalues different from λ_j; and

(2) the eigenvalue λ_j with partial multiplicities $n_{j,2},\dots,n_{j,g_j}$.

If by chance the pair (u_0,v_0) is not in Ω', then we slightly perturb (u_0,v_0) to obtain a new pair $(u_0',v_0')\in\Omega'$ such that (1) and (2) are still valid for the matrix $A+u_0'(v_0')^T$. Such choice of (u_0',v_0') is possible because Ω' is generic, the property of eigenvalues being simple persists under small perturbations of $A+u_0v_0^T$, and the total number of eigenvalues of $A+uv^T$ within $D(\lambda_j,\epsilon^{2/n})$, counted with multiplicities, is equal to $n_{j,1}+\cdots+n_{j,g_j}$, for every $(u,v)\in\mathbb{C}^n$, $\|u\|,\|v\|<\epsilon$. Since Ω' is open, clearly there exists $\delta>0$ such that (1) and (2) are valid for every $A+uv^T$, where $(u,v)\in\mathbb{C}^n\times\mathbb{C}^n$ and $\|u-u_0\|,\|v-v_0\|<\delta$. Since the set of all such pairs of vectors (u,v) is open in $\mathbb{C}^n\times\mathbb{C}^n$, it follows from the properties of the set $Q(\lambda_j)$ established in the preceding paragraph that in fact we have

$$q(\lambda_j,u,v)=q_{\max}(\lambda_j)=n_{j,1}+1,\quad\text{for all}\ \ u,v\in\mathbb{C}^n,\quad\|u-u_0\|,\|v-v_0\|<\delta.$$

So for the following open set

$$\Omega_j^{(1)}:=Q(\lambda_j)\cap\Omega'$$

which is dense in $\{(u,v) \in \mathbb{C}^n \times \mathbb{C}^n : \|u\|, \|v\| < \epsilon\}$, we have that the part of the Jordan form of $A + uv^T$, where $(u,v) \in \Omega_j^{(1)}$, corresponding to the eigenvalues within $D(\lambda_j, \epsilon^{2/n})$ consists of

$$\mathcal{J}_{n_{j,2}}(\lambda_j) \oplus \cdots \oplus \mathcal{J}_{n_{j,p_j}}(\lambda_j)$$

and $n_{j,1}$ simple eigenvalues different from λ_j.

Now let

$$\Omega'' = \left(\bigcap_{j=1}^{p} \Omega_j^{(1)} \right) \cap \Omega'$$

to satisfy Lemma 2.5. Note that Ω'' is nonempty as the intersection of finitely many open dense (in $\{(u,v) \in \mathbb{C}^n \times \mathbb{C}^n : \|u\|, \|v\| < \epsilon\}$) sets. □

Proof of Theorem 2.3. As noted above, in view of the main result of [28], we only need to prove part (b). Let $\chi(u,v)$ be the characteristic polynomial (in the independent variable x) of $A + B$. Then the number of distinct roots of $\chi(u,v)$ is given by the rank of the Sylvester resultant matrix $S(\chi(u,v), \frac{\partial}{\partial x}\chi(u,v))$ minus $n-1$ (cf. the proof of Lemma 2.5). Therefore, the set Ω_0 of all pairs of vectors (u,v) on which the number of distinct roots of $\chi(u,v)$ is maximal, is a generic set. By Lemma 2.5, the maximal number of distinct roots of $\chi(u,v)$ is equal to

$$n_{1,1} + \cdots + n_{p,1} + \sum_{j=1}^{p} \min\{g_j - 1, 1\}.$$

Thus, for the generic set $U = \Omega_0 \cap \Omega'$ the Jordan structure of $A + uv^T$ is described by (a) and (b), as required. □

We will re-prove the part (a) of Theorem 2.3 in Section 2.4, using the Brunovsky canonical form.

2.3 Structured canonical forms

In the following we will recall the canonical forms for J-Hamiltonian and H-symmetric matrices which is available in many sources, see e.g. [21, 24], or [22, 35] in the framework of pairs of symmetric and skew-symmetric matrices.

Theorem 2.6 *Let $H \in \mathbb{C}^{n\times n}$ be symmetric and invertible and let $A \in \mathbb{C}^{n\times n}$ be H-symmetric. Then there exists an invertible matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^{-1}AP = \mathcal{J}_{n_1}(\lambda_1) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda_m), \quad P^T HP = R_{n_1} \oplus \cdots \oplus R_{n_m}, \tag{2.10}$$

where $n_1, \ldots, n_m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ are not necessarily pairwise distinct. The form (2.10) *is uniquely determined by the pair (A, H), up to a simultaneous permutation of diagonal blocks in the right hand sides of* (2.10).

Theorem 2.7 *Let $J \in \mathbb{C}^{n\times n}$ be skew-symmetric and invertible (i.e., n is even), and let $A \in \mathbb{C}^{n\times n}$ be J-Hamiltonian. Then there exists an invertible matrix $P \in \mathbb{C}^{n\times n}$ such that $P^{-1}AP$ and P^TJP are block diagonal matrices*

$$P^{-1}AP = A_1 \oplus A_2 \oplus A_3, \quad P^TJP = J_1 \oplus J_2 \oplus J_3, \tag{2.11}$$

where the blocks have the following forms.

(i) $$A_1 = \mathcal{J}_{2n_1}(0) \oplus \cdots \oplus \mathcal{J}_{2n_p}(0), \quad J_1 = \Sigma_{2n_1} \oplus \cdots \oplus \Sigma_{2n_p},$$

with $n_1, \ldots, n_p \in \mathbb{N}$;

(ii) $$A_2 = \begin{bmatrix} \mathcal{J}_{2m_1+1}(0) & 0 \\ 0 & \mathcal{J}_{2m_1+1}(0) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{2m_q+1}(0) & 0 \\ 0 & \mathcal{J}_{2m_q+1}(0) \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 0 & \Sigma_{2m_1+1} \\ -\Sigma_{2m_1+1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \Sigma_{2m_q+1} \\ -\Sigma_{2m_q+1} & 0 \end{bmatrix},$$

with $m_1, \ldots, m_q \in \mathbb{N} \cup \{0\}$;

(iii) $$A_3 = A_{3,1} \oplus \cdots \oplus A_{3,k}, \quad J_3 = J_{3,1} \oplus \cdots \oplus J_{3,k}\ ,$$

where

$$A_{3,j} = \begin{bmatrix} \mathcal{J}_{\ell_{j,1}}(\lambda_j) & 0 \\ 0 & -\mathcal{J}_{\ell_{j,1}}(\lambda_j)^T \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\ell_{j,q_j}}(\lambda_j) & 0 \\ 0 & -\mathcal{J}_{\ell_{j,q_j}}(\lambda_j)^T \end{bmatrix},$$

$$J_{3,j} = \begin{bmatrix} 0 & I_{\ell_{j,1}} \\ -I_{\ell_{j,1}} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\ell_{j,q_j}} \\ -I_{\ell_{j,q_j}} & 0 \end{bmatrix},$$

with $\ell_{j,1}, \ldots, \ell_{j,q_j} \in \mathbb{N}$ and $\lambda_j \in \mathbb{C}$ with $\mathrm{Re}(\lambda_j) > 0$ or $\mathrm{Re}(\lambda_j) = 0$ and $\mathrm{Im}(\lambda_j) > 0$ for $j = 1, \ldots, k$. Moreover, $\lambda_1, \ldots, \lambda_k$ are pairwise distinct.

The form (2.11) *is uniquely determined by the pair (A, J), up to a simultaneous permutation of diagonal blocks in the right hand sides of* (2.11).

2.4 The Brunovsky form

To analyze the effect of rank one perturbations, we will make use of the following theorem, which follows directly from the *Brunovsky canonical form*, [9], see also [10] or [14] for example, of general *multi-input* control systems $\dot{x} = Ax + Bu$ under transformations

$$(A, \ B) \quad \mapsto \quad (C^{-1}(A + BR)C, \ \ C^{-1}BD)$$

with invertible C, D, and arbitrary R of suitable sizes.

Theorem 2.8 *Let $A \in \mathbb{C}^{n\times n}$ be a matrix in Jordan canonical form*

$$A = \mathcal{J}_{n_1}(\lambda_1) \oplus \cdots \oplus \mathcal{J}_{n_g}(\lambda_g) \oplus \mathcal{J}_{n_{g+1}}(\lambda_{g+1}) \oplus \cdots \oplus \mathcal{J}_{n_\nu}(\lambda_\nu), \tag{2.12}$$

where $\lambda_1 = \cdots = \lambda_g =: \widehat{\lambda} \in \mathbb{C}$, $\lambda_{g+1}, \ldots, \lambda_\nu \in \mathbb{C} \setminus \{\widehat{\lambda}\}$, $n_1 \geq \cdots \geq n_g$. Moreover, let $B = uv^T$, where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_\nu \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_\nu \end{bmatrix}, \quad u_i, v_i \in \mathbb{C}^{n_i},\ i = 1, \ldots, \nu.$$

Assume that the first component of each vector v_i, $i = 1, \ldots, \nu$ is nonzero. Then the matrix $\mathrm{Toep}\,(v_1) \oplus \cdots \oplus \mathrm{Toep}\,(v_\nu)$ *is invertible, and if we denote its inverse by S, then $S^{-1}AS = A$ and*

$$S^{-1}BS = \left[we_{1,n_1}^T, \ldots, we_{1,n_\nu}^T\right], \tag{2.13}$$

where $w = S^{-1}u$. Moreover, the matrix $S^{-1}(A+B)S$ has at least $g-1$ Jordan chains associated with $\widehat{\lambda}$ of lengths at least $n_2, \ldots, n_g$ given by

$$\begin{array}{lll} e_1 - e_{n_1+1}, & \ldots, & e_{n_2} - e_{n_1+n_2}; \\ e_1 - e_{n_1+n_2+1}, & \ldots, & e_{n_3} - e_{n_1+n_2+n_3}; \\ \vdots & \ddots & \vdots \\ e_1 - e_{n_1+\cdots+n_{g-1}+1}, & \ldots, & e_{n_g} - e_{n_1+\cdots+n_{g-1}+n_g}. \end{array} \tag{2.14}$$

Proof. Clearly $\mathrm{Toep}\,(v_i)$ is invertible if the first component of v_i is nonzero, so S exists. Moreover, S commutes with A, and $e_{1,n_i}^T(\mathrm{Toep}\,(v_i)) = v_i^T$, so we have

$$S^{-1}BS = S^{-1}uv^TS = \left[we_{1,n_1}^T, \ldots, we_{1,n_\nu}^T\right].$$

It is then straightforward to check that the given chains are indeed Jordan chains associated with $\widehat{\lambda}$. □

We emphasize that in Theorem 2.8 there is no claim whether the Jordan chains (2.14) associated with $\widehat{\lambda}$ can be extended to a longer chain or not, nor is there a claim whether (2.14) form a full basis of the corresponding root subspace or not.

Example 2.9 If $\widehat{\lambda} = 0$, $\nu = g = 3$, $n_1 = 4$, $n_2 = 3$, $n_3 = 2$, then the Brunovsky form of $A+B$ and the corresponding Jordan chains associated with $\widehat{\lambda} = 0$ of length 3 and 2 are given by

$$\left[\begin{array}{cccc|ccc|cc} w_1 & 1 & 0 & 0 & w_1 & 0 & 0 & w_1 & 0 \\ w_2 & 0 & 1 & 0 & w_2 & 0 & 0 & w_2 & 0 \\ w_3 & 0 & 0 & 1 & w_3 & 0 & 0 & w_3 & 0 \\ w_4 & 0 & 0 & 0 & w_4 & 0 & 0 & w_4 & 0 \\ \hline w_5 & 0 & 0 & 0 & w_5 & 1 & 0 & w_5 & 0 \\ w_6 & 0 & 0 & 0 & w_6 & 0 & 1 & w_6 & 0 \\ w_7 & 0 & 0 & 0 & w_7 & 0 & 0 & w_7 & 0 \\ \hline w_8 & 0 & 0 & 0 & w_8 & 0 & 0 & w_8 & 1 \\ w_9 & 0 & 0 & 0 & w_9 & 0 & 0 & w_9 & 0 \end{array}\right], \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

In the following, we want to apply Theorem 2.8 to the canonical forms in Section 2.3 which are close to but not quite in Jordan canonical form. Therefore, we will introduce in the next theorem the so called *partial Brunovsky form* with respect to a particular eigenvalue $\widehat{\lambda}$. With this form, the characteristic polynomial associated with the eigenvalue $\widehat{\lambda}$ can be conveniently characterized. In the next section, we will need explicit formulas for some of the coefficients of the characteristic polynomial of the perturbed matrix. We establish those in the next theorem as well. For the ease of future reference, we group together Jordan blocks of the same size in the Jordan canonical form of A.

