Diametral Pairs of Linear Extensions

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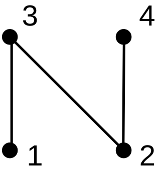
Research Training Group "Methods for Discrete Structures"
TU Berlin

September 2008

Def: Diametral Pair of Linear Extensions

$\mathcal{P}$



- Poset $\mathcal{P}$: Finite set with order relation
- Linear extension (LE) of $\mathcal{P}$:
  Complete ordering of $\mathcal{P}$ respecting the relations
- Ex: $L_1 = 1234$, $L_2 = 2413$

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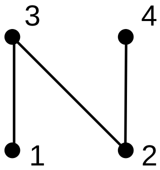

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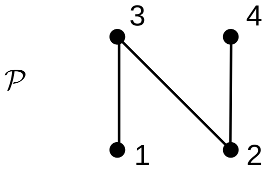

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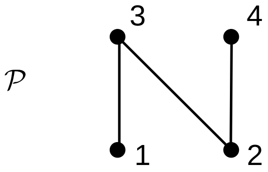

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- Distance between LEs L_1, L_2 $:= \#$ of reversals
  $:= \#$ pairs $\{x, y\}$ with $x < y$ in L_1 and $x > y$ in L_2
- L_1, L_2 diametral pair of LEs:
  L_1, L_2 maximize distance over all pairs of LEs of $\mathcal{P}$

The Linear Extension Diameter

Linear extension diameter led($\mathcal{P}$):
Distance between a diametral pair of LEs of $\mathcal{P}$

LINEAR EXTENSION DIAMETER
Input: A finite poset $\mathcal{P}$, integer k
Question: Are there two LEs of $\mathcal{P}$ with distance $\geq k$?

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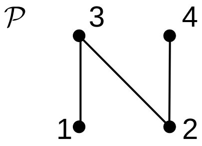
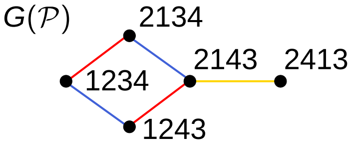
Theorem
LINEAR EXTENSION DIAMETER is polynomially solvable for posets of width 3.

The Linear Extension Graph of a Poset

- The LE-graph $G(\mathcal{P})$ is a graph on the LEs of $\mathcal{P}$.
- Each LE of $\mathcal{P}$ represents a vertex of $G(\mathcal{P})$.
- L_1 and L_2 are adjacent in $G(\mathcal{P})$ iff they differ only in the order of one pair of elements.





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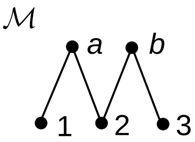

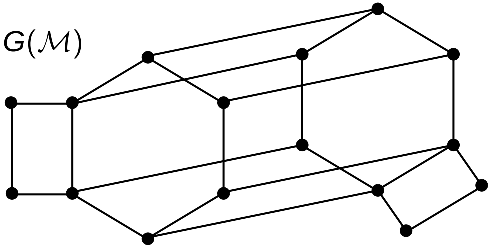

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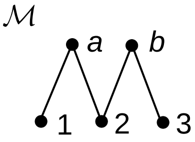

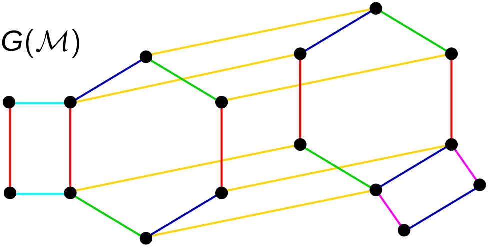

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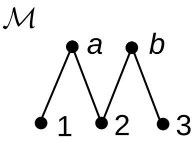

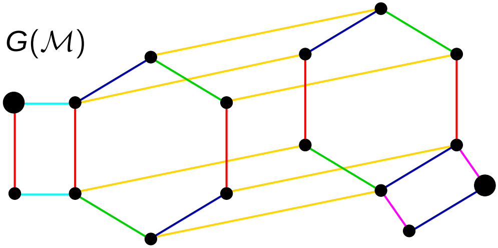

Conjecture: Diametral LEs are Reversing

- $x, y \in \mathcal{P}$ form a critical pair (x,y) if:
  - x, y incomparable in $\mathcal{P}$
  - $\text{Pred}(x) \subseteq \text{Pred}(y)$
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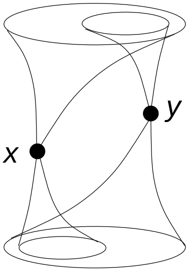

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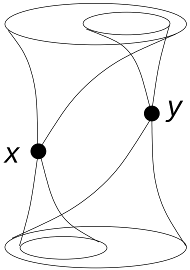
- An LE L is reversing if $y < x$ in L for some critical pair (x, y).



