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A CENSUS OF TIGHT TRIANGULATIONS

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Abstract

A triangulation of a manifold (or pseudomanifold) is called a *tight triangulation* if any simplexwise linear embedding into any Euclidean space is tight. Tightness of an embedding means that the inclusion of any sublevel selected by a linear functional is injective in homology and, therefore, topologically essential. Tightness is a generalization of convexity, and the tightness of a triangulation is a fairly restrictive property. We give a review on all known examples of tight triangulations and formulate a (computer-aided) enumeration theorem for the case of at most 15 vertices and the presence of a vertex-transitive automorphism group. Altogether, six new examples of tight triangulations are presented, a vertex-transitive 15-vertex triangulation of the non-orientable 4-manifold $(S^3 \underline{\times} S^1) \# (\mathbb{C}\mathbf{P}^{\,2})^{\#5}$, a 13-vertex triangulation of the simply connected homogeneous 5-manifold $SU(3)/SO(3)$ with vertex-transitive action, two non-symmetric 12-vertex triangulations of $S^3 \times S^2$, and two non-symmetric triangulations of $S^3 \times S^3$ on 13 vertices.

1. Introduction: The concept of tightness

Tightness is a generalization of the notion of convexity that applies to objects other than topological balls and their boundary manifolds. It provides a way to say that an embedding of a space into Euclidean space E^n is 'as convex as possible' with respect to its topological properties. There are many characterizations of convexity and thus many ways for possible generalizations. We use the simple property that the intersection of a convex body with a closed half space is either empty or again a convex body (of some dimension). This leads to the following definition.

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DEFINITION 1.1. (Tightness): *A compact and connected subset $M \subset E^n$ is called **tight** with respect to a field $\mathbb{F}$ if for every open or closed half space $h \subset E^n$ the induced homomorphism*

$$H_*(h \cap M; \mathbb{F}) \to H_*(M; \mathbb{F})$$

*is injective, where H_* denotes an appropriate homology theory. M is called **tight** if it is tight with respect to at least one field, where the standard case is the field $\mathbb{Z}_2$ with two elements.*

In differential geometry, tightness is closely related to minimum total absolute curvature, a concept which was first studied by A.D. Aleksandrov, J.W. Milnor, S.-s. Chern, R.K. Lashof, and N.H. Kuiper. In fact, Definition 1.1 has another origin in Morse theory for height functions. On the one hand, in a suitable normalization the total absolute curvature of a smooth immersion equals the average of the number of critical points of all nondegenerate height functions. On the other hand, the Morse inequality states that the number of critical points of any Morse function on a compact smooth manifold M is greater or equal to the rank of the homology $H_*(M)$, with respect to any field of coefficients. Therefore, the normalized total absolute curvature is greater or equal to the rank of the total homology $H_*(M)$, where tightness just characterizes the case of equality. Even for polyhedra it is possible to talk about critical points of height functions in an adapted Morse type critical point theory; compare [K, Sect. 3] for an outline. In this case, one has to take care about the multiplicities of critical points. On the other hand, Definition 1.1 does not refer to any regularity assumption. Therefore Morse type arguments are not absolutely necessary for studying tightness but are often very useful. The theory of tightness can be developed in an analogous way for both the smooth and the polyhedral case. Nevertheless, there are some important differences; see the survey articles [Kui] and [BaK].

It is obvious from the definition that any convex body in E^n is a tightly embedded n-ball and that its boundary is a tightly embedded $(n-1)$-sphere. Polyhedral examples are convex polytopes and their boundaries, respectively. Vice versa, any tightly embedded polyhedral $(n-1)$-sphere (or homology sphere) in any E^m is the boundary of a convex body in some n-space; for a short proof see [K, 3.6].

The first examples of non-convex tight polyhedral embeddings in high codimension were found by neighborly triangulations, [Ba1], [Ba2], in particular by the unique 6-vertex triangulation of $\mathbb{R}\mathbf{P}^2$ and the unique 7-vertex triangulation of the 2-torus. These can be regarded as a tightly embedded subcomplex of the 5-dimensional simplex and of the 6-dimensional simplex, respectively. These are also known to be the unique tight polyhedral embeddings of these surfaces in these particular codimensions. In general, any triangulated 2-manifold without boundary can be treated in the same way if it can be triangulated with n vertices and $\binom{n}{2}$ edges (the so-called *regular cases* in Heawood's map color problem; compare 2.1). It then can be regarded as a tightly embedded subcomplex of the $(n-1)$-dimensional simplex. This leads to the notion of a *tight triangulation* [K] as follows.