Theorem 2.10 *Let*

$$A = \left(\mathcal{J}_{n_1}(\widehat{\lambda})^{\oplus \ell_1}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(\widehat{\lambda})^{\oplus \ell_m}\right) \oplus \widetilde{A} \in \mathbb{C}^{n \times n}, \tag{2.15}$$

where $n_1 > \cdots > n_m$ *and* $\sigma(\widetilde{A}) \subseteq \mathbb{C} \setminus \{\widehat{\lambda}\}$. *Moreover, let* $a = \ell_1 n_1 + \cdots + \ell_m n_m$ *denote the algebraic multiplicity of* $\widehat{\lambda}$ *and let* $B = uv^T$, *where* $u, v \in \mathbb{C}^n$ *and*

$$v = \begin{bmatrix} v^{(1)} \\ \vdots \\ v^{(m)} \\ \widetilde{v} \end{bmatrix}, \quad v^{(i)} = \begin{bmatrix} v^{(i,1)} \\ \vdots \\ v^{(i,\ell_i)} \end{bmatrix}, \quad v^{(i,j)} \in \mathbb{C}^{n_i}, \quad j = 1, \ldots, \ell_i, \quad i = 1, \ldots, m.$$

Assume that the first component of each vector $v^{(i,j)}$, $j = 1, \ldots, \ell_i$, $i = 1, \ldots, m$ *is nonzero. Then the following statements hold:*

(1) *The matrix* $S := \left(\bigoplus_{j=1}^{\ell_1} \mathrm{Toep}(v^{(1,j)}) \oplus \cdots \oplus \bigoplus_{j=1}^{\ell_m} \mathrm{Toep}(v^{(m,j)})\right)^{-1} \oplus I_{n-a}$ *exists and satisfies*

$$S^{-1}AS = A, \quad S^{-1}BS = w \left[\underbrace{e_{1,n_1}^T, \ldots, e_{1,n_1}^T}_{\ell_1 \text{ times}}, \ldots, \underbrace{e_{1,n_m}^T, \ldots, e_{1,n_m}^T}_{\ell_m \text{ times}}, z^T\right] \tag{2.16}$$

where $w = S^{-1}u$ *and for some appropriate vector* $z \in \mathbb{C}^{n-a}$.

(2) *The matrix* $S^{-1}(A+B)S$ *has at least* $\ell_1 + \cdots + \ell_m - 1$ *Jordan chains associated with* $\widehat{\lambda}$ *given as follows:*

a) $\ell_1 - 1$ *Jordan chains of length at least* n_1:

$$\begin{array}{lll} e_1 - e_{n_1+1}, & \ldots, & e_{n_1} - e_{2n_1}; \\ \vdots & \ddots & \vdots \\ e_1 - e_{(\ell_1-1)n_1+1}, & \ldots, & e_{n_1} - e_{\ell_1 n_1}; \end{array} \tag{2.17}$$

b) ℓ_i *Jordan chains of length at least* n_i *for* $i = 2, \ldots, m$*:*

$$\begin{array}{lll} e_1 - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + 1}, & \ldots, & e_{n_i} - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + n_i}; \\ e_1 - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + n_i + 1}, & \ldots, & e_{n_i} - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + 2n_i}; \\ \vdots & \ddots & \vdots \\ e_1 - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + (\ell_i - 1) n_i + 1}, & \ldots, & e_{n_i} - e_{\ell_1 n_1 + \cdots + \ell_{i-1} n_{i-1} + \ell_i n_i}; \end{array} \tag{2.18}$$

(3) *Partition* $w = S^{-1} u$ *as*

$$w = \begin{bmatrix} w^{(1)} \\ \vdots \\ w^{(m)} \\ \widetilde{w} \end{bmatrix}, \quad w^{(i)} = \begin{bmatrix} w^{(i,1)} \\ \vdots \\ w^{(i,\ell_i)} \end{bmatrix}, \quad w^{(i,j)} = \begin{bmatrix} w_1^{(i,j)} \\ \vdots \\ w_{n_i}^{(i,j)} \end{bmatrix} \in \mathbb{C}^{n_i},$$

and let $\lambda_1, \ldots, \lambda_q$ *be the pairwise distinct eigenvalues of* A *different from* $\widehat{\lambda}$ *having the algebraic multiplicities* $r_1, \ldots, r_q$*, respectively. Set* $\mu_i = \lambda_i - \widehat{\lambda}$*,* $i = 1, 2, \ldots, q$*. Then the characteristic polynomial* $p_{\widehat{\lambda}}$ *of* $A + B - \widehat{\lambda} I$ *is given by*

$$p_{\widehat{\lambda}}(\lambda) = (-\lambda)^a q(\lambda) + \left(\prod_{i=1}^{q} (\mu_i - \lambda)^{r_i} \right) \cdot \left((-\lambda)^a + (-1)^{a-1} \sum_{i=1}^{m} \sum_{j=1}^{\ell_i} \sum_{k=1}^{n_i} w_k^{(i,j)} \lambda^{a-k} \right), \tag{2.19}$$

where $q(\lambda)$ *is some polynomial;*

(4) *Write* $p_{\widehat{\lambda}}(\lambda) = c_n \lambda^n + \cdots + c_{a-n_1+1} \lambda^{a-n_1+1} + c_{a-n_1} \lambda^{a-n_1}$*. Then*

$$c_{a-n_1} = (-1)^{a-1} \left(\prod_{i=1}^{q} \mu_i^{r_i} \right) \left(\sum_{j=1}^{\ell_1} w_{n_1}^{(1,j)} \right); \tag{2.20}$$

and in the case $n_1 > 1$ *we have in addition that*

$$c_{a-n_1+1} = (-1)^a \left(\sum_{\nu=1}^{q} r_\nu \mu_\nu^{r_\nu - 1} \prod_{\substack{i=1 \\ i \neq \nu}}^{q} \mu_i^{r_i} \right) \left(\sum_{j=1}^{\ell_1} w_{n_1}^{(1,j)} \right) + (-1)^{a-1} \left(\prod_{i=1}^{q} \mu_i^{r_i} \right) \left(\sum_{j=1}^{\ell_1} w_{n_1-1}^{(1,j)} \right), \tag{2.21}$$

if $n_1 - 1 > n_2$ *or, if* $n_1 - 1 = n_2$*, then*

$$\begin{aligned} c_{a-n_1+1} &= (-1)^a \left(\sum_{\nu=1}^{q} r_\nu \mu_\nu^{r_\nu - 1} \prod_{\substack{i=1 \\ i \neq \nu}}^{q} \mu_i^{r_i} \right) \left(\sum_{j=1}^{\ell_1} w_{n_1}^{(1,j)} \right) \\ &+ (-1)^{a-1} \left(\prod_{i=1}^{q} \mu_i^{r_i} \right) \left(\sum_{j=1}^{\ell_1} w_{n_1-1}^{(1,j)} + \sum_{j=1}^{\ell_2} w_{n_2}^{(2,j)} \right). \end{aligned} \tag{2.22}$$

Proof. The parts (1) and (2) follow exactly as in Theorem 2.8.

For the proof of part (3), we need to work with individual Jordan blocks rather than with groups of blocks. We simplify somewhat the notation, and denote

$$(w_1,\dots,w_g)=(w^{(1,1)},w^{(1,2)},\dots,w^{(1,\ell_1)},\dots,w^{(m,\ell_m)}),\quad g=\ell_1+\dots+\ell_m;$$

where the w_i's are column vectors

$$w_i=\begin{bmatrix} w_{i,1}\\ \vdots \\ w_{i,s_i}\end{bmatrix}\in\mathbb{C}^{s_i},\quad i=1,\dots,g.$$

We may assume without loss of generality that $A+B$ is in Brunovsky form. Indeed, all that is needed is another similarity transformation with a matrix of the form $I_a\oplus\widetilde{S}^{-1}\in\mathbb{C}^{n\times n}$ which leaves the vectors $w_1,\dots,w_g$ invariant in $(I_a\oplus\widetilde{S}^{-1})w$. Thus, we let

$$\widetilde{A}=\mathcal{J}_{s_{g+1}}(\gamma_{g+1})\oplus\dots\oplus\mathcal{J}_{s_\nu}(\gamma_\nu),$$

where $\gamma_{g+1},\dots,\gamma_\nu$ are not necessarily distinct. (Clearly, $\{\gamma_{g+1},\dots,\gamma_\nu\}=\{\lambda_1,\dots,\lambda_q\}$, but we may have $\nu-g>q$.) Denote also $\kappa_i=\gamma_i-\widehat{\lambda}$, $i=g+1,g+2,\dots,\nu$.

With $A+B$ also $A+B-\widehat{\lambda}I_n$ is in Brunovsky form, and the list of the diagonal elements of $A-\widehat{\lambda}I$ is given by $(0,\dots,0,\kappa_{g+1},\dots,\kappa_\nu)$. Let M denote the matrix that is obtained from $(A+B-\widehat{\lambda}I)-\lambda I$ by subtracting the first column from the columns s_1+1, s_1+s_2+1,..., $s_1+s_2+\dots+s_{\nu-1}+1$. Note that the column $s_1+\dots+s_i+1$ then becomes zero except for λ in the first entry, for $-\lambda$ in the $(s_1+\dots+s_i+1)$-st entry if $i=1,2,\dots,g-1$, and for $\gamma_{i+1}-\lambda$ in the $(s_1+\dots+s_i+1)$-st entry if $i=g,\dots,\nu-1$. Then clearly $p_{\widehat{\lambda}}(\lambda)=\det M$. If $\nu>g$, then partition M as

$$M=\begin{bmatrix} w_{1,1}-\lambda & \beta^T & \lambda & 0 & \dots & 0\\ \alpha & T & 0 & 0 & \dots & 0\\ w_{\nu,1} & 0 & \kappa_\nu-\lambda & 1 & & 0\\ w_{\nu,2} & 0 & 0 & \kappa_\nu-\lambda & \ddots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & 1\\ w_{\nu,s_\nu} & 0 & 0 & \dots & 0 & \kappa_\nu-\lambda \end{bmatrix},$$

where $T\in\mathbb{C}^{(n-s_\nu-1)\times(n-s_\nu-1)}$, $n=s_1+\ldots+s_\nu$, is an upper triangular matrix whose first $a-1$ diagonal elements are equal to $-\lambda$. Thus, applying Laplace expansion successively, we obtain that

$$\det M=\lambda^a\widetilde{q}(\lambda)+(\kappa_\nu-\lambda)^{s_\nu}\det\begin{bmatrix} w_{1,1}-\lambda & \beta^T\\ \alpha & T\end{bmatrix}\tag{2.23}$$

for some polynomial $\widetilde{q}(\lambda)$. Indeed, for $s_\nu=1$ this is obvious and for $s_\nu>1$ we obtain

$$\begin{aligned}\det M &= (-1)^{n-1}w_{\nu,s_\nu}\cdot\underbrace{1\cdot\ldots\cdot 1}_{s_\nu-1\text{ times}}\cdot(-1)^{n-s_\nu-1}\lambda\det T+(\kappa_\nu-\lambda)\det M_{n-1}\\ &= \lambda^a\breve{q}(\lambda)+(\kappa_\nu-\lambda)\det M_{n-1},\end{aligned}$$

where $\breve{q}(\lambda)$ is some polynomial and M_{n-1} is the principal $(n-1)\times(n-1)$ submatrix of M. Note that M_{n-1} has the same structure as M just with s_ν replaced with $s_{\nu-1}$. The claim then follows by induction. By further induction, we then obtain from (2.23) that

$$\det M = \lambda^a q(\lambda) + \left(\prod_{i=g+1}^{\nu} (\kappa_i - \lambda)^{s_i}\right) \det M_a, \tag{2.24}$$

where M_a is the principal $a\times a$ submatrix of M. It remains to compute the determinant of M_a. To this end, partition M_a as

$$M_a = \begin{bmatrix} w_{1,1}-\lambda & \widehat{\beta} & \lambda & 0 & \dots & 0 \\ \widehat{\alpha} & \widehat{T} & 0 & 0 & \dots & 0 \\ w_{g,1} & 0 & -\lambda & 1 & & 0 \\ w_{g,2} & 0 & 0 & -\lambda & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ w_{g,s_g} & 0 & 0 & \dots & 0 & -\lambda \end{bmatrix},$$

Applying the cofactor expansion of $\det M_a$ by the first column, and using $\det \widehat{T} = (-\lambda)^{a-s_g-1}$, we obtain that

$$\begin{aligned} &\det M_a \\ = \; & (-1)^{a-1} w_{g,s_g} (-1)^{a-s_g-1}\lambda \det \widehat{T} + (-\lambda)(-1)^{a-2} w_{g,s_g-1}(-1)^{a-s_g-1}\lambda \det \widehat{T} \\ & + \cdots + (-\lambda)^{s_g-1}(-1)^{a-s_g} w_{g,1}(-1)^{a-s_g-1}\lambda \det \widehat{T} + (-\lambda)^{s_g} \det \begin{bmatrix} w_{1,1}-\lambda & \widehat{\beta} \\ \widehat{\alpha} & \widehat{T} \end{bmatrix} \\ = \; & (-1)^{a-1}\left(\sum_{k=1}^{s_g} w_{g,k}\lambda^{a-k}\right) + (-\lambda)^{s_g} \det \begin{bmatrix} w_{1,1}-\lambda & \widehat{\beta} \\ \widehat{\alpha} & \widehat{T} \end{bmatrix}. \end{aligned}$$

By induction, we finally obtain

$$\det M_a = (-\lambda)^a + (-1)^{a-1} \sum_{i=1}^{g}\sum_{k=1}^{s_i} w_{i,k}\lambda^{a-k}, \tag{2.25}$$

where the extra term $(-\lambda)^a$ appears due to the fact that the first entry of the first column of M_a is not $w_{1,1}$, but $w_{1,1}-\lambda$. Combining (2.25) with (2.24), formula (2.19) follows.

