Theorem: Let (E, A) be a pair of matrix functions and let the strangeness index μ of (E, A) be well defined. Then with the notation from 3.1., the pair (E, A) is globally equivalent to the pair

$$\left(\begin{bmatrix} I_{d_\mu} & 0 & W \\ 0 & 0 & F \\ 0 & 0 & G \end{bmatrix}, \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{a_\mu} \end{bmatrix} \right),$$

with

$$F = \begin{array}{c} \\ w_\mu \\ \vdots \\ w_1 \\ w_0 \end{array} \begin{array}{c} \begin{array}{cccc} c_\mu & c_{\mu-1} & \cdots & c_0 \end{array} \\ \begin{bmatrix} 0 & F_\mu & & * \\ & \ddots & \ddots & \\ & & \ddots & F_1 \\ & & & 0 \end{bmatrix} \end{array}, \quad G = \begin{array}{c} \\ c_\mu \\ \vdots \\ c_1 \\ c_0 \end{array} \begin{array}{c} \begin{array}{cccc} c_\mu & c_{\mu-1} & \cdots & c_0 \end{array} \\ \begin{bmatrix} 0 & F_\mu & & * \\ & \ddots & \ddots & \\ & & \ddots & F_1 \\ & & & 0 \end{bmatrix} \end{array},$$

$$W = {}_{d_\mu} \overset{\begin{array}{cccc} c_\mu & c_{\mu-1} & \cdots & c_0 \end{array}}{\begin{bmatrix} 0 & * & \cdots & * \end{bmatrix}},$$

and rank $\begin{bmatrix} F_i \\ G_i \end{bmatrix} = w_i + c_i = s_{i-1}$.

Proof: (E, A) is globally equivalent to the global canonical form:

$$(E,A) \sim \left(\begin{array}{c} \\ \begin{array}{c} s_0 \\ d_0 \\ a_0 \\ s_0 \\ v_0 \end{array} \end{array} \begin{array}{c} \begin{array}{cccc} s_0 & d_0 & a_0 & u_0 \end{array} \\ \begin{bmatrix} I_{s_0} & 0 & 0 & 0 \\ 0 & I_{d_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ \begin{array}{c} s_0 \\ d_0 \\ a_0 \\ s_0 \\ v_0 \end{array} \end{array} \begin{array}{c} \begin{array}{cccc} s_0 & d_0 & a_0 & u_0 \end{array} \\ \begin{bmatrix} 0 & A_{12}^{(0)} & 0 & A_{14}^{(0)} \\ 0 & 0 & 0 & A_{24}^{(0)} \\ 0 & 0 & I_{a_0} & 0 \\ I_{s_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \right)$$

Proof: (E,A) is globally equivalent to the global canonical form:

$$(E,A) \sim \left(\begin{array}{c} \\ s_0 \\ d_0 \\ a_0 \\ s_0 \\ v_0 \end{array} \begin{array}{c} \begin{array}{cccc} s_0 & d_0 & a_0 & u_0 \end{array} \\ \begin{bmatrix} I_{s_0} & 0 & 0 & 0 \\ 0 & I_{d_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ s_0 \\ d_0 \\ a_0 \\ s_0 \\ v_0 \end{array} \begin{array}{c} \begin{array}{cccc} s_0 & d_0 & a_0 & u_0 \end{array} \\ \begin{bmatrix} 0 & A_{12}^{(0)} & 0 & A_{14}^{(0)} \\ 0 & 0 & 0 & A_{24}^{(0)} \\ 0 & 0 & I_{a_0} & 0 \\ I_{s_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \right)$$

Group the columns 1 and 3 together $(a_0 + s_0 = c_0)$ as well as the rows 3 and 4 and permute the form such that those blocks come last:

$$(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_0 \\ s_0 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & u_0 & c_0 \end{array} \\ \begin{bmatrix} I_{d_0} & 0 & 0 \\ 0 & 0 & \widetilde{U}_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_0 \\ s_0 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & u_0 & c_0 \end{array} \\ \begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right),$$

where $U_0 = \left[\, I_{s_0} \;\; 0 \, \right]$, $A_{12} = A_{24}^{(0)}$, $A_{21} = A_{12}^{(0)}$, $A_{22} = A_{14}^{(0)}$.

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_0 \\ s_0 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & u_0 & c_0 \end{array} \\ \begin{bmatrix} I_{d_0} & 0 & 0 \\ 0 & 0 & \widetilde{U}_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_0 \\ s_0 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & u_0 & c_0 \end{array} \\ \begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right)
$$

Next, perform a smooth 'SVD' on A_{22}. We have $b_1 = \text{rank}\, A_{22}$ by definition of b_1.