DEFINITION 1.2. (Tight triangulations): *A subcomplex M of the boundary complex of an n-dimensional simplex is called a* ***tight simplicial complex*** *if M is tight when regarded as a polyhedral subset of the ambient Euclidean space. If, in addition, it is a triangulation of a manifold with or without boundary, then we call this simplicial complex a* ***tight triangulation*** *of the manifold. Since linear projections are known to preserve tightness, this essentially means that any simplexwise linear embedding of a tight triangulation is tight. The embedding into the n-simplex is also called the* ***canonical embedding*** *if $n+1$ is the number of vertices of the triangulation. Note that for a subcomplex of the regular simplex the combinatorial automorphism group coincides with the geometric symmetry group. Furthermore, it is well known that each full subcomplex of a tight subcomplex is again tight. Here,* ***full*** *means that it is the span of a certain subset of vertices in the given complex.*

The tightness of a triangulations can be interpreted as a kind of a *topological rigidity.* If *any* simplexwise linear embedding into any Euclidean space is tight, then the behaviour of the sublevels of linear functions is always the following: The topology gets more and more complicated (measured by the homology) if we pass through the manifold from bottom to top. This can be best visualized by the simple case of an empty triangle and an empty quadrilateral. The former one is a tight triangulation, the latter one is not, because it admits convex (i.e., tight) embeddings as well as non-convex positions. This type of 'rigidity' does not concern metrical properties of the space but its combinatorial structure in connection with its topology.

Throughout this paper we will consider only such triangulations which are combinatorial in the following sense. A simplicial complex is called a *combinatorial manifold* of dimension d if the link (or vertex figure) of any k-dimensional simplex is a triangulated $(d-k-1)$-dimensional sphere. This condition implies that this manifold carries a PL structure, and that each vertex star is PL homeomorphic to a d-ball. Conversely, every PL manifold admits a triangulation which is a combinatorial manifold in this sense.

CONJECTURE 1.3.

(i) *Every tight triangulation of a manifold M has the minimum number of vertices among all combinatorial triangulations of M.*

(ii) *The f-vector of any tight triangulation M is minimal (in each dimension separately) among all f-vectors of any combinatorial triangulation of M.*

The *f-vector* $f = (f_0, f_1, \ldots, f_d)$ is the vector defined by the numbers f_i of i-dimensional simplices. The conjecture above is true at least for 2-manifolds, Part 1 is also true for simply connected 4-manifolds [K]. The converse of Part 1 is certainly not true in general: The minimum number of vertices for a triangulation of the Klein bottle is 8. On the other hand, there is no tight triangulation of the Klein bottle. By simple counting arguments it would necessarily have 7 vertices; see 2.1. Note that the minimum number of vertices for any combinatorial triangulation is a PL invariant for any given PL manifold. However, it seems that this invariant is known only for very few cases. Lower bounds for the number n of combinatorial manifolds were given in [BrK1] as follows:

THEOREM 1.4.

(i) *Assume that a combinatorial d-manifold with n vertices is given which is not a sphere. Then $n \geq 3\frac{d}{2}+3$ if d is even and $n \geq 3\frac{d+1}{2}+3$ if d is odd. Furthermore, any combinatorial non-sphere with $n = 3\frac{d}{2}+3$ vertices is a tight triangulation of a manifold with the same cohomology ring as a projective plane over the reals, complex numbers, quaternions, or Cayley numbers.*

(ii) *Any combinatorial triangulation of the manifold $S^k \times S^m$ with $k \geq m$ (or of any manifold with the same homology as $S^k \times S^m$) has $n \geq 2k+m+4$ vertices.*

The case of equality in Part 1 of Theorem 1.4 is interesting because there exist in fact examples in dimension 2, 4, 8; see [BrK2]. The 16-dimensional case is still open. Concerning the case of equality in Part 2 of Theorem 1.4, we feel confident to propose the following conjecture.

CONJECTURE 1.5. *An n-vertex triangulation of $S^k \times S^m$ (or of any manifold with the same homology as $S^k \times S^m$) with $k \geq m$ is tight if and only if equality $n = 2k+m+4$ is attained.*

For $k = m$, this conjecture is true for $(k-1)$-connected manifolds with the homology of $S^k \times S^k$; see [K, 4.7]. Equality is attained for $k = m = 3$ and for $k = 3$, $m = 2$; for the tightness of the latter case see 3.5 below. It is open whether equality is attained for infinitely many cases $k \geq m \geq 2$.

