Linear Extension Diameter of Downset Lattices of 2-Dimensional Posets

Stefan Felsner and Mareike Massow [1,2]

Institut für Mathematik
TU Berlin
Berlin, Germany

Abstract

The linear extension diameter of a finite poset $\mathcal{P}$ is the maximum distance between a pair of linear extensions of $\mathcal{P}$, where the distance between two linear extensions is the number of pairs of elements of $\mathcal{P}$ appearing in different orders in the two linear extensions. We prove a formula for the linear extension diameter of Boolean Lattices and characterize all pairs of linear extensions attaining the maximum distance. These results can be extended to all downset lattices of 2-dimensional posets.

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1 Introduction

With a finite poset $\mathcal{P}$ consider its *graph of linear extensions* $G(\mathcal{P})$, which has the linear extensions of $\mathcal{P}$ as vertices with two of them being adjacent iff they differ only in an adjacent transposition. The graph of linear extensions is implicitly used in many investigations around finite posets, see e.g. [2]. Explicit research of properties of the graph was carried out in [5].

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[1] Email: `{felsner, massow}@math.tu-berlin.de`

[2] A full version of this paper can be found at www.math.tu-berlin.de/~massow.

The diameter of $G(\mathcal{P})$ is the *linear extension diameter* of $\mathcal{P}$, denoted by $\text{led}(\mathcal{P})$, see [3]. It equals the maximum number of pairs of elements of $\mathcal{P}$ that can be in different orders in two linear extensions of $\mathcal{P}$. A *diametral pair of linear extensions* of $\mathcal{P}$ is a diametral pair of $G(\mathcal{P})$.

A realizer of $\mathcal{P}$ is a set $\mathcal{R}$ of linear extensions of $\mathcal{P}$ such that the comparabilities of $\mathcal{P}$ are exactly the intersection of the comparabilities of the linear extensions in $\mathcal{R}$ (cf. [6]). The dimension of a poset $\mathcal{P}$ is the minimum size of a realizer. If $\mathcal{P}$ is 2-dimensional, i.e., if it has a realizer $\mathcal{R} = \{L_1, L_2\}$, then every incomparable pair $x||y$ appears in different orders in L_1 and L_2. It follows that $\mathcal{P}$ is 2-dimensional exactly if $\text{led}(\mathcal{P})$ equals the number of incomparable pairs of $\mathcal{P}$. Figure 1 shows a six-element poset $\mathcal{P}$ (often called the chevron) with its graph of linear extensions. Note that $\mathcal{P}$ has seven incomparable pairs, but the diameter of its linear extension graph is only six. Hence, the dimension of $\mathcal{P}$ must be at least three.

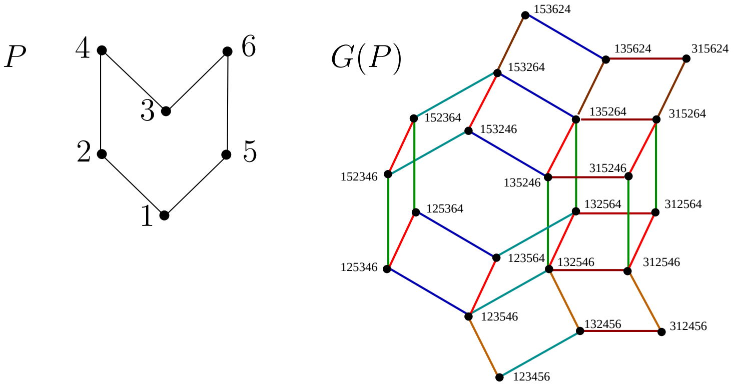

Fig. 1. The chevron and its graph of linear extensions.

A diametral pair L_1, L_2 can be used to obtain a drawing of $\mathcal{P}$ which is in some sense optimal: Use L_1 and L_2 on the two coordinate axes to get a position in the plane for each element of $\mathcal{P}$. Since the number of incomparable pairs which appear in different orders in L_1 and L_2 is maximized, the resulting drawing has a minimal number of pairs of elements which are comparable in the dominance order, but incomparable in $\mathcal{P}$. See Figure 2 for an example.

In [1] it was shown that it is NP-complete to determine the linear extension diameter of a given poset $\mathcal{P}$. In Section 2 we prove a formula for the linear extension diameter of the Boolean lattice. In fact we can characterize all diametral pairs of linear extensions of the Boolean lattice. In Section 3 we generalize these results to downset lattices of 2-dimensional posets. The proofs turn out to be a surprisingly direct generalization of the proofs for the Boolean lattices.