For part (4) observe that the lowest possible power of λ associated with a nonzero coefficient in $p_{\widehat{\lambda}}(\lambda)$ (given by (2.19)) is clearly $a-n_1$, and a calculation shows that the corresponding coefficient c_{a-n_1} is as in (2.20), while the coefficient c_{a-n_1+1} of λ^{a-n_1+1} in $p_{\widehat{\lambda}}(\lambda)$ is as in (2.21) or (2.22) depending on whether $n_1-1>n_2$ or $n_1-1=n_2$. ∎

2.5 Extension of Jordan chains

In this section, we discuss the extension of some Jordan chains of a matrix in Brunovsky form. This will be needed to prove our main results. However, it is not always possible to extend a given set of Jordan chains to a set of Jordan chains that forms a basis as we will illustrate in the following example.

Example 2.11 Consider the rank one perturbation of $A = \mathcal{J}_2(0) \oplus \mathcal{J}_2(0) \oplus \mathcal{J}_1(0)$ given by

$$\widetilde{A} = \begin{bmatrix} a & 1 & a & 0 & a \\ b & 0 & b & 0 & b \\ c & 0 & c & 1 & c \\ -b & 0 & -b & 0 & -b \\ d & 0 & d & 0 & d \end{bmatrix}, \quad b, d \neq 0$$

which is obviously in Brunovsky form. By Theorem 2.8 we know that $\widetilde{A}$ has at least two Jordan chains associated with zero of lengths at least 2 and 1, given by

$$e_1 - e_3,\ e_2 - e_4, \quad \text{and} \quad e_1 - e_5, \tag{2.26}$$

respectively. Let us check whether the first chain can be extended to a Jordan chain of length three. For this, we would have to show that $e_2 - e_4$ is in the range of $\widetilde{A}$. However, the linear system

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & 1 & a & 0 & a \\ b & 0 & b & 0 & b \\ c & 0 & c & 1 & c \\ -b & 0 & -b & 0 & -b \\ d & 0 & d & 0 & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 + a(x_1 + x_3 + x_5) \\ b(x_1 + x_3 + x_5) \\ x_4 + c(x_1 + x_3 + x_5) \\ -b(x_1 + x_3 + x_5) \\ d(x_1 + x_3 + x_5) \end{bmatrix}$$

with unknowns $x_1, \ldots, x_5$ does not have a solution, because $d \neq 0$, so the chain $e_1 - e_3$, $e_2 - e_4$ cannot be extended to a Jordan chains of length 3. Nevertheless, it can be shown that $\widetilde{A}$ does have a Jordan chain of length at least 3 associated with the eigenvalue zero. To this end, consider the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha \\ 1 \\ 0 \\ -1 \\ -\alpha \end{bmatrix}$$

that form a Jordan chain of $\widetilde{A}$ associated with zero of length 2. We now show that this chain can be extended for a particular choice of α. Indeed, for $\alpha = -d/b$ the linear system

$$\begin{bmatrix} \alpha \\ 1 \\ 0 \\ -1 \\ -\alpha \end{bmatrix} = \begin{bmatrix} a & 1 & a & 0 & a \\ b & 0 & b & 0 & b \\ c & 0 & c & 1 & c \\ -b & 0 & -b & 0 & -b \\ d & 0 & d & 0 & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 + a(x_1+x_3+x_5) \\ b(x_1+x_3+x_5) \\ x_4 + c(x_1+x_3+x_5) \\ -b(x_1+x_3+x_5) \\ d(x_1+x_3+x_5) \end{bmatrix}$$

has $x_1 = 1/b$, $x_2 = -d/b - a/b$, $x_3 = 0$, $x_4 = -c/b$, $x_5 = 0$ as a solution.

Note that the Jordan chain that could be extended in Example 2.11 can be considered as a "linear combination" of the two Jordan chains in (2.26). We will need similar constructions later in this paper and therefore, we introduce the following "sum" of Jordan chains.

Definition 2.12 *Let $A \in \mathbb{C}^{n\times n}$ and let $X = (x_1, \ldots, x_p)$ and $Y = (y_1, \ldots, y_q)$ be two Jordan chains of A associated with the same eigenvalue $\widehat{\lambda}$ of (possibly different) lengths p and q. Then the sum $X+Y$ of X and Y is defined to be the chain $Z = (z_1, \ldots, z_{\max(p,q)})$, where*

$$z_j = \begin{cases} x_j & \text{if } p \geq q \\ y_j & \text{if } p < q \end{cases}, \quad j = 1, \ldots, |p-q|$$

and

$$z_j = \begin{cases} x_j + y_{j-p+q} & \text{if } p \geq q \\ y_j + x_{j-q+p} & \text{if } p < q \end{cases}, \quad j = |p-q|+1, \ldots, \max(p,q).$$

To illustrate this construction, consider e.g. $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2)$, then $X + Y = (x_1, x_2, x_3 + y_1, x_4 + y_2)$.

It is straightforward to check that the sum $Z = X + Y$ of two Jordan chains associated with an eigenvalue $\widehat{\lambda}$ is again a Jordan chain associated with $\widehat{\lambda}$ of the given matrix A, but it should be noted that this sum is not commutative.

With these preliminary results, we have now set the stage to derive the desired perturbation theorems for structured matrices under generic rank one perturbations in the following sections.

3 Generic structured rank one perturbations for general classes of matrices with symmetries

In this section we state and prove general theorems concerning generic structured rank one perturbations. Although we focus on symmetry structures with respect to bilinear forms in this paper, the theorems cover a much wider class of structured matrices including matrices that are structured with respect to sesquilinear forms. To this end in the next two theorems, we will use the notation * to denote either the transpose ^{T} or the conjugate transpose *.

Theorem 3.1 *Let $A \in \mathbb{F}^{n\times n}$ and let $T, H \in \mathbb{F}^{n\times n}$ be invertible such that*

$$T^{-1}AT = \left(\mathcal{J}_{n_1}(\widehat{\lambda})^{\oplus \ell_1}\right) \oplus \left(\mathcal{J}_{n_2}(\widehat{\lambda})^{\oplus \ell_2}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(\widehat{\lambda})^{\oplus \ell_m}\right) \oplus \widetilde{A}, \tag{3.1}$$

$$T^{\star}HT = \left(\bigoplus_{j=1}^{\ell_1} H^{(1,j)}\right) \oplus H^{(2)} \oplus \cdots \oplus H^{(m)} \oplus \widetilde{H}, \tag{3.2}$$

where $\widehat{\lambda} \in \mathbb{F}$ and the decompositions (3.1) *and* (3.2) *have the following properties:*

(1) $n_1 > n_2 > \cdots > n_m$;

(2) $H^{(j)} \in \mathbb{F}^{\ell_j n_j \times \ell_j n_j}$, $j = 2, \ldots, m$ *and the matrices*

$$H^{(1,j)} = \begin{bmatrix} 0 & \ldots & 0 & h^{(1,j)}_{1,n_1} \\ \vdots & \iddots & h^{(1,j)}_{2,n_1-1} & h^{(1,j)}_{2,n_1} \\ 0 & \iddots & \iddots & \vdots \\ h^{(1,j)}_{n_1,1} & h^{(1,j)}_{n_1,2} & \ldots & h^{(1,j)}_{n_1,n_1} \end{bmatrix}, \quad j = 1, 2, \ldots, \ell_1;$$

are anti-triangular (necessarily invertible);

(3) $\widetilde{H}, \widetilde{A} \in \mathbb{F}^{(n-a)\times(n-a)}$, *where* $a = \sum_{j=1}^{m} \ell_j n_j$ *and* $\sigma(\widetilde{A}) \subseteq \mathbb{C} \setminus \{\widehat{\lambda}\}$.

If $B \in \mathbb{F}^{n\times n}$ is a rank one matrix of the form $B = uu^{\star}H$, then generically (with respect to the components of u if $\star = T$, and with respect to the real and imaginary parts of the components of u if $\star = *$) $A + B$ has the Jordan canonical form

$$\left(\mathcal{J}_{n_1}(\widehat{\lambda})^{\oplus (\ell_1-1)}\right) \oplus \left(\mathcal{J}_{n_2}(\widehat{\lambda})^{\oplus \ell_2}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(\widehat{\lambda})^{\oplus \ell_m}\right) \oplus \widetilde{\mathcal{J}}, \tag{3.3}$$

where $\widetilde{\mathcal{J}}$ contains all the Jordan blocks of $A + B$ associated with eigenvalues different from $\widehat{\lambda}$.

Proof. Without loss of generality, let A, H be in the forms (3.1) and (3.2) already. In view of Theorem 2.10 it is sufficient to show that the algebraic multiplicity of the eigenvalue $\widehat{\lambda}$ of $A + B$ is $a - n_1$ generically. Let

$$u = \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(m)} \\ \widetilde{u} \end{bmatrix}, \quad u^{(i)} = \begin{bmatrix} u^{(i,1)} \\ \vdots \\ u^{(i,\ell_i)} \end{bmatrix}, \quad u^{(i,j)} = \begin{bmatrix} u^{(i,j)}_1 \\ \vdots \\ u^{(i,j)}_{n_i} \end{bmatrix} \in \mathbb{F}^{n_i}, \quad \widetilde{u} \in \mathbb{F}^{n-a},$$

and

$$v = H^\star u = \begin{bmatrix} v^{(1)} \\ \vdots \\ v^{(m)} \\ \widetilde{v} \end{bmatrix}, \quad v^{(i)} = \begin{bmatrix} v^{(i,1)} \\ \vdots \\ v^{(i,\ell_i)} \end{bmatrix}, \quad v^{(i,j)} = \begin{bmatrix} v_1^{(i,j)} \\ \vdots \\ v_{n_i}^{(i,j)} \end{bmatrix} \in \mathbb{F}^{n_i}, \quad \widetilde{v} \in \mathbb{F}^{n-a}.$$

Generically (in the sense of the theorem), we have $v_1^{(i,j)} \neq 0$, because H is invertible. In particular, we have

$$v_1^{(1,j)} = \begin{cases} h_{n_1,1}^{(1,j)} u_{n_1}^{(1,j)} & \text{if } \star = T, \\ \overline{h_{n_1,1}^{(1,j)}} u_{n_1}^{(1,j)} & \text{if } \star = *. \end{cases} \tag{3.4}$$