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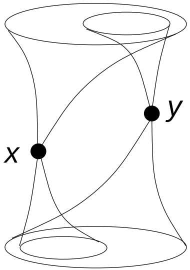

Weak Conjecture (Felsner, Reuter '99)

Let $\mathcal{P}$ be a poset which is not a chain. In every diametral pair of LEs of $\mathcal{P}$, at least one LE is reversing.

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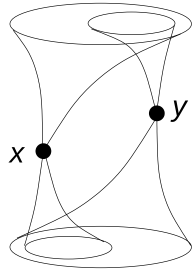

Strong Conjecture

Let $\mathcal{P}$ be a poset which is not a chain. In every diametral pair of LEs of $\mathcal{P}$, both LEs are reversing.

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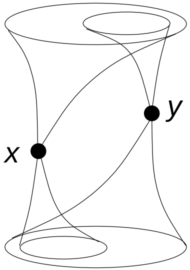

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A poset fulfilling the Strong Conjecture is diametrally reversing.

Positive Results

Definition

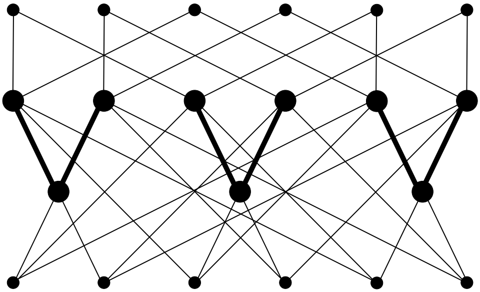
A poset $\mathcal{P}$ is diametrally reversing if in every diametral pair of LEs of $\mathcal{P}$, both LEs are reversing.

Theorem

The following posets are diametrally reversing:

- *Posets of dimension 2*
- *Posets of height 2*
- *Interval orders*
- *Random posets (with probability 1 for large n)*

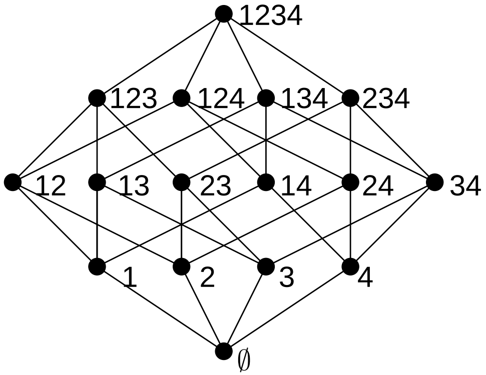
The Counterexample



Theorem

There is a poset $\mathcal{P}$ such that no LE contained in a diametral pair of $\mathcal{P}$ is reversing.

Diametral Pairs of LEs for Boolean Lattice



Conjecture (Felsner, Reuter '99)

Reverse lexicographic LE and reverse antilexicographic LE form a diametral pair for the Boolean Lattice B_n.

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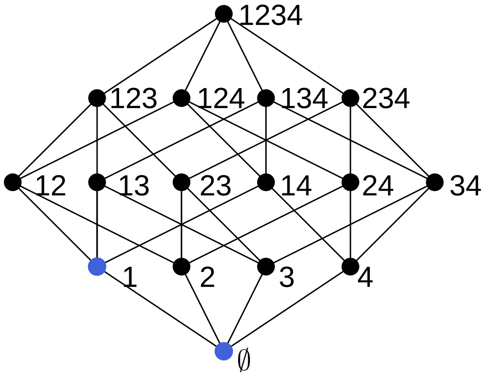

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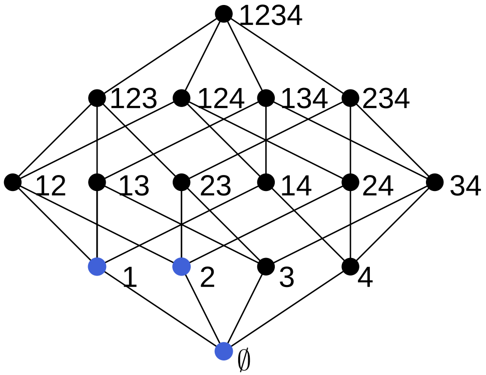