2 Boolean Lattices

Let B_n denote the n-dimensional Boolean lattice, that is, the poset on all subsets of $[n]$, ordered by inclusion. In this section we will prove a formula for the linear extension diameter of B_n, confirming a conjecture from [3].

Definition 2.1 Let σ be a permutation of $[n]$ and let S, T be subsets of $[n]$. Define a relation on the subsets of $[n]$ by setting $S <_\sigma T$ if $\max_\sigma S \triangle T \in T$. If $S <_\sigma T$, we say that S and T are in *σ-revlex order.*

The relation of being in σ-revlex order defines a linear extension of the Boolean lattice which we denote by L_σ. Let us denote the reverse of a permutation σ by $\overline{\sigma}$. We will see that the pairs $L_\sigma, L_{\overline{\sigma}}$ are exactly the diametral pairs of B_n. Figure 2 shows the induced drawing of B_4 for $\sigma = \text{id}$.

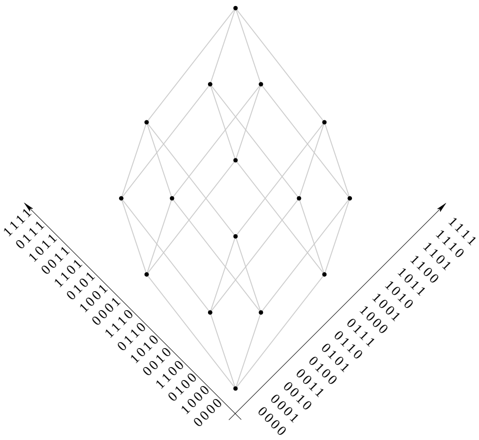

Fig. 2. The drawing of B_4 based on a diametral pair $L_\sigma, L_{\overline{\sigma}}$.

For the proof of the conjecture, we will need Kleitman's Lemma [4]:

Kleitman's Lemma 1 *If $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of $[d]$ which are closed downwards, then $|\mathcal{A}| \cdot |\mathcal{B}| \leq 2^d |\mathcal{A} \cap \mathcal{B}|$.*

Theorem 2.2 *led*$(B_n) = 2^{2n-2} - (n+1) \cdot 2^{n-2}$

Proof. It was proved in [3] that for a permutation σ of $[n]$, the distance between L_σ and $L_{\overline{\sigma}}$ as linear extensions of B_n is $2^{2n-2} - (n+1) \cdot 2^{n-2}$. To prove that this formula is also an upper bound on $\text{led}(B_n)$, we partition the pairs $\{S, T\}$ of subsets of $[n]$ with $|S \triangle T| \geq 2$ into equivalence classes $\mathcal{C}$. We fix a class $\mathcal{C}$ by choosing a set $D \subseteq [n]$ with $|D| = d \geq 2$, which is the symmetric difference of each pair in the class, and a set $I \subseteq [n] \setminus D$, which is the intersection of each pair in the class. Our plan is to bound the contribution that each class can make to the distance between two linear extensions of B_n.

Let L_1 and L_2 be two linear extensions of B_n. The distance between them is the number of *reversals*, that is, the number of pairs of elements of B_n appearing in different orders in L_1 and L_2. Let $\mathcal{C}$ be a class defined by D and I. Each subset X of D determines a pair $\{S, T\} \in \mathcal{C}$ by setting $S = X \cup I$ and $T = X^c \cup I$, with $X^c = D - X$. We assume $I = \emptyset$, thus $\{X, X^c\}$ itself is a pair of $\mathcal{C}$. For the case $I \neq \emptyset$, one only needs to replace each X by $X \cup I$ and each X^c by $X^c \cup I$ in the following argument.

We say that $X \subseteq D$ is *down* in an linear extension L if $X < X^c$ in L. Let $\mathcal{F}_i$ be the family of subsets of D which are down in L_i. A pair $\{X, X^c\} \in \mathcal{C}$ is a reversal between L_1 and L_2 exactly if X is down in one L_i, but not in the other. Thus our aim is to find an upper bound on $|\mathcal{F}_1 \triangle \mathcal{F}_2|$.