So by Theorem 2.10 we can compute $S^{-1} = \widehat{S} \oplus I_{n-a}$, where

$$\begin{aligned} \widehat{S} &= \bigoplus_{j=1}^{\ell_1} \mathrm{Toep}\left(v^{(1,j)}\right) \oplus \cdots \oplus \bigoplus_{j=1}^{\ell_m} \mathrm{Toep}\left(v^{(m,j)}\right) \in \mathbb{F}^{a\times a}, \quad \text{if } \star = T, \\ \widehat{S} &= \bigoplus_{j=1}^{\ell_1} \mathrm{Toep}\left(\overline{v^{(1,j)}}\right) \oplus \cdots \oplus \bigoplus_{j=1}^{\ell_m} \mathrm{Toep}\left(\overline{v^{(m,j)}}\right) \in \mathbb{C}^{a\times a}, \quad \text{if } \star = *. \end{aligned}$$

Thus, we obtain that

$$S^{-1}(A+B)S = S^{-1}(A+uv^\star)S$$

is in partial Brunovsky form (2.16) with respect to $\widehat{\lambda}$ and

$$w := S^{-1}u = \begin{bmatrix} w^{(1)} \\ \vdots \\ w^{(m)} \\ \widetilde{w} \end{bmatrix}, \quad w^{(i)} = \begin{bmatrix} w^{(i,1)} \\ \vdots \\ w^{(i,\ell_i)} \end{bmatrix}, \quad w^{(i,j)} = \begin{bmatrix} w_1^{(i,j)} \\ \vdots \\ w_{n_i}^{(i,j)} \end{bmatrix} \in \mathbb{F}^{n_i}, \quad \widetilde{w} \in \mathbb{F}^{n-a},$$

where

$$w_{n_1}^{(1,j)} = (v_1^{(1,j)})^\star u_{n_1}^{(1,j)} = \begin{cases} h_{n_1,1}^{(1,j)} \left(u_{n_1}^{(1,j)}\right)^2 & \text{if } \star = T, \\ h_{n_1,1}^{(1,j)} |u_{n_1}^{(1,j)}|^2 & \text{if } \star = *. \end{cases}$$

By Theorem 2.10, and taking into account formula (2.13), the characteristic polynomial of $A + B - \widehat{\lambda} I$ is given by

$$p_{\widehat{\lambda}}(\lambda) = \sum_{i=a-n_1}^{n} c_i \lambda^i,$$

where

$$c_{a-n_1} = M \sum_{j=1}^{\ell_1} w_{n_1}^{(1,j)} = M \cdot \begin{cases} \sum_{j=1}^{\ell_1} h_{n_1,1}^{(1,j)} \left(u_{n_1}^{(1,j)}\right)^2 & \text{if } \star = T, \\ \sum_{j=1}^{\ell_1} h_{n_1,1}^{(1,j)} |u_{n_1}^{(1,j)}|^2 & \text{if } \star = *; \end{cases}$$

here $M \neq 0$ is a constant independent of B. Clearly, c_{a-n_1} is generically (in the sense stated in the theorem) nonzero and hence the algebraic multiplicity of the eigenvalue $\widehat{\lambda}$ of $A+B$ is $a-n_1$. Together with Theorem 2.10, we obtain that the only possible Jordan canonical forms for $A+B$ are given by (3.3). □

Theorem 3.2 *Let $A \in \mathbb{F}^{n\times n}$ and let $T, H \in \mathbb{F}^{n\times n}$ be invertible matrices such that*

$$T^{-1}AT = \widehat{A} \oplus \breve{A} \oplus \widetilde{A}, \quad T^{\star}HT = \begin{bmatrix} 0 & I_a \\ \widehat{H} & 0 \end{bmatrix} \oplus \widetilde{H}, \tag{3.5}$$

where the decomposition (3.5) *has the following properties:*

(a)
$$\widehat{A} = \left(\mathcal{J}_{n_1}(\widehat{\lambda})^{\oplus \ell_1}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(\widehat{\lambda})^{\oplus \ell_m}\right),$$
where $n_1 > n_2 > \cdots > n_m$ and $\widehat{\lambda} \in \mathbb{F}$;

(b) $a = \sum_{j=1}^{m} \ell_j n_j$ *and* $\widehat{H}, \breve{A} \in \mathbb{F}^{a\times a}$, $\widetilde{H} \in \mathbb{F}^{(n-2a)\times(n-2a)}$;

(c) $\sigma(\breve{A}), \sigma(\widetilde{A}) \subseteq \mathbb{C} \setminus \{\widehat{\lambda}\}$.

If $B \in \mathbb{F}^{n\times n}$ is a rank one perturbation of the form $B = uu^{\star}H$, $u \in \mathbb{F}^n$, then generically (with respect to the components of u if $\star = T$, and with respect to the real and imaginary parts of the components of u if $\star = *$) $A+B$ has the Jordan canonical form (3.3)*.*

Note that $\widehat{H}, \widetilde{H}$ are necessarily invertible.

Proof. As in the proof of Theorem 3.1, we may assume that A and H are in the forms (3.5). Partition

$$u = \begin{bmatrix} \widehat{u} \\ \breve{u} \\ \widetilde{u} \end{bmatrix}, \quad \widehat{u} = \begin{bmatrix} \widehat{u}^{(1)} \\ \vdots \\ \widehat{u}^{(m)} \end{bmatrix}, \quad \widehat{u}^{(i)} = \begin{bmatrix} \widehat{u}^{(i,1)} \\ \vdots \\ \widehat{u}^{(i,\ell_i)} \end{bmatrix}, \quad \widehat{u}^{(i,j)} = \begin{bmatrix} \widehat{u}_1^{(i,j)} \\ \vdots \\ \widehat{u}_{n_i}^{(i,j)} \end{bmatrix} \in \mathbb{F}^{n_i},$$

and

$$\breve{u} = \begin{bmatrix} \breve{u}^{(1)} \\ \vdots \\ \breve{u}^{(m)} \end{bmatrix}, \quad \breve{u}^{(i)} = \begin{bmatrix} \breve{u}^{(i,1)} \\ \vdots \\ \breve{u}^{(i,\ell_i)} \end{bmatrix}, \quad \breve{u}^{(i,j)} = \begin{bmatrix} \breve{u}_1^{(i,j)} \\ \vdots \\ \breve{u}_{n_i}^{(i,j)} \end{bmatrix} \in \mathbb{F}^{n_i}.$$

Observe that the vector $v = H^{\star}u$ has the form

$$v = \begin{bmatrix} \breve{u} \\ \widehat{H}\widehat{u} \\ \widetilde{H}\widetilde{u} \end{bmatrix}.$$

Generically (in the sense of the theorem), we can now form the matrix $S = \widehat{S} \oplus I_{n-a}$, where

$$\widehat{S}^{-1} = \left(\bigoplus_{j=1}^{\ell_1} \mathrm{Toep}\,(\breve{u}^{(1,j)}) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{\ell_m} \mathrm{Toep}\,(\breve{u}^{(m,j)}) \right) \quad \text{if } \star = T,$$

$$\widehat{S}^{-1} = \left(\bigoplus_{j=1}^{\ell_1} \mathrm{Toep}\,(\overline{\breve{u}^{(1,j)}}) \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{\ell_m} \mathrm{Toep}\,(\overline{\breve{u}^{(m,j)}}) \right) \quad \text{if } \star = *.$$

Then $S^{-1}(A+B)S = S^{-1}(A+uv^{\star})S$ is in partial Brunovsky form (2.16) as in Theorem 2.10. Next, consider the vector

$$w := S^{-1}u = \begin{bmatrix} \widehat{w} \\ \breve{w} \\ \widetilde{w} \end{bmatrix}, \ \widehat{w} = \begin{bmatrix} \widehat{w}^{(1)} \\ \vdots \\ \widehat{w}^{(m)} \end{bmatrix}, \ \widehat{w}^{(i)} = \begin{bmatrix} \widehat{w}^{(i,1)} \\ \vdots \\ \widehat{w}^{(i,\ell_i)} \end{bmatrix}, \ \widehat{w}^{(i,j)} = \begin{bmatrix} \widehat{w}_1^{(i,j)} \\ \vdots \\ \widehat{w}_{n_i}^{(i,j)} \end{bmatrix} \in \mathbb{F}^{n_i}.$$

Then we obtain

$$\widehat{w}_{n_i}^{(i,j)} = (\breve{u}_1^{(i,j)})^{\star} \widehat{u}_{n_i}^{(i,j)}.$$

By Theorem 2.10, the characteristic polynomial $p_{\widehat{\lambda}}$ of $A + B - \widehat{\lambda} I$ has the form

$$p_{\widehat{\lambda}}(\lambda) = c_n \lambda^n + \cdots + c_{a-n_1+1}\lambda^{a-n_1+1} + c_{a-n_1}\lambda^{a-n_1},$$

where

$$c_{a-n_1} = M \cdot \left(\sum_{j=1}^{\ell_1} \widehat{w}_{n_1}^{(1,j)} \right) = M \left(\sum_{j=1}^{\ell_1} (\breve{u}_1^{(1,j)})^{\star} \widehat{u}_{n_1}^{(1,j)} \right);$$

$M \neq 0$ is a constant independent of B. Clearly, c_{a-n_1} is generically (in the sense indicated in the statement of Theorem 3.2) nonzero and thus $\widetilde{a} = a - n_1$ is generically the algebraic multiplicity of the eigenvalue $\widehat{\lambda}$ of $A + B$. Together with Theorem 2.10, it follows that the only possible Jordan canonical forms for $A + B$ are as in (3.3). □

Note that the scenario in Theorems 3.1 and 3.2 corresponds exactly to the scenario under arbitrary unstructured rank one perturbations; cf. Theorem 2.3.

Observe that the case when $\mathbb{F} = \mathbb{R}$ and $\widehat{\lambda}$ nonreal is covered in these theorems: Just apply the complex version of the theorems to this particular case.

The particular forms of the matrix H in Theorems 3.1 and 3.2 are set with a view for applications to many types of structured matrices. The two theorems apply to the cases of symmetric complex matrix H and H-symmetric matrices (see Theorem 2.6) discussed in Section 5, and also to the case of H-selfadjoint matrices discussed in [25]. Finally, they apply to the case where J is skew-symmetric, and A is J-Hamiltonian and invertible (case (iii) in Theorem 2.7). Thus, for J-Hamiltonian matrices it remains to study the case of the eigenvalue zero. This will be done in the next section.

4 Generic structured rank one perturbations for complex J-Hamiltonian matrices

In this section we state and prove one of the main results of the paper concerning perturbations of complex J-Hamiltonian matrices. According to Theorem 2.7, if $\lambda \neq 0$ is an eigenvalue of a complex J-Hamiltonian matrix A, then so is $-\lambda$ (with the same partial multiplicities), and for every odd k, the number of Jordan blocks in the Jordan form of A of size k corresponding to the zero eigenvalue is even.

As Theorem 4.2 shows, in the case the largest partial multiplicity of the zero eigenvalue is odd, the generic behavior of the Jordan structure of the perturbed matrix contrasts sharply with the unstructured situation (Theorem 2.3). To motivate the main result, consider an example first:

Example 4.1 Consider the matrix

$$Z(w) = \begin{bmatrix} \mathcal{J}_{2m+1}(0) & 0 \\ 0 & \mathcal{J}_{2m+1}(0) \end{bmatrix} + ww^T \begin{bmatrix} 0 & \Sigma_{2m+1} \\ -\Sigma_{2m+1} & 0 \end{bmatrix} \in \mathbb{C}^{(4m+2)\times(4m+2)}.$$

We will show that generically (with respect to the components of $w \in \mathbb{C}^{4m+2}$) $Z(w)$ has the Jordan from of type $\mathcal{J}_{2m+2}(0) \oplus k_1 \oplus k_2 \oplus \cdots \oplus k_{2m}$, where the k_j's are distinct nonzero complex numbers.

A standard transformation allows us to consider the J-Hamiltonian (see (1.1)) matrix

$$M := M(u,v) := \begin{bmatrix} \mathcal{J}_{2m+1}(0) & 0 \\ 0 & -\mathcal{J}_{2m+1}(0)^T \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} -v^T & u^T \end{bmatrix}$$

instead of $Z(w)$. Indeed, one verifies that

$$\begin{bmatrix} I_{2m+1} & 0 \\ 0 & \Sigma_{2m+1} \end{bmatrix} \cdot \begin{bmatrix} 0 & \Sigma_{2m+1} \\ -\Sigma_{2m+1} & 0 \end{bmatrix} \cdot \begin{bmatrix} I_{2m+1} & 0 \\ 0 & \Sigma_{2m+1} \end{bmatrix} = J,$$

and

$$\begin{bmatrix} I_{2m+1} & 0 \\ 0 & \Sigma_{2m+1} \end{bmatrix} \cdot Z(w) \cdot \begin{bmatrix} I_{2m+1} & 0 \\ 0 & \Sigma_{2m+1} \end{bmatrix} = M(u', \Sigma_{2m+1}v'),$$

where we have put $w = \begin{bmatrix} u' \\ v' \end{bmatrix}$, $u', v' \in \mathbb{C}^{2m+1}$. (Note that $\Sigma_{2m+1} = \Sigma_{2m+1}^* = \Sigma_{2m+1}^{-1}$.)