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_0 \\ s_0 - b_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{cccc} d_0 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} I_{d_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_0 \\ s_0 - b_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{cccc} d_0 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} 0 & A_{12} & A_{13} & 0 \\ A_{21} & 0 & 0 & 0 \\ A_{31} & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).
$$

$$(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_0 \\ s_0-b_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{cccc} d_0 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} I_{d_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_0 \\ s_0-b_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{cccc} d_0 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} 0 & A_{12} & A_{13} & 0 \\ A_{21} & 0 & 0 & 0 \\ A_{31} & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).$$

A_{21} has rank s_1, so perform a smooth 'SVD' on A_{21}:

$$(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_0 \\ s_0-b_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{cccc} d_0 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} I_{d_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_0 \\ s_0-b_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{cccc} d_0 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} 0 & A_{12} & A_{13} & 0 \\ A_{21} & 0 & 0 & 0 \\ A_{31} & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).$$

A_{21} has rank s_1, so perform a smooth 'SVD' on A_{21} (use $s_0 - b_1 - s_1 = s_0 - c_1 = w_1$):

$$(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & s_1 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} I_{d_1} & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 & U_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & s_1 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & 0 \\ A_{51} & A_{52} & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).$$

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c|ccccc|} & d_1 & s_1 & u_1 & b_1 & c_0 \\ d_1 & I_{d_1} & 0 & 0 & 0 & 0 \\ s_1 & 0 & I_{s_1} & 0 & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & U_1 \\ s_1 & 0 & 0 & 0 & 0 & U_2 \\ b_1 & 0 & 0 & 0 & 0 & U_3 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 & 0 & 0 \end{array}, \quad \begin{array}{c|ccccc|} & d_1 & s_1 & u_1 & b_1 & c_0 \\ d_1 & A_{11} & A_{12} & A_{13} & A_{14} & 0 \\ s_1 & A_{21} & A_{22} & A_{23} & A_{24} & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & I_{s_1} & 0 & 0 & 0 \\ b_1 & A_{51} & A_{52} & 0 & I_{b_1} & 0 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 & 0 & I_{c_0} \end{array} \right).
$$

The blocks $A_{11}, A_{12}, A_{21}, A_{22}$ appeared due to a '$\dot{Q}$-transformation'.

$$(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ \begin{matrix} d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} d_1 & s_1 & u_1 & b_1 & c_0 \end{matrix} \\ \begin{bmatrix} I_{d_1} & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 & U_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ \begin{matrix} d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} d_1 & s_1 & u_1 & b_1 & c_0 \end{matrix} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & 0 \\ A_{51} & A_{52} & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).$$

Next, we can eliminate the **red blocks** by smooth Gauss transformations using I_{b_1}.

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ \begin{array}{c} d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & s_1 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} I_{d_1} & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & * \\ 0 & 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 & U_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ \begin{array}{c} d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & s_1 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & 0 \\ A_{51} & A_{52} & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).
$$

Next, we can eliminate the **red blocks** by smooth Gauss transformations using I_{b_1}.

$$
(E, A) \overset{new}{\sim} \left(\begin{array}{c|ccccc|} & d_1 & s_1 & u_1 & b_1 & c_0 \\ d_1 & I_{d_1} & 0 & 0 & 0 & * \\ s_1 & 0 & I_{s_1} & 0 & 0 & * \\ w_1 & 0 & 0 & 0 & 0 & U_1 \\ s_1 & 0 & 0 & 0 & 0 & U_2 \\ b_1 & 0 & 0 & 0 & 0 & U_3 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 & 0 & 0 \end{array}, \begin{array}{c|ccccc|} & d_1 & s_1 & u_1 & b_1 & c_0 \\ d_1 & A_{11} & A_{12} & A_{13} & 0 & 0 \\ s_1 & A_{21} & A_{22} & A_{23} & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & I_{s_1} & 0 & 0 & 0 \\ b_1 & A_{51} & A_{52} & 0 & I_{b_1} & 0 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 & 0 & I_{c_0} \end{array} \right).
$$

Next, we can eliminate the **red blocks** by smooth Gauss transformations using I_{b_1}.

$$(E,A) \overset{new}{\sim} \left(\begin{array}{c|ccccc|} & d_1 & s_1 & u_1 & b_1 & c_0 \\ d_1 & I_{d_1} & 0 & 0 & 0 & * \\ s_1 & 0 & I_{s_1} & 0 & 0 & * \\ w_1 & 0 & 0 & 0 & 0 & U_1 \\ s_1 & 0 & 0 & 0 & 0 & U_2 \\ b_1 & 0 & 0 & 0 & 0 & U_3 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 & 0 & 0 \end{array}, \begin{array}{c|ccccc|} & d_1 & s_1 & u_1 & b_1 & c_0 \\ d_1 & A_{11} & A_{12} & A_{13} & 0 & 0 \\ s_1 & A_{21} & A_{22} & A_{23} & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & I_{s_1} & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & I_{b_1} & 0 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 & 0 & I_{c_0} \end{array} \right).$$

Next, we can eliminate the **red blocks** by smooth Gauss transformations using I_{b_1}. (There will be no new '$\dot{Q}$-blocks', because the forth block column of E is zero.)