We observe that for every L and every set $X \subseteq D$, either itself or its complement is down in L. Hence we have $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{d-1}$. Similarly, $X \in \mathcal{F}_1 \cap \mathcal{F}_2 \iff X^c \in B_d \setminus (\mathcal{F}_2 \cup \mathcal{F}_1)$ and thus $|\mathcal{F}_1 \cap \mathcal{F}_2| = |B_d \setminus (\mathcal{F}_2 \cup \mathcal{F}_1)|$.

The following key observation captures the essence of transitive forcing between different pairs: If $X < X^c$ in L_i and $Y \subseteq D$ is a subset of X, then $X^c \subseteq Y^c$, and hence by transitivity $Y < X < X^c < Y^c$ in L_i. Thus $\mathcal{F}_1$ and $\mathcal{F}_2$ each form a family of subsets of $[d]$ which is closed downwards. We can now apply Kleitman's Lemma which yields $|\mathcal{F}_1 \cap \mathcal{F}_2| \geq 2^{d-2}$. It follows that

$$|\mathcal{F}_1 \triangle \mathcal{F}_2| = |B_d| - |\mathcal{F}_1 \cap \mathcal{F}_2| - |B_d \setminus (\mathcal{F}_2 \cup \mathcal{F}_1)| \leq 2^d - 2^{d-2} - 2^{d-2} = 2^{d-1}.$$

In $\mathcal{F}_1 \triangle \mathcal{F}_2$, every reversal is counted twice. Therefore the number of reversals that one class $\mathcal{C}$ can contribute is at most 2^{d-2}.

How many reversals does this yield in total? Each set $D \subseteq [n]$ with $d \geq 2$ forms a class together with each set $I \subseteq [n] \setminus D$. So each such D contributes at most $2^{n-d} \cdot 2^{d-2} = 2^{n-2}$ reversals. There are $2^n - n - 1$ possibilities to choose a set $D \subseteq [n]$ with $d \geq 2$, which yields the desired formula.

Again using the equivalence classes defined above, we can characterize the diametral pairs of linear extensions of B_n as follows:

Theorem 2.3 *If $L, \overline{L}$ is a diametral pair of linear extensions of B_n and σ is the order of the atoms in L, then $L = L_\sigma$ and $\overline{L} = L_{\overline{\sigma}}$.*

It was conjectured in [3] that in every diametral pair of linear extensions of a poset $\mathcal{P}$, at least one of the two linear extensions reverses a critical pair of elements of $\mathcal{P}$. Critical pairs appear in the dimension theory of posets, see [6]. In [1] it was shown that the conjecture is false in general, but that many classes of posets have the stronger property of being *diametrally reversing*, which means that *every* linear extension contained in a diametral pair reverses a critical pair. Still for Boolean lattices it remained open even whether they have the weaker property. As a consequence of Theorem 2.3, we can settle this question now:

Corollary 2.4 *Boolean lattices are diametrally reversing.*

3 Downset Lattices of 2-Dimensional Posets

The Boolean lattice can be viewed as the lattice of downsets of an antichain. Now let $\mathcal{P}$ be an arbitrary 2-dimensional poset, and let σ be a linear extension of $\mathcal{P}$ which is contained in a diametral pair. Since $\mathcal{P}$ is 2-dimensional, σ has a unique partner $\overline{\sigma}$ with which it forms a diametral pair. All incomparable pairs of $\mathcal{P}$ are reversals between σ and $\overline{\sigma}$.

Denote with $\mathcal{D}_{\mathcal{P}}$ the downset lattice of $\mathcal{P}$, that is, the poset on all downsets of $\mathcal{P}$, ordered by inclusion. Again let us define a linear extension L_σ of $\mathcal{D}_{\mathcal{P}}$ by setting $S <_\sigma T$ for two downsets S, T of $\mathcal{P}$ if $\max_\sigma S \triangle T \in T$.

Theorem 3.1 *Let $\mathcal{P}$ be a 2-dimensional poset $\mathcal{P}$, and let $L, \overline{L}$ be a diametral pair of linear extensions of $\mathcal{D}_{\mathcal{P}}$. Let σ be the order of the downsets $\downarrow x$ for $x \in \mathcal{P}$ in L. Then $L = L_\sigma$ and $L = L_{\overline{\sigma}}$.*

The proofs of Section 2 can be generalized by identifying each downset with the antichain of its maxima. The crucial observation is that these maxima are completely reversed between σ and $\overline{\sigma}$.

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