We shall denote the entries of u and v by $u_1, \ldots, u_{2m+1}$ and $v_1, \ldots, v_{2m+1}$, respectively. Clearly, M is singular for all u and v. It is easy to see that for some choice of u and v the rank of M is equal to $4m+1$, and therefore there exists a generic (with respect to the entries of u and v) set Ω such that for every $(u,v) \in \Omega$ the rank of M is equal to $4m+1$ (cf. Lemma 2.1).

Next, we introduce the $(2m+1) \times (2m+1)$ matrix Υ:

$$\Upsilon = \Sigma_{2m+1} R_{2m+1} = 1 \oplus (-1) \oplus 1 \oplus (-1) \oplus \cdots \oplus (-1) \oplus 1.$$

It is useful to note that $\mathcal{J}_{2m+1} := \mathcal{J}_{2m+1}(0)$ and Υ anti-commute:

$$\Upsilon \mathcal{J}_{2m+1} = -\mathcal{J}_{2m+1}\Upsilon.$$

Our first observation is that the vector

$$x_1 = \begin{bmatrix} \mathcal{J}_{2m+1}^{2m} u \\ (\mathcal{J}_{2m+1}^{2m})^T v \end{bmatrix} \in \text{Ker } M.$$

Indeed, $Ax_1 = 0$, where

$$A = \begin{bmatrix} \mathcal{J}_{2m+1}(0) & 0 \\ 0 & -\mathcal{J}_{2m+1}(0)^T \end{bmatrix},$$

and $\begin{bmatrix} -v^T & u^T \end{bmatrix} x_1 = 0$ as well. Now define for $j = 2, \dots, 2m+1$ the vectors

$$x_j = (-1)^{j+1} \begin{bmatrix} \Upsilon \mathcal{J}_{2m+1}^{2m+1-j} u \\ (\mathcal{J}_{2m+1}^{2m+1-j})^T \Upsilon v \end{bmatrix}.$$

Note that for all j we have $\begin{bmatrix} -v^T & u^T \end{bmatrix} x_j = 0$, and so

$$Mx_j = Ax_j = (-1)^{j+1} \begin{bmatrix} \mathcal{J}_{2m+1} \Upsilon \mathcal{J}_{2m+1}^{2m+1-j} u \\ -(\mathcal{J}_{2m+1}^{2m+2-j})^T \Upsilon v \end{bmatrix} = (-1)^j \begin{bmatrix} \Upsilon \mathcal{J}_{2m+1}^{2m+2-j} u \\ (\mathcal{J}_{2m+1}^{2m+2-j})^T \Upsilon v \end{bmatrix} = x_{j-1}.$$

Thus we see that $x_1, \dots, x_{2m+1}$ is a Jordan chain of M corresponding to zero.

Next, note that

$$x_{2m+1} = \begin{bmatrix} \Upsilon u \\ \Upsilon v \end{bmatrix}.$$

We now define for some complex numbers a and b, still to be determined, the vector

$$x_{2m+2} = ae_1 + be_{4m+2} + \begin{bmatrix} -(I+\Upsilon)\mathcal{J}_{2m+1}^T u \\ (I+\Upsilon)\mathcal{J}_{2m+1} v \end{bmatrix}.$$

Then

$$\begin{aligned} Ax_{2m+2} &= A \begin{bmatrix} -(I+\Upsilon)\mathcal{J}_{2m+1}^T u \\ (I+\Upsilon)\mathcal{J}_{2m+1} v \end{bmatrix} = \begin{bmatrix} -\mathcal{J}_{2m+1}(I+\Upsilon)\mathcal{J}_{2m+1}^T u \\ -\mathcal{J}_{2m+1}^T (I+\Upsilon)\mathcal{J}_{2m+1} v \end{bmatrix} \\ &= \begin{bmatrix} -(I-\Upsilon)\mathcal{J}_{2m+1}\mathcal{J}_{2m+1}^T u \\ -(I-\Upsilon)\mathcal{J}_{2m+1}^T \mathcal{J}_{2m+1} v \end{bmatrix} = \begin{bmatrix} -(I-\Upsilon)u \\ -(I-\Upsilon)v \end{bmatrix} = x_{2m+1} - \begin{bmatrix} u \\ v \end{bmatrix}. \end{aligned}$$

So

$$\begin{aligned} Mx_{2m+2} &= \left(x_{2m+1} - \begin{bmatrix} u \\ v \end{bmatrix} \right) + \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} -v^T & u^T \end{bmatrix} x_{2m+2} \\ &= x_{2m+1} + \begin{bmatrix} u \\ v \end{bmatrix} (\begin{bmatrix} -v^T & u^T \end{bmatrix} x_{2m+2} - 1). \end{aligned}$$

If we can choose a and b so that

$$\begin{bmatrix} -v^T & u^T \end{bmatrix} x_{2m+2} = -av_1 + bu_{2m+1} + 2u^T(I+\Upsilon)\mathcal{J}_{2m+1}v = 1,$$

then we have constructed a Jordan chain of length $2m+2$ for M corresponding to the eigenvalue zero. But it is easily seen that a and b can be chosen as desired, whenever not both $u_{2m+1} = 0$ and $v_1 = 0$. So, generically this can be done. Note that $x_1, \dots, x_{2m+2}$ are linearly independent as one easily verifies using the properties $x_1 \neq 0$ (generically), $Mx_{j+1} = x_j$ for $j = 1, 2, \dots, m+1$, and $Mx_1 = 0$.

The next step is to see that generically the Jordan block with eigenvalue zero of M has size $2m+2$. Here we make essential use of the fact that we already know that the rank of M generically is $4m+1$, and hence there can be at most one Jordan block with eigenvalue zero in the Jordan normal form of M. Then for any Jordan chain it must be possible to extend it to a Jordan chain of length equal to the algebraic multiplicity (this follows, for example, from general results on marked invariant subspaces in [8]). So, it suffices to show that the Jordan chain we have constructed cannot be extended further. For this, observe that vectors $\begin{bmatrix} z \\ y \end{bmatrix}$ in the range of M are such that $\begin{bmatrix} z_{2m+1} \\ y_1 \end{bmatrix}$ is a multiple of $\begin{bmatrix} u_{2m+1} \\ v_1 \end{bmatrix}$. So, in order for x_{2m+2} to be in the range of M it is necessary and sufficient that $\begin{bmatrix} -u_{2m} \\ v_2 \end{bmatrix}$ is a multiple of $\begin{bmatrix} u_{2m+1} \\ v_1 \end{bmatrix}$. Obviously, generically this will not be the case.

Next, we show that generically all nonzero eigenvalues are simple eigenvalues. The characteristic polynomial of M is, by what we have shown, generically of the form

$$x^{2m+2}(x^{2m} + x^{2m-2}a_{2m-2} + \dots + x^2 a_2 + a_0)$$

(we also use that M is J-Hamiltonian matrix, and so its characteristic polynomial is a polynomial in x^2), and generically, $a_0 \neq 0$.

Now we find particular vectors u_0 and v_0 such that for the characteristic polynomial of $M(u_0, v_0)$ we have $a_2 = a_4 = \dots = a_{2m-2} = 0$. Indeed, take u_0, v_0 with zero entries, except for $(u_0)_{2m}, (u_0)_{2m+1}, (v_0)_1, (v_0)_2$. Then

$$
\begin{aligned}
& \det\left(A - xI + \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \begin{bmatrix} -v_0^T & u_0^T \end{bmatrix}\right) = \\
= \; & \det\left\{(A-xI)\left(I + (A-xI)^{-1}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \begin{bmatrix} -v_0^T & u_0^T \end{bmatrix}\right)\right\} = \\
= \; & \det(A-xI)\det\left(I + (A-xI)^{-1}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \begin{bmatrix} -v_0^T & u_0^T \end{bmatrix}\right) = \\
= \; & \det(A-xI)\left(1 + \begin{bmatrix} -v_0^T & u_0^T \end{bmatrix}(A-xI)^{-1}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right) = \\
= \; & x^{4m+2}\left(1 + \begin{bmatrix} -v_0^T & u_0^T \end{bmatrix}(A-xI)^{-1}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right) = \\
= \; & x^{4m+2}(1 - v_0^T(\mathcal{J}_{2m+1} - xI)^{-1}u_0 - u_0^T(\mathcal{J}_{2m+1}^T + xI)^{-1}v_0).
\end{aligned}
$$

Now take u_0 and v_0 as above, so v_0 having only the first two entries nonzero and u_0 having only the last two entries nonzero. Then it is clear that we are interested in the 2×2 block in the right upper corner of $(\mathcal{J}_{2m+1} - xI)^{-1}$, and the 2×2 block in the left lower corner of $(\mathcal{J}_{2m+1}^T + xI)^{-1}$. It is easily computed that

$$
\begin{aligned}
-v_0^T(\mathcal{J}_{2m+1} - xI)^{-1}u_0 &= \frac{(v_0)_1(u_0)_{2m} + (v_0)_2(u_0)_{2m+1}}{x^{2m}} + \frac{(v_0)_2(u_0)_{2m}}{x^{2m-1}} + \frac{(v_0)_1(u_0)_{2m+1}}{x^{2m+1}}, \\
-u_0^T(\mathcal{J}_{2m+1}^T + xI)^{-1}v_0 &= \frac{(v_0)_1(u_0)_{2m} + (v_0)_2(u_0)_{2m+1}}{x^{2m}} - \frac{(v_0)_2(u_0)_{2m}}{x^{2m-1}} - \frac{(v_0)_1(u_0)_{2m+1}}{x^{2m+1}}.
\end{aligned}
$$

Because the terms with odd powers cancel, the characteristic polynomial of $M(u_0, v_0)$ is given by

$$\det\big(M(u_0, v_0) - xI\big) = x^{2m+2}\Big(x^{2m} + 2\big((v_0)_1(u_0)_{2m} + (v_0)_2(u_0)_{2m+1}\big)\Big),$$

and so for such a perturbation the nonzero eigenvalues are all simple.

Now, there is an open neighborhood U of the pair (u_0, v_0) such that for all matrices $M(u, v)$ with $(u, v) \in U$ all nonzero eigenvalues are simple. Choosing $(u, v) \in U$ so that also the multiplicity of zero of $M(u, v)$ is equal to $2m+2$, we have found an open set of vectors w with the property that $Z(w)$ has the Jordan form of the required type. But then the set of all vectors w for which $Z(w)$ has the Jordan form of the required type is generic; to see that use the Sylvester resultant matrix of the characteristic polynomial of $Z(w)$ and of its derivative, as it was done in the proof of Lemma 2.5. □

The next theorem shows that the situation of Example 4.1 is typical for the case of odd largest partial multiplicity corresponding to the zero eigenvalue. We assume in the next theorem that A has zero as an eigenvalue; if A is invertible, then all statements concerning the zero eigenvalue should be considered as void.

Theorem 4.2 *Let $J \in \mathbb{C}^{n\times n}$ be skew-symmetric and invertible, let $A \in \mathbb{C}^{n\times n}$ be J-Hamiltonian, with pairwise distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_p, \lambda_{p+1} = 0$ and let B be a rank one perturbation of the form $B = uu^T J \in \mathbb{C}^{n\times n}$.*

For every λ_j, $j = 1, 2, \ldots, p+1$, let $n_{1,j} > n_{2,j} > \ldots > n_{m_j,j}$ be the sizes of Jordan blocks in the Jordan form of A associated with the eigenvalue λ_j, and let there be exactly $\ell_{k,j}$ Jordan blocks of size $n_{k,j}$ associated with λ_j in the Jordan form of A, for $k = 1, 2, \ldots, m_j$.

(1) *If $n_{1,p+1}$ is even (in particular, if A is invertible), then generically with respect to the components of u, the matrix $A + B$ has the Jordan canonical form*

$$\bigoplus_{j=1}^{p+1} \left(\left(\mathcal{J}_{n_{1,j}}(\lambda_j)^{\oplus \ell_{1,j}-1}\right) \oplus \left(\mathcal{J}_{n_{2,j}}(\lambda_j)^{\oplus \ell_{2,j}}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_{m_j,j}}(\lambda_j)^{\oplus \ell_{m_j,j}}\right)\right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}}$ contains all the Jordan blocks of $A + B$ associated with eigenvalues different from any of $\lambda_1, \ldots, \lambda_{p+1}$.