2. Tight triangulations: Combinatorial conditions

By definition, the tightness of a triangulation depends only on the triangulation itself. So it is given by the combinatorial data which determine the triangulation. However, since the homology is involved in the definition, it is not obvious whether there is a simple combinatorial criterion which characterizes tightness. Fortunately, for triangulations of compact 2-manifolds the tightness condition can easily be transformed into a purely combinatorial condition as follows.

PROPOSITION 2.1. (Tight triangulations of surfaces, [K, 2.16]): *Let M be a surface without boundary, triangulated with n vertices. Then the following conditions are equivalent:*

(i) *The triangulation is tight with respect to the field $\mathbb{Z}_2$.*

(ii) *The triangulation is 2-neighborly, i.e., its edge graph is the complete graph K_n.*

(iii) $\binom{n-3}{2} = 3\big(2-\chi(M)\big)$.

Conversely, given an abstract surface M (which is not topologically equivalent to the Klein bottle) and any number n satisfying (iii)*, then there is a tight triangulation of M with n vertices.*

The equivalence (i) $\Leftrightarrow$ (ii) is not difficult to verify. It is based on the standard lemma [Ba1] that any connected and tight polyhedron must contain the 1-skeleton of its convex hull. (ii) $\Leftrightarrow$ (iii) is an elementary calculation from $\chi = n - f_1 + f_2$ and $2f_1 = 3f_2$. The converse direction follows from the work of G. Ringel and his collaborators on the solution of the Heawood map color problem [R].

For surfaces with boundary, (i) $\Leftrightarrow$ (ii) is still true if we replace (ii) by

(ii') *The triangulation is 2-neighborly and each vertex lies on the boundary.*

Moreover, in the case of surfaces M with one boundary component, (iii) can be replaced by the similar condition

(iii') $\binom{n-2}{2} = 3\big(1-\chi(M)\big)$ *and each vertex lies on the boundary.*

The construction of tightly triangulated surfaces with one boundary component is given by cutting out one open vertex star from a tightly triangulated surface without boundary. Conversely, one can add a cone with respect to one additional vertex.

Similarly, for highly connected manifolds, classified in the smooth case in [W], the tightness condition can be transformed into a purely combinatorial condition. This is based on the vanishing of the homology $H_1, H_2, \ldots, H_{k-1}$. Therefore, the tightness of a triangulation implies that the span of any subset of vertices has again vanishing homology $H_1, H_2, \ldots, H_{k-1}$. This means that any $k+1$ vertices span a simplex of the triangulation, a property usually called $(k+1)$-*neighborliness.*

THEOREM 2.2. [K, 4.7]: *Let M be an n-vertex triangulation of a $(k-1)$-connected $2k$-manifold. Then the following conditions are equivalent:*

(i) *The triangulation is tight.*

(ii) *The triangulation is $(k+1)$-neighborly.*

(iii) $\binom{n-k-2}{k+1} = (-1)^k \binom{2k+1}{k+1} \big(\chi(M) - 2\big).$

In particular, any 3-neighborly triangulation of a 4-manifold is automatically tight because 3-neighborliness implies that the manifold is simply connected. The very first example of this type is the unique 9-vertex triangulation of the complex projective plane, $\mathbb{C}\mathbf{P}^2_9$, which was constructed by the first author by starting with the knowledge of all possible 2-neighborly vertex links [KL1]. Note that for $k \geq 2$ condition (i) is independent of the field of coefficients for the homology.

For odd-dimensional manifolds and for non-simply-connected 4-manifolds tightness is substantially harder to verify. From the Definitions 1.1 and 1.2 it is clear that one has to examine the homology of all intersections with half spaces. In the case of the canonical embedding this just means that one has to examine the homology of the subcomplexes spanned by any subset of vertices. The non-bounding cycles in these subcomplexes have to be non-bounding in the entire manifold, too.

Otherwise, the tightness is violated. By duality arguments this has to be verified only for the homology $H_0, H_1, \dots, H_{k-1}$ if the manifold is $2k$-dimensional or $(2k-1)$-dimensional and if Poincaré duality holds for the particular choice of a field of coefficients. The injectivity of H_0 for tight triangulations is equivalent to the 2-neighborliness. Similarly, if the manifold is simply connected, injectivity for H_1 is equivalent to 3-neighborliness. More generally, a triangulation is called *k-tight* if in Definition 1.1 the induced morphism

$$H_j(h \cap M; \mathbb{F}) \to H_j(M; \mathbb{F})$$

is injective for $j = 0, \dots, k$. The duality implies the following lemma:

LEMMA 2.3. [K, 3.18]: *Let M be a triangulation of a $2k$- or $(2k-1)$-manifold satisfying Poincaré duality. Then the following conditions are equivalent:*

(i) *The triangulation is tight.*
(ii) *The triangulation is $(k-1)$-tight.*

3. Tight triangulations: The list of old and new examples

Even though the tightness condition for embeddings is restrictive, there are many tight polyhedral embeddings of manifolds, and a number of general constructions are known; see [K]. However, the tightness of a triangulation is even more restrictive. As a matter of fact, in dimensions at least 3 only very few examples are known so far. In this section we present new examples, after a brief discussion of all previously known ones.