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c|ccccc} & d_1 & s_1 & u_1 & b_1 & c_0 \\ \hline d_1 & I_{d_1} & 0 & 0 & 0 & * \\ s_1 & 0 & I_{s_1} & 0 & 0 & * \\ w_1 & 0 & 0 & 0 & 0 & U_1 \\ s_1 & 0 & 0 & 0 & 0 & U_2 \\ b_1 & 0 & 0 & 0 & 0 & U_3 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 & 0 & 0 \end{array} \;,\; \begin{array}{c|ccccc} & d_1 & s_1 & u_1 & b_1 & c_0 \\ \hline d_1 & A_{11} & A_{12} & A_{13} & 0 & 0 \\ s_1 & A_{21} & A_{22} & A_{23} & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & I_{s_1} & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & I_{b_1} & 0 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 & 0 & I_{c_0} \end{array} \right).
$$

Next, eliminate A_{12}, A_{22} with I_{s_1}.

$$(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & s_1 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} I_{d_1} & 0 & 0 & 0 & * \\ 0 & I_{s_1} & 0 & 0 & * \\ 0 & 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 & U_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & s_1 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} A_{11} & 0 & A_{13} & 0 & 0 \\ A_{21} & 0 & A_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).$$

Next, eliminate A_{12}, A_{22} with I_{s_1}.

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ \begin{matrix} d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} d_1 & s_1 & u_1 & b_1 & c_0 \end{matrix} \\ \begin{bmatrix} I_{d_1} & 0 & 0 & 0 & * \\ 0 & I_{s_1} & 0 & 0 & * \\ 0 & 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 & U_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ \begin{matrix} d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} d_1 & s_1 & u_1 & b_1 & c_0 \end{matrix} \\ \begin{bmatrix} 0 & 0 & A_{13} & 0 & 0 \\ A_{21} & 0 & A_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).
$$

Next, eliminate A_{11} using the '$\dot{Q}_1 = A_{11}Q_1$'-trick.

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & s_1 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} I_{d_1} & 0 & 0 & 0 & * \\ 0 & I_{s_1} & 0 & 0 & * \\ 0 & 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 & U_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_1 \\ s_1 \\ w_1 \\ s_1 \\ b_1 \\ w_0 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & s_1 & u_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} 0 & 0 & A_{13} & 0 & 0 \\ A_{21} & 0 & A_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).
$$

Permute the rows....

$$
(E, A) \overset{new}{\sim} \left(\begin{array}{c|ccccc|} & d_1 & s_1 & u_1 & b_1 & c_0 \\ \hline d_1 & I_{d_1} & 0 & 0 & 0 & * \\ s_1 & 0 & I_{s_1} & 0 & 0 & * \\ w_1 & 0 & 0 & 0 & 0 & U_1 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & 0 & 0 & 0 & U_2 \\ b_1 & 0 & 0 & 0 & 0 & U_3 \\ c_0 & 0 & 0 & 0 & 0 & 0 \end{array}, \begin{array}{c|ccccc|} & d_1 & s_1 & u_1 & b_1 & c_0 \\ \hline d_1 & 0 & 0 & A_{13} & 0 & 0 \\ s_1 & A_{21} & 0 & A_{23} & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & I_{s_1} & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & I_{b_1} & 0 \\ c_0 & 0 & 0 & 0 & 0 & I_{c_0} \end{array} \right).
$$

... and columns ...

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_1 \\ s_1 \\ w_1 \\ w_0 \\ s_1 \\ b_1 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & u_1 & s_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} I_{d_1} & 0 & 0 & 0 & * \\ 0 & 0 & I_{s_1} & 0 & * \\ 0 & 0 & 0 & 0 & U_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & U_2 \\ 0 & 0 & 0 & 0 & U_3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_1 \\ s_1 \\ w_1 \\ w_0 \\ s_1 \\ b_1 \\ c_0 \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & u_1 & s_1 & b_1 & c_0 \end{array} \\ \begin{bmatrix} 0 & A_{12} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{s_1} & 0 & 0 \\ 0 & 0 & 0 & I_{b_1} & 0 \\ 0 & 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).
$$