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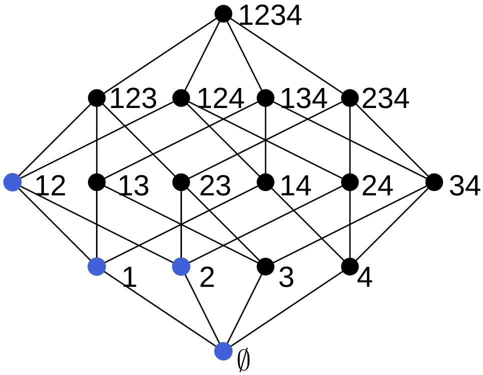

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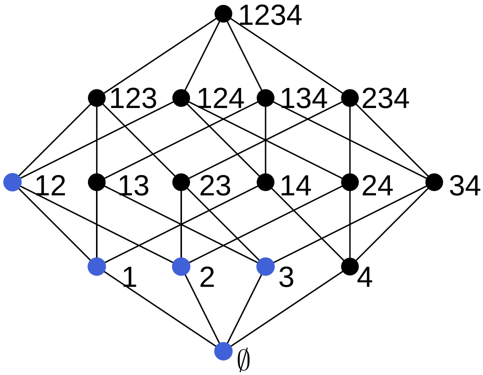

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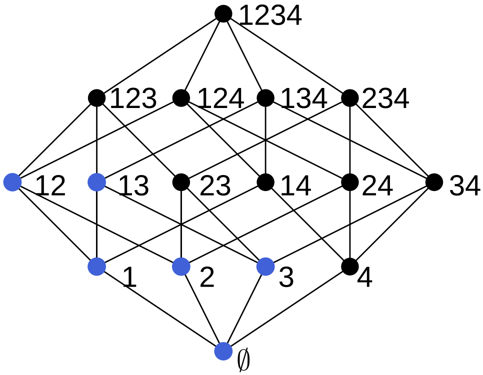

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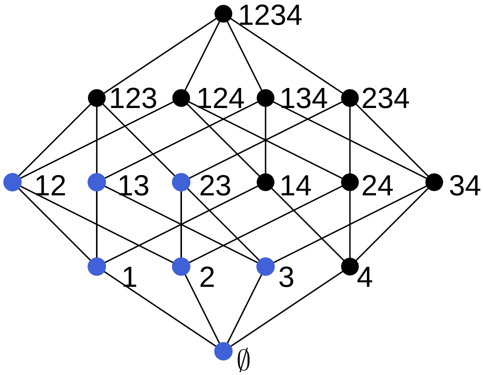

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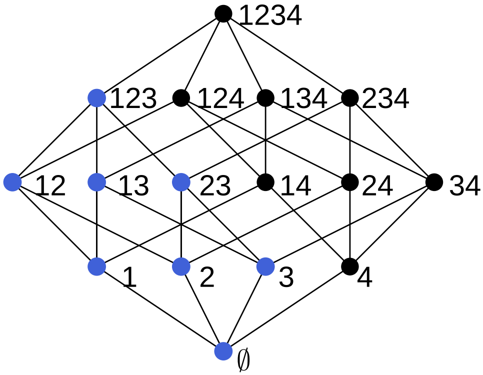

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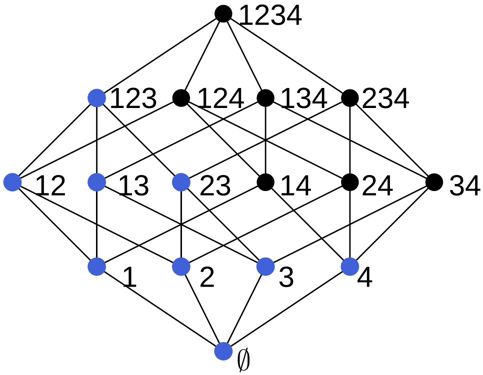

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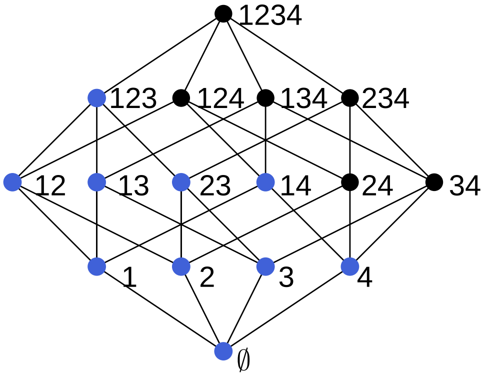