(2) *If $n_{1,p+1}$ is odd (in this case $\ell_{1,p+1}$ is even), then generically with respect to the components of u, the matrix $A + B$ has the Jordan canonical form*

$$\begin{aligned} &\bigoplus_{j=1}^{p} \left(\left(\mathcal{J}_{n_{1,j}}(\lambda_j)^{\oplus \ell_{1,j}-1}\right) \oplus \left(\mathcal{J}_{n_{2,j}}(\lambda_j)^{\oplus \ell_{2,j}}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_{m_j,j}}(\lambda_j)^{\oplus \ell_{m_j,j}}\right)\right) \\ &\oplus \left(\mathcal{J}_{n_{1,p+1}}(0)^{\oplus \ell_{1,p+1}-2}\right) \oplus \left(\mathcal{J}_{n_{2,p+1}}(0)^{\oplus \ell_{2,p+1}}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_{m_{p+1},p+1}}(0)^{\oplus \ell_{m_{p+1},p+1}}\right) \\ &\oplus \mathcal{J}_{n_{1,p+1}+1}(0) \oplus \widetilde{\mathcal{J}}, \end{aligned} \tag{4.1}$$

where $\widetilde{\mathcal{J}}$ contains all the Jordan blocks of $A + B$ associated with eigenvalues different from any of $\lambda_1, \ldots, \lambda_{p+1}$.

(3) *In either case (1) or (2), generically the part $\widetilde{\mathcal{J}}$ has simple eigenvalues.*

Proof. If (1) holds, then it follows from Theorem 2.7 that we can apply Theorem 3.1 or Theorem 3.2, and we immediately obtain the desired result; here we also use the easily verifiable fact that the intersection of finitely many generic sets is again generic.

Consider the case (2). In this case, generically the part of the Jordan form of $A + B$ that involves nonzero eigenvalues has again the form as given in (2), in view of Theorems 2.7, 3.1, and 3.2. It remains to prove that generically the part of the Jordan form of $A + B$ corresponding to the zero eigenvalue has the form

$$\mathcal{J}_{n_1+1}(0) \oplus \left(\mathcal{J}_{n_1}(0)^{\oplus \ell_1 - 2}\right) \oplus \left(\mathcal{J}_{n_2}(0)^{\oplus \ell_2}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(0)^{\oplus \ell_m}\right) \oplus \widehat{\mathcal{J}}. \tag{4.2}$$

Here, we let $m = m_{p+1}$; $n_k = n_{k,p+1}$ for $k = 1, 2, \ldots, m$; and $\ell_k = \ell_{k,p+1}$ for $k = 1, 2, \ldots, m$, and $\widehat{\mathcal{J}}$ contains all the Jordan blocks of $A + B$ associated with nonzero eigenvalues.

To this end, we may assume without loss of generality that A and J are in the form (2.11), where we assume in addition that the diagonal blocks of A and J have been permuted in such a way that the blocks associated with the eigenvalue zero appear first and that they are ordered with decreasing sizes. Thus, we assume that A and J have the forms

$$A = \left(\mathcal{J}_{n_1}(0)^{\oplus \ell_1}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_m}(0)^{\oplus \ell_m}\right) \oplus \widetilde{A}, \tag{4.3}$$

where $\sigma(\widetilde{A}) \subseteq \mathbb{C} \setminus \{0\}$ and

$$J = \begin{bmatrix} 0 & \Sigma_{n_1} \\ -\Sigma_{n_1} & 0 \end{bmatrix}^{\oplus \ell_1/2} \oplus J_2 \oplus \cdots \oplus J_m \oplus \widetilde{J}. \tag{4.4}$$

Then the algebraic and geometric multiplicity a and g of the eigenvalue zero of A are given by

$$a = \sum_{s=1}^{m} \ell_s n_s, \quad g = \sum_{s=1}^{m} \ell_s,$$

respectively. The corresponding J-Hamiltonian rank one perturbation B has the form $B = uv^T = uu^T J$, where we partition

$$u = \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(m)} \\ \widetilde{u} \end{bmatrix}, \quad u^{(i)} = \begin{bmatrix} u^{(i,1)} \\ \vdots \\ u^{(i,\ell_i)} \end{bmatrix}, \quad u^{(i,s)} = \begin{bmatrix} u_1^{(i,s)} \\ \vdots \\ u_{n_i}^{(i,s)} \end{bmatrix} \in \mathbb{C}^{n_i},$$

for $s = 1, \ldots, \ell_i$; $i = 1, \ldots, m$. Thus, $\widetilde{u} \in \mathbb{C}^{n-a}$. We will now show in two steps that generically $A + B$ has the Jordan canonical form (4.2). By Theorem 2.10 we know that generically $A + B$ has $\ell_1 - 1$ Jordan chains of length n_1 and ℓ_j Jordan chains of length n_j, $j = 2, \ldots, m$ associated with the eigenvalue zero. (Theorem 2.10 is applicable because the hypothesis that the first component of each vector $v^{(i,j)}$, in the notation of Theorem 2.10, is nonzero, is satisfied in our situation.) In the first step, we will show that generically there exists a Jordan chain of length $n_1 + 1$. In the second step, we will show that the algebraic multiplicity of the eigenvalue zero of $A + B$ generically is $\widetilde{a} = (\sum_{s=1}^{m} \ell_s n_s) - n_1 + 1 = a - n_1 + 1$. Both steps together obviously imply that (4.2) represents the only possible Jordan canonical forms for $A + B$.

Step 1: Existence of a Jordan chain of length $n_1 + 1$.

Generically, the hypothesis of Theorem 2.10 is satisfied (i.e., specific entries of vectors are nonzero), so generically the matrix S as in Theorem 2.10 exists so that $S^{-1}(A + B)S$ is in partial Brunovsky form. We first investigate the structure of the vector $v^T = u^T J$. From (4.4), we obtain that v has the form

$$v = (u^T J)^T = -Ju = \begin{bmatrix} v^{(1)} \\ \vdots \\ v^{(m)} \\ \widetilde{v} \end{bmatrix}, \quad v^{(i)} = \begin{bmatrix} v^{(i,1)} \\ \vdots \\ v^{(i,\ell_i)} \end{bmatrix}, \quad v^{(i,s)} = \begin{bmatrix} v_1^{(i,s)} \\ \vdots \\ v_{n_i}^{(i,s)} \end{bmatrix} \in \mathbb{C}^{n_i}, \tag{4.5}$$

for $s = 1, \ldots, \ell_i$ and $i = 1, \ldots, m$, where

$$v^{(1,2s-1)} = -\Sigma_{n_1} u^{(1,2s)} = \begin{bmatrix} -u_{n_1}^{(1,2s)} \\ u_{n_1-1}^{(1,2s)} \\ \mp \quad \vdots \\ -u_1^{(1,2s)} \end{bmatrix}, \quad v^{(1,2s)} = \Sigma_{n_1} u^{(1,2s-1)} = \begin{bmatrix} u_{n_1}^{(1,2s-1)} \\ -u_{n_1-1}^{(1,2s-1)} \\ \pm \quad \vdots \\ u_1^{(1,2s-1)} \end{bmatrix}$$

for $s = 1, \ldots, \ell_1/2$. Thus, S^{-1} takes the form

$$S^{-1} = \left(\bigoplus_{s=1}^{\ell_1} \mathrm{Toep}\left(v^{(1,s)}\right) \right) \oplus \cdots \oplus \left(\bigoplus_{s=1}^{\ell_m} \mathrm{Toep}\left(v^{(m,s)}\right) \right) \oplus I_{n-a},$$

and it follows that

$$S^{-1}BS = w(\underbrace{e_{1,n_1}^T, \ldots, e_{1,n_1}^T}_{\ell_1 \text{ times}}, \ldots, \underbrace{e_{1,n_m}^T, \ldots, e_{1,n_m}^T}_{\ell_m \text{ times}}, z^T) \tag{4.6}$$

for some $z \in \mathbb{C}^{n-a}$. Thus,

$$w = S^{-1}u = \begin{bmatrix} w^{(1)} \\ \vdots \\ w^{(m)} \\ \widetilde{w} \end{bmatrix}, \quad w^{(i)} = \begin{bmatrix} w^{(i,1)} \\ \vdots \\ w^{(i,\ell_i)} \end{bmatrix}, \quad w^{(i,s)} = \begin{bmatrix} w_1^{(i,s)} \\ \vdots \\ w_{n_i}^{(i,s)} \end{bmatrix} \in \mathbb{C}^{n_i}, \tag{4.7}$$

for $s = 1, \ldots, \ell_i$ and $i = 1, \ldots, m$, where

$$w_{n_1}^{(1,2s-1)} = -u_{n_1}^{(1,2s)} u_{n_1}^{(1,2s-1)}, \qquad w_{n_1}^{(1,2s)} = u_{n_1}^{(1,2s-1)} u_{n_1}^{(1,2s)} = -w_{n_1}^{(1,2s-1)} \tag{4.8}$$

and, provided that $n_1 > 1$,

$$w_{n_1-1}^{(1,2s-1)} = u_{n_1-1}^{(1,2s)} u_{n_1}^{(1,2s-1)} - u_{n_1}^{(1,2s)} u_{n_1-1}^{(1,2s-1)}, \tag{4.9}$$

$$w_{n_1-1}^{(1,2s)} = -u_{n_1}^{(1,2s)} u_{n_1-1}^{(1,2s-1)} + u_{n_1-1}^{(1,2s)} u_{n_1}^{(1,2s-1)} = w_{n_1-1}^{(1,2s-1)}, \tag{4.10}$$

for $s = 1, \ldots, \ell_1/2$. Consider the following Jordan chains associated with the eigenvalue zero of $S^{-1}(A+B)S$ and denoted by $C_{i,s}$:

$$\begin{array}{lll} \text{length } n_1: & C_{1,s}: \quad e_{2(s-1)n_1+1} - e_{(2s-1)n_1+1}, \ldots, e_{(2s-1)n_1} - e_{2sn_1}, & s = 1, \ldots, \frac{\ell_1}{2} \\ \text{length } n_i: & C_{i,s}: \quad -e_1 + e_{\Sigma_{k=1}^{i-1}\ell_k n_k + (s-1)n_i + 1}, \ldots, -e_{n_i} + e_{\Sigma_{k=1}^{i-1}\ell_k n_k + sn_i}, & s = 1, \ldots, \ell_i, \end{array}$$

where $i = 2, \ldots, m$. Observe that $C_{i,s}$, $i \neq 1$, are just the Jordan chains from Theorem 2.10 multiplied by -1 while the chains $C_{1,s}$ are linear combinations of the Jordan chains from Theorem 2.10. Namely, in the notation of (2.14), and numbering the chains in (2.14) first, second, etc., from the top to the bottom, we see that the chains $C_{1,1}, \ldots, C_{1,\ell_1/2}$ are the first chain, the negative of the second chain plus the third chain, ..., the negative of the $(\ell_1 - 2)$-th chain plus the $(\ell_1 - 1)$-th chain, respectively. Now consider the Jordan chain

$$C := \left(\sum_{s=1}^{\ell_1/2} \alpha_{1,s} C_{1,s} \right) + \sum_{i=2}^{m} \sum_{s=1}^{\ell_i} \alpha_{i,s} C_{i,s}$$

of length n_1 (see Definition 2.12), and let y denote the n_1-th (and thus last) vector of this chain. We next show that the Jordan chain C can be extended by a certain vector to a Jordan chain of length $n_1 + 1$ associated with the eigenvalue zero, for some particular choice of the parameters $\alpha_{i,s}$ (depending on u) such that generically at least one of $\alpha_{1,1}, \ldots \alpha_{1,\ell_1/2}$ is nonzero. To see this, we have to show that y is in the range of $S^{-1}(A + B)S$. First, partition

$$y = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(m)} \\ \widetilde{y} \end{bmatrix}, \quad y^{(i)} = \begin{bmatrix} y^{(i,1)} \\ \vdots \\ y^{(i,\ell_i)} \end{bmatrix}, \quad y^{(i,s)} = \begin{bmatrix} y_1^{(i,s)} \\ \vdots \\ y_{n_i}^{(i,s)} \end{bmatrix} \in \mathbb{C}^{n_i},$$

for $s = 1, \ldots, \ell_i$; $i = 1, \ldots, m$. Then by the definition of y, we have $\widetilde{y} = 0 \in \mathbb{C}^{n-a}$,

$$\begin{aligned} y_{n_1}^{(1,2s-1)} &= \alpha_{1,s}, \quad y_{n_1}^{(1,2s)} = -\alpha_{1,s}, \quad s = 1, \ldots, \ell_1/2, \\ y_{n_i}^{(i,s)} &= \alpha_{i,s}, \quad s = 1, \ldots, \ell_i; \quad i = 2, \ldots, m. \end{aligned}$$