$$(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ \begin{matrix} & d_1 & u_1 & s_1 & b_1 & c_0 \\ d_1 & I_{d_1} & 0 & 0 & 0 & * \\ s_1 & 0 & 0 & I_{s_1} & 0 & * \\ w_1 & 0 & 0 & 0 & 0 & U_1 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & 0 & 0 & 0 & U_2 \\ b_1 & 0 & 0 & 0 & 0 & U_3 \\ c_0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \\ \end{array} , \begin{array}{c} \\ \begin{matrix} & d_1 & u_1 & s_1 & b_1 & c_0 \\ d_1 & 0 & A_{12} & 0 & 0 & 0 \\ s_1 & A_{21} & A_{22} & 0 & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 & 0 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & 0 & I_{s_1} & 0 & 0 \\ b_1 & 0 & 0 & 0 & I_{b_1} & 0 \\ c_0 & 0 & 0 & 0 & 0 & I_{c_0} \end{matrix} \\ \\ \end{array} \right).$$

Then merge the s_1 and r_1 columns and rows adjacent to each other - and remember that the matrix

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

has full row rank.

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ \begin{matrix} d_1 \\ s_1 \\ w_1 \\ w_0 \\ c_1 \\ c_0 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} d_1 & u_1 & c_1 & c_0 \end{matrix} \\ \begin{bmatrix} I_{d_1} & 0 & 0 & * \\ 0 & 0 & \widetilde{U}_1 & * \\ 0 & 0 & 0 & F_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ \begin{matrix} d_1 \\ s_1 \\ w_1 \\ w_0 \\ c_1 \\ c_0 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} d_1 & u_1 & c_1 & c_0 \end{matrix} \\ \begin{bmatrix} 0 & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{c_1} & 0 \\ 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right),
$$

where $\widetilde{U}_1 = \left[\ I_{s_1} \ \ 0 \ \right]$, and where $\begin{bmatrix} F_1 \\ G_1 \end{bmatrix}$ has full row rank.

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ \begin{matrix} & d_1 & u_1 & c_1 & c_0 \\ d_1 & I_{d_1} & 0 & 0 & * \\ s_1 & 0 & 0 & \widetilde{U}_1 & * \\ w_1 & 0 & 0 & 0 & F_1 \\ w_0 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & G_1 \\ c_0 & 0 & 0 & 0 & 0 \end{matrix} \\ \\ \end{array}, \begin{array}{c} \\ \begin{matrix} & d_1 & u_1 & c_1 & c_0 \\ d_1 & 0 & A_{12} & 0 & 0 \\ s_1 & A_{21} & A_{22} & 0 & 0 \\ w_1 & 0 & 0 & 0 & 0 \\ w_0 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & I_{c_1} & 0 \\ c_0 & 0 & 0 & 0 & I_{c_0} \end{matrix} \\ \\ \end{array} \right).
$$

Recall the first step:

$$
(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ \begin{matrix} & d_0 & u_0 & c_0 \\ d_0 & I_{d_0} & 0 & 0 \\ s_0 & 0 & 0 & \widetilde{U}_0 \\ w_0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 \end{matrix} \\ \\ \end{array}, \begin{array}{c} \\ \begin{matrix} & d_0 & u_0 & c_0 \\ d_0 & 0 & A_{12} & 0 \\ s_0 & A_{21} & A_{22} & 0 \\ w_0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & I_{c_0} \end{matrix} \\ \\ \end{array} \right).
$$

Observe that the first three block columns of the first two block rows look identical. Thus, we can iterate the procedure.

After the second iteration, we obtain the following form:

$$(E,A) \overset{new}{\sim} \left(\begin{array}{c} \\ d_1 \\ s_1 \\ w_2 \\ w_1 \\ w_0 \\ c_2 \\ c_1 \\ c_0 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} d_1 & u_1 & c_2 & c_1 & c_0 \end{array} \\ \begin{bmatrix} I_{d_1} & 0 & 0 & * & * \\ 0 & 0 & \widetilde{U}_2 & * & * \\ 0 & 0 & 0 & F_2 & * \\ 0 & 0 & 0 & 0 & F_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_2 & * \\ 0 & 0 & 0 & 0 & G_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \begin{array}{c} \\ d_1 \\ s_1 \\ w_2 \\ w_1 \\ w_0 \\ c_2 \\ c_1 \\ c_0 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} d_1 & u_1 & c_2 & c_1 & c_0 \end{array} \\ \begin{bmatrix} 0 & A_{12} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{c_2} & 0 & 0 \\ 0 & 0 & 0 & I_{c_1} & 0 \\ 0 & 0 & 0 & 0 & I_{c_0} \end{bmatrix} \end{array} \right).$$

Then the statement of the theorem follows by induction after μ steps, because $\widetilde{U}_\mu$ is empty.