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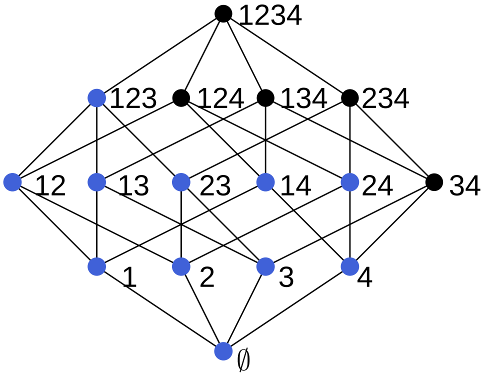

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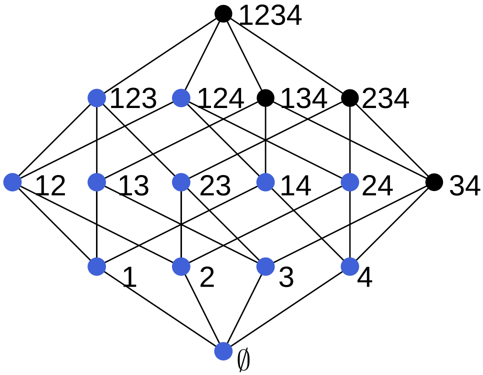

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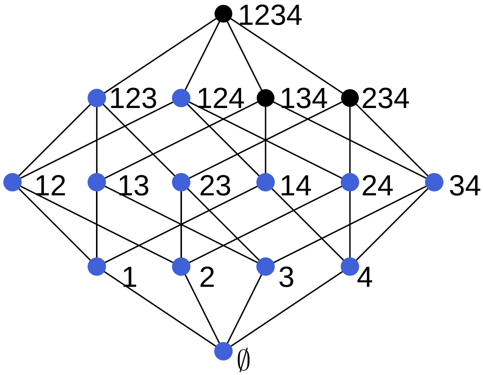

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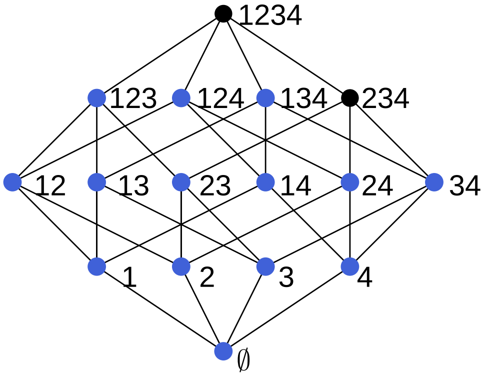

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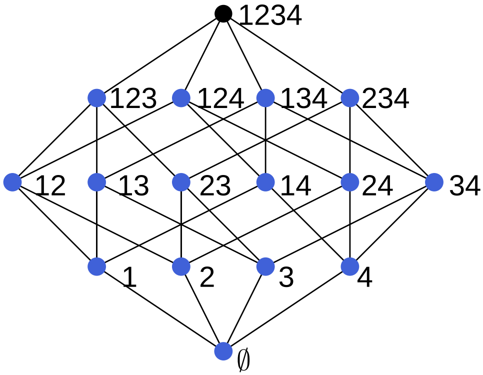

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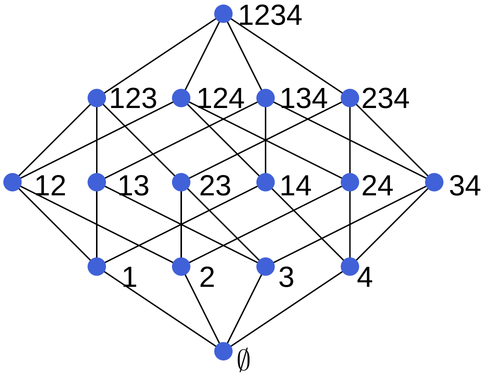

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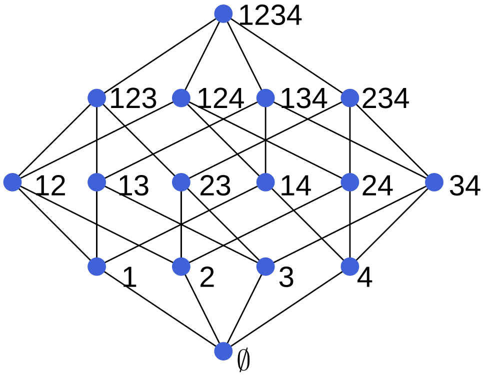

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Note: This follows if the Weak Conjecture is true for B_n.

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Thank you!