We have to solve the linear system

$$S^{-1}(A + B)Sx = y. \tag{4.11}$$

Partitioning

$$x = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(m)} \\ \widetilde{x} \end{bmatrix}, \quad x^{(i)} = \begin{bmatrix} x^{(i,1)} \\ \vdots \\ x^{(i,\ell_i)} \end{bmatrix}, \quad x^{(i,s)} = \begin{bmatrix} x_1^{(i,s)} \\ \vdots \\ x_{n_i}^{(i,s)} \end{bmatrix} \in \mathbb{C}^{n_i},$$

and making the ansatz $\widetilde{x}=0$, then equation (4.11) becomes (we use here (4.6, (4.7))):

$$w_k^{(i,s)}\left(\sum_{\nu=1}^{m}\sum_{\mu=1}^{\ell_\nu}x_1^{(\nu,\mu)}\right)+x_{k+1}^{(i,s)}=y_k^{(i,s)},\qquad k{=}1,...,n_i{-}1;\ s{=}1,...,\ell_i;\ i{=}1,...,m,\quad(4.12)$$

$$w_{n_i}^{(i,s)}\left(\sum_{\nu=1}^{m}\sum_{\mu=1}^{\ell_\nu}x_1^{(\nu,\mu)}\right)=\alpha_{i,s},\qquad s{=}1,...,\ell_i;\ i{=}2,...,m,\qquad(4.13)$$

$$w_{n_1}^{(1,2s-1)}\left(\sum_{\nu=1}^{m}\sum_{\mu=1}^{\ell_\nu}x_1^{(\nu,\mu)}\right)=\alpha_{1,s},\qquad s{=}1,...,\ell_1/2,\qquad(4.14)$$

$$w_{n_1}^{(1,2s)}\left(\sum_{\nu=1}^{m}\sum_{\mu=1}^{\ell_\nu}x_1^{(\nu,\mu)}\right)={-}\alpha_{1,s},\quad s{=}1,...,\ell_1/2.\qquad(4.15)$$

Set $x_1^{(1,1)}=1$ and $x_1^{(\nu,\mu)}=0$, for $\mu=1,\ldots,\ell_\nu$; $\nu=1,\ldots,m$; $(\nu,\mu)\neq(1,1)$, as well as $\alpha_{i,s}=w_{n_i}^{(i,s)}$ for $s=1,\ldots,\ell_i$; $i=2,\ldots,m$ and $\alpha_{1,s}=w_{n_1}^{(1,2s-1)}$ for $s=1,\ldots,\ell_1/2$. Then (4.13) and (4.14) are satisfied and so is (4.15), because by (4.8) we have

$$w_{n_1}^{(1,2s)}=u_{n_1}^{(1,2s)}u_{n_1}^{(1,2s-1)}=-w_{n_1}^{(1,2s-1)}=-\alpha_{1,s},\quad s=1,\ldots,\ell_1/2.$$

Finally, (4.12) can be solved by choosing $x_{k+1}^{(i,s)}=y_k^{(i,s)}-w_k^{(i,s)}$ for $k=1,\ldots,n_i-1$; $s=1,\ldots,\ell_i$; $i=1,\ldots,m$.

Step 2: We show that the algebraic multiplicity of the eigenvalue zero of $A+B$ generically is $\widetilde{a}=(\sum_{s=1}^m\ell_s n_s)-n_1+1=a-n_1+1$.

Let $\mu_1,\ldots,\mu_q$ denote the pairwise distinct nonzero eigenvalues of A and let $r_1,\ldots,r_q$ be their algebraic multiplicities. By Theorem 2.10, the lowest possible power of λ associated with a nonzero coefficient in $p_0(\lambda)$ is $a-n_1$ and the corresponding coefficient c_{a-n_1} is

$$c_{a-n_1}=(-1)^{a-1}\left(\prod_{i=1}^{q}\mu_i^{r_i}\right)\left(\sum_{s=1}^{\ell_1}w_{n_1}^{(1,s)}\right)=0,$$

because of (4.8). If $n_1=1$ then $\widetilde{a}=a$ and there is nothing to show as the algebraic multiplicity of the eigenvalue zero cannot increase when a generic perturbation is applied. Otherwise, we distinguish the cases $n_2<n_1-1$ and $n_2=n_1-1$. If $n_2<n_1-1$,

then by Theorem 2.10 the coefficient c_{a-n_1+1} of λ^{a-n_1+1} in $p_0(\lambda)$ is

$$\begin{aligned} c_{a-n_1+1} &= (-1)^a \left(\sum_{\nu=1}^{q} r_\nu \mu_\nu^{r_\nu - 1} \prod_{\substack{i=1 \\ i\neq\nu}}^{q} \mu_i^{r_i} \right) \underbrace{\left(\sum_{s=1}^{\ell_1} w_{n_1}^{(1,s)} \right)}_{=0 \text{ using } (4.8)} \\ & \quad + (-1)^{a-1} \left(\prod_{i=1}^{q} \mu_i^{r_i} \right) \left(\sum_{s=1}^{\ell_1} w_{n_1-1}^{(1,s)} \right) \\ &\overset{\text{using } (4.10)}{=} (-1)^{a-1} \left(\prod_{i=1}^{q} \mu_i^{r_i} \right) \left(\sum_{s=1}^{\ell_1/2} 2(u_{n_1-1}^{(1,2s)} u_{n_1}^{(1,2s-1)} - u_{n_1}^{(1,2s)} u_{n_1-1}^{(1,2s-1)}) \right) \end{aligned}$$

which generically is nonzero. If, on the other hand, $n_2 = n_1 - 1$, then again by Theorem 2.10 the coefficient c_{a-n_1+1} of λ^{a-n_1+1} in $p_0(\lambda)$ is

$$c_{a-n_1+1} = (-1)^{a-1} \left(\prod_{i=1}^{q} \mu_i^{r_i} \right) \left(\sum_{s=1}^{\ell_1} w_{n_1-1}^{(1,s)} + \sum_{s=1}^{\ell_2} w_{n_2}^{(2,s)} \right).$$

Since $n_1 > 1$ is odd, $n_2 \geq 2$ is even and the block J_2 in (4.4) takes the form

$$J_2 = \Sigma_{n_2} \oplus \cdots \oplus \Sigma_{n_2}.$$

Hence, for the component v_2 in (4.5) we obtain that

$$v^{(2,s)} = -\Sigma_{n_2} u^{(2,s)} = \begin{bmatrix} -u_{n_2}^{(2,s)} \\ u_{n_2-1}^{(2,s)} \\ \mp \vdots \\ u_1^{(2,s)} \end{bmatrix}, \quad s = 1, \ldots, \ell_2$$

and thus

$$w_{n_2}^{(2,s)} = -u_{n_2}^{(2,s)} v_1^{(2,s)} = -(u_{n_2}^{(2,s)})^2,$$

which gives

$$c_{a-n_1+1} = (-1)^{a-1} \left(\prod_{i=1}^{q} \mu_i^{r_i} \right) \left(\sum_{s=1}^{\ell_1/2} 2(u_{n_1-1}^{(1,2s)} u_{n_1}^{(1,2s-1)} - u_{n_1}^{(1,2s)} u_{n_1-1}^{(1,2s-1)}) - \sum_{s=1}^{\ell_2} (u_{n_2}^{(2,s)})^2 \right).$$

Again, this is nonzero generically. In all cases, we have shown that zero is a root of $p_0(\lambda)$ with multiplicity $a - n_1 + 1$. Thus, the algebraic multiplicity of the eigenvalue zero of $A + B$ is $a - n_1 + 1$. Together with Step 1, we obtain that (4.2) generically are the only possible Jordan canonical forms of $A + B$.

Finally, we prove part (3) by following the arguments of the proof of part (b) of Theorem 2.3, and using Examples 4.3–4.5 and Lemma 4.6 (instead of Lemma 2.5 that was used the proof of Theorem 2.3) presented in the remainder of the section. □

Example 4.3 Let

$$Z^{(2)}(\alpha) = \mathcal{J}_{2m}(0) + (\alpha e_{2m})(\alpha e_{2m}^T)\Sigma_{2m} \in \mathbb{C}^{2m\times 2m}, \quad \alpha \in \mathbb{C}\setminus\{0\}.$$

Analogously to Example 2.4, we have $\chi(Z^{(2)}(\alpha)) = x^{2m} + \alpha^2$, in particular, $Z^{(2)}(\alpha)$ has $2m$ distinct nonzero eigenvalues. ∎

Example 4.4 Consider the $(4m+2)\times(4m+2)$ matrix

$$Z^{(3)}(\alpha, w) = \begin{bmatrix} \mathcal{J}_{2m+1}(0) & 0 \\ 0 & \mathcal{J}_{2m+1}(0) \end{bmatrix} + (\alpha w)(\alpha w^T)\begin{bmatrix} 0 & \Sigma_{2m+1} \\ -\Sigma_{2m+1} & 0 \end{bmatrix},$$

where $\alpha \in \mathbb{C}\setminus\{0\}$. It follows from Example 4.1 that there exist a nonzero vector w and $\epsilon > 0$ with the property that the matrix $Z^{(3)}(\alpha, w)$ has the Jordan form $\mathcal{J}_{2m+2}(0)\oplus\mathcal{K}$, where $\mathcal{K}$ is a diagonal invertible matrix with distinct diagonal entries, for every α in the punctured disc $0 < |\alpha| < \epsilon$. ∎

Example 4.5 Let

$$\begin{aligned} Z^{(4)}(\lambda, \alpha) \quad = \quad & \begin{bmatrix} \mathcal{J}_m(\lambda) & 0 \\ 0 & -\mathcal{J}_m(\lambda)^T \end{bmatrix} + (\alpha u)(\alpha u)^T \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} \in \mathbb{C}^{2m\times 2m}, \\ & \lambda \in \mathbb{C}\setminus\{0\}, \quad \alpha \in \mathbb{C}\setminus\{0\}. \end{aligned}$$

Let

$$u = \begin{bmatrix} e_m \\ e_1 \end{bmatrix}.$$

We shall prove that there exists $\epsilon > 0$ which depends only on λ and on m, such that for all α with $0 < |\alpha| < \epsilon$, the matrix $Z^{(4)}(\lambda, \alpha)$ has $2m$ distinct eigenvalues and none of them is equal to $\pm\lambda$.

Using the Laplace theorem for determinants with respect to the first m rows of $\det(xI - Z^{(4)}(\lambda, \alpha))$, and omitting terms that are obviously zeros, we easily compute

$$\chi(Z^{(4)}(\lambda, \alpha)) = ((x-\lambda)^m + \alpha^2)((x+\lambda)^m + (-1)^m\alpha^2) + (-1)^{m+1}\alpha^4 =$$

$$(x-\lambda)^m(x+\lambda)^m + \alpha^2(x+\lambda)^m + (-1)^m\alpha^2(x-\lambda)^m.$$

Clearly, $\pm\lambda$ are not zeros of $\chi(Z^{(4)}(\lambda, \alpha)$ because $\lambda \neq 0$, $\alpha \neq 0$. Assuming that $\chi(Z^{(4)}(\lambda, \alpha))$ and $\frac{\partial}{\partial x}\chi(Z^{(4)}(\lambda, \alpha))$ have a common root x_0, we have the equalities

$$(x_0-\lambda)^m(x_0+\lambda)^m + \beta(x_0+\lambda)^m + (-1)^m\beta(x_0-\lambda)^m = 0, \tag{4.16}$$

$$(x_0-\lambda)^{m-1}(x_0+\lambda)^m + (x_0-\lambda)^m(x_0+\lambda)^{m-1} + \beta(x_0+\lambda)^{m-1} + (-1)^m\beta(x_0-\lambda)^{m-1} = 0, \tag{4.17}$$

where $\beta = \alpha^2$. Multiplying (4.17) by $x_0 - \lambda$ and using (4.16) yields after simple algebra

$$(x_0-\lambda)^{m+1} = 2\beta\lambda.$$

Analogously $(x_0+\lambda)^{m+1} = (-1)^{m+1}2\beta\lambda$ is obtained. These equalities are contradictory if $|\alpha|$ is sufficiently small. ∎

Using Examples 4.3, 4.4, and 4.5, and the already proved parts (1) and (2) of Theorem 4.2, the following lemma is proved in the same way as Lemma 2.5. We omit the details of proof.

Lemma 4.6 *Let Ω' be the (open) generic set of vectors $u \in \mathbb{C}^n$ for which* (1) *or* (2) *of Theorem* 4.2 *holds. Then there is $\epsilon > 0$ and an open dense (in the ball $\{u \in \mathbb{C}^n : \|u\| < \epsilon\}$) set*

$$\Omega'' \subseteq \Omega' \cap \{u \in \mathbb{C}^n : \|u\| < \epsilon\}$$

such that for every $u \in \Omega''$, the Jordan form of $A + uu^T J$ is of the type described in items (1) - (3) *of Theorem* 4.2.

We conclude that in case (1) of Theorem 4.2 generically all Jordan blocks associated with eigenvalues $\lambda_1, \ldots, \lambda_{p+1}$ remain unchanged except for one block of the largest size for every eigenvalue λ_j which disappears (leading to eigenvalues different from λ_j). In the case (2), the generic behavior of Jordan blocks of nonzero eigenvalues is the same as in the case (1), whereas all Jordan blocks associated with the zero eigenvalue remain unchanged except for two of the largest size ones of which one of them disappears (leading to nonzero eigenvalues), while the other one increases its size by one.

5 Generic structured rank one perturbations for complex H-symmetric matrices

Our next result concerns perturbations of H-symmetric matrices. In view of Theorem 2.6 every matrix $X \in \mathbb{C}^{n\times n}$ is similar to an H-symmetric matrix, for any fixed symmetric invertible matrix H. Indeed, assuming (without loss of generality) that X is in the Jordan form as in (2.10), we see that X is R-symmetric, where $R = R_{n_1} \oplus \cdots \oplus R_{n_m}$; on the other hand, there exist an invertible (complex) matrices S_1 and S_2 such that $S_1^T H S_1 = I = S_2^T R S_2$ (as follows by applying Theorem 2.6 to the R-symmetric and H-symmetric zero matrix), and so $S_1 S_2^{-1} X S_2 S_1^{-1}$ is H-symmetric.

Theorem 5.1 *Let $H \in \mathbb{C}^{n\times n}$ be symmetric and invertible, $A \in \mathbb{C}^{n\times n}$ be H-symmetric, with pairwise distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_p$ and having the Jordan canonical form*

$$\bigoplus_{j=1}^{p} \Big(\big(\mathcal{J}_{n_{1,j}}(\lambda_j)^{\oplus \ell_{1,j}}\big) \oplus \big(\mathcal{J}_{n_{2,j}}(\lambda_j)^{\oplus \ell_{2,j}}\big) \oplus \cdots \oplus \Big(\mathcal{J}_{n_{m_j,j}}(\lambda_j)^{\oplus \ell_{m_j,j}}\Big)\Big)$$

where $n_{1,j} > n_{2,j} > \ldots > n_{m_j,j}$, $j = 1, \ldots p$. Let $B \in \mathbb{C}^{n\times n}$ be a rank one perturbation of the form $B = uu^T H$, $u \in \mathbb{C}^n$. Then:

(1) *generically (with respect to the components of u), the matrix $A+B$ has the Jordan canonical form*

$$\bigoplus_{j=1}^{p} \Big(\big(\mathcal{J}_{n_{1,j}}(\lambda_j)^{\oplus \ell_{1,j}-1}\big) \oplus \big(\mathcal{J}_{n_{2,j}}(\lambda_j)^{\oplus \ell_{2,j}}\big) \oplus \cdots \oplus \Big(\mathcal{J}_{n_{m_j,j}}(\lambda_j)^{\oplus \ell_{m_j,j}}\Big)\Big) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}}$ contains all the Jordan blocks of $A + B$ associated with eigenvalues different from any of $\lambda_1, \ldots, \lambda_p$;

(2) *generically, all eigenvalues of $A + B$ different from any of $\lambda_1, \ldots, \lambda_p$, are simple.*

Proof. Part (1) follows immediately from Theorem 2.6 and Theorem 3.1.

Part (2) is proved completely analogously to the proofs of part (b) of Theorem 2.3 and part (3) of Theorem 4.2 by using Lemma 5.2 below which is based on Example 2.4. We omit details. □

Lemma 5.2 *Let Ω' be the open generic set of vectors $u \in \mathbb{C}^n$ for which* (1) *of Theorem* 5.1 *holds. Then there is $\epsilon > 0$ and an open dense (in the ball $\{u \in \mathbb{C}^n \,:\, \|u\| < \epsilon\}$) set*

$$\Omega'' \subseteq \Omega' \cap \{u \in \mathbb{C}^n \,:\, \|u\| < \epsilon\}$$

such that for every $u \in \Omega''$ the Jordan form of $A + uu^T H$ is of the type described in items (1) *and* (2) *of Theorem* 5.1.

6 Conclusion

We have presented several results on Jordan structures of matrices under structured and unstructured rank one perturbations in a general context, and studied the perturbation analysis for the Jordan structures of complex J-Hamiltonian and complex H-symmetric matrices under structured rank one perturbations. We have shown that as in the case of unstructured perturbations, generically only (one of) the largest Jordan blocks is destroyed; genericity here is understood in the sense of structured rank one perturbations. However in the structured case, there is a particular situation, where the effect of generic structured perturbation differs from the effect of generic unstructured perturbations. If the largest Jordan block associated with the eigenvalue zero of a complex J-Hamiltonian matrix has odd size, then this Jordan block must occur an even number of times. As the result of a generic rank one complex J-Hamiltonian perturbation, one of the largest Jordan blocks is destroyed and the size of one other largest Jordan block is increased by one.

In subsequent papers, this perturbation analysis will be extended to the cases of H-selfadjoint matrices under generic H-selfadjoint rank one perturbations [25], and real H-symmetric matrices under real perturbations.

Acknowledgments. We thank the referees for very careful reading of the manuscript and many useful suggestions, and S.V. Savchenko for pointing out to us the paper [18].

References

[1] B. Adhikari. *Backward perturbation and sensitivity analysis of structured polynomial eigenvalue problem.* PhD thesis, IIT Guwahati, Dept. of Mathematics, 2009.

[2] T. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM Publications, Philadelphia, PA, USA, 2005.

[3] S. Barnett. Greatest common divisor of several polynomials. *Proc. Cambridge Philos. Soc.*, 70:263–268, 1971.

[4] M.A. Beitia, I. de Hoyos, and I. Zaballa. The change of the Jordan structure under one row perturbations. *Linear Algebra Appl.*, 401:119–134, 2005.

[5] P. Benner, R. Byers, V. Mehrmann, and H. Xu. A robust numerical method for the γ-iteration in H_∞-control. *Linear Algebra Appl.*, 425:548–570, 2007.

[6] P. Benner, D. Kressner, and V. Mehrmann. Structure preservation: a challenge in computational control. *Future Generation Comp. Syst.*, 19:1243–1252, 2003.

[7] S. Bora and V. Mehrmann. Linear perturbation theory for structured matrix pencils arising in control theory. *SIAM J. Matrix Anal. Appl.*, 28:148–169, 2006.

[8] R. Bru, L. Rodman, and H. Schneider. Extensions of Jordan bases for invariant subspaces of a matrix. *Linear Algebra Appl.*, 150:209–225, 1991.

[9] P. Brunovsky. A classification of linear controllable systems. *Kybernetika (Prague)*, 6:173–188, 1970.

[10] B.M. Chen. *Robust and H_∞ Control*. Springer-Verlag, London, UK, 2000.

[11] D. Chu, X. Liu, and V. Mehrmann. A numerical method for computing the Hamiltonian Schur form. *Numer. Math.*, 105:375–412, 2007.

[12] C.P. Coelho, J.R. Phillips, and L.M. Silveira. Robust rational function approximation algorithm for model generation. In *Proceedings of the 36th DAC*, pages 207–212, New Orleans, Louisiana, USA, 1999.

[13] R.W. Freund and F. Jarre. An extension of the positive real lemma to descriptor systems. *Optimization methods and software*, 19:69–87, 2004.

[14] P.A. Fuhrmann. *Linear Systems and Operators in Hilbert Space*. McGraw Hill, New York, 1981.

[15] I. Gohberg and G. Heinig. The resultant matrix and its generalizations. I. The resultant operator of matrix polynomials. *Acta Sci. Math. (Szeged)*, 37:41–61, 1975. (Russian).

[16] S. Grivet-Talocia. Passivity enforcement via perturbation of Hamiltonian matrices. *IEEE Trans. Circuits Systems*, 51:1755–1769, 2004.

[17] B. Gustavsen and A. Semlyen. Enforcing passivity for admittance matrices approximated by rational functions. *IEEE Trans. on Power Systems*, 16:97–104, 2001.

[18] L. Hörmander and A. Melin. A remark on perturbations of compact operators. *Math. Scand.*, 75:255–262, 1994.

[19] M.M. Konstantinov, D.W. Gu, V. Mehrmann, and P.Hr. Petkov. *Perturbation Theory for Matrix Equations*. Elsevier, North Holland, 2003.

[20] M. Krupnik. Changing the spectrum of an operator by perturbation. *Linear Algebra Appl.*, 167:113–118, 1992.

[21] P. Lancaster and L. Rodman. *The Algebraic Riccati Equation*. Oxford University Press, Oxford, 1995.

[22] P. Lancaster and L. Rodman. Canonical forms for symmetric/skew-symmetric real matrix pairs under strict equivalence and congruence. *Linear Algebra Appl.*, 406:1–76, 2005.

[23] P. Lancaster and M. Tismenetsky. *The Theory of Matrices*. Academic Press, Orlando, FL, 2nd edition, 1985.

[24] W.-W. Lin, V. Mehrmann, and H. Xu. Canonical forms for Hamiltonian and symplectic matrices and pencils. *Linear Algebra Appl.*, 301–303:469–533, 1999.

[25] C. Mehl, V. Mehrmann, A. Ran, and L. Rodman. Perturbation theory of selfadjoint matrices and sign characteristics under generic structured rank one perturbations. *Linear Algebra Appl.*, To appear, 2010. Preprint 675 Matheon DFG Research Center *Mathematics for key technologies* in Berlin `url: http://www.matheon.de/` `DOI: http://dx.doi.org/10.1016/j.laa.2010.04.008`.

[26] V. Mehrmann and H. Xu. Numerical methods in control. *J. Comput. Appl. Math.*, 123:371–394, 2000.

[27] V. Mehrmann and H. Xu. Perturbation of purely imaginary eigenvalues of Hamiltonian matrices under structured perturbations. *Electr. J. Lin. Alg.*, 17:234–257, 2008.

[28] J. Moro and F. Dopico. Low rank perturbation of Jordan structure. *SIAM J. Matrix Anal. Appl.*, 25:495–506, 2003.

[29] D. Saraswat, R. Achar, and M. Nakhia. On passivity check and compensation of macromodels from tabulated data. In *7th IEEE Workshop on Signal Propagation on Interconnects*, pages 25–28, Siena, Italy, 2003.

[30] S.V. Savchenko. Typical changes in spectral properties under perturbations by a rank-one operator. *Mat. Zametki*, 74:590–602, 2003. (Russian). Translation in Mathematical Notes. 74:557–568, 2003.

[31] S.V. Savchenko. On the change in the spectral properties of a matrix under a perturbation of a sufficiently low rank. *Funktsional. Anal. i Prilozhen*, 38:85–88, 2004. (Russian). Translation in Funct. Anal. Appl. 38:69–71, 2004.

[32] S.V. Savchenko. Laurent expansion for the determinant of the matrix of scalar resolvents. *Mathematicheskii Sbornik*, 196:121–144, 2005. (Russian). Translation in Sbornik: Mathematics 196:743–764, 2005.

[33] C. Schröder and T. Stykel. Passivation of LTI systems. Technical Report 368, DFG Research Center Matheon, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany, 2007.

[34] G.W. Stewart and J.-G. Sun. *Matrix Perturbation Theory*. Academic Press, New York, 1990.

[35] R. C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 147:323–371, 1991.

[36] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, NJ, 1995.