3.1 *Trivial examples*

A trivial sequence of examples is given by the boundary complexes of simplices. These are the only tight triangulations of spheres (and homology spheres): For any tight triangulation of S^d with n vertices the tightness condition implies the $(d+1)$-neighborliness by the vanishing of $H_j(S)$ for $j = 1, \dots, d-1$. This is known to be impossible for $n > d+2$.

3.2 *2-manifolds*

For 2-manifolds tightness and 2-neighborliness are equivalent conditions by 2.1. So an exhaustive list of examples can be given only by an exhaustive list of all triangulated surfaces with a complete edge graph. The existence is well known by the solution of the Heawood map color problem [R]. However, an enumeration is far beyond reach. For small n the number of distinct combinatorial types is the following:

n	# orientable surfaces	# non-orientable surfaces
4	1	0
6	–	1
7	1	0
9	–	2 [ABr]
10	–	14 [ABr]
12	59 [ABS]	$\geq$ 40,615 [A]

In each of these cases of 4, 6, 7, 9, 10, and 12 vertices it happens that exactly one of the tight triangulations has a vertex-transitive automorphism group. In contrast to this observation, among the 59 orientable examples with 12 vertices there are some with a trivial automorphism group [ABS]. It thus makes sense to consider the case of *tight triangulations of* 2*-manifolds with a vertex-transitive automorphism group* separately. Here, for small n the number of distinct combinatorial types is the following [Lu1]:

n	# orientable surfaces	# non-orientable surfaces
4	1	0
6	–	1
7	1	0
9	–	1
10	–	1
12	1	0
13	–	2
15	0	0
16	46	36
18	–	0
19	32	46
21	–	913
22	–	0

It is especially surprising that there is no example with 15 vertices admitting a vertex-transitive automorphism group although there are many distinct transitive permutation groups of degree 15.

3.3 *Sphere bundles over* S^1

There is one particular example of a non-spherical tight combinatorial manifold in every dimension [K, 5B]. It is a tight triangulation of a d-manifold with $2d+3$ vertices, which is homeomorphic to the product $S^{d-1} \times S^1$ for even d and is homeomorphic to the twisted product $S^{d-1} \mathbin{\underline{\times}} S^1$ for odd d. The automorphism group of this triangulation is the dihedral group on $2d+3$ vertices. This combinatorial manifold is nothing but the boundary of an ordinary 1-handle, which is defined as the orbit of the simplex $\langle 1\ 2\ 3 \ldots d+2\rangle$ under the natural action of the cyclic group $\mathbb{Z}_n$ for $n = 2d+3$. For the case $d = 3$ compare the paper [Wa]. It seems to be the first occurrence of this 9-vertex triangulation.

3.4 *Sporadic examples*

Besides this infinite series and examples in dimension 2, only five sporadic (non-spherical) examples were known until recently: The unique 9-vertex triangulation of the complex projective plane [KBa], [KL1], a tight 16-vertex triangulation of the K3 surface [CK], and three distinct 15-vertex triangulations of an 8-manifold 'like the quaternionic projective plane' [BrK2]. Three of them (the one with 9, the one with 16, and one with 15 vertices) admit a vertex-transitive automorphism group. The 16-vertex triangulation of the K3 surface has an edge transitive automorphism group. In this section, we present six new sporadic examples of tight triangulations.

3.5 *New examples*

There is a tight 15*-vertex triangulation of the non-orientable* 4*-manifold* $(S^3 \underline{\times} S^1)\#(\mathbb{C}\mathbf{P}^2)^{\#5}$*, and there is a tight* 13*-vertex triangulation of the simply connected homogeneous* 5*-manifold* $SU(3)/SO(3)$*. Both examples have a vertex-transitive automorphism group. Furthermore, there are two distinct tight* 12*-vertex triangulations of the product manifold* $S^3 \times S^2$*, and there are are two distinct tight* 13*-vertex triangulations of the product manifold* $S^3 \times S^3$*. The four latter examples have the minimum numbers of vertices of any non-spherical combinatorial* 5*- or* 6*-manifold according to* 1.4.

The example $SU(3)/SO(3)$ on 13 vertices is particularly interesting: it is simply connected with 2-torsion in the second homology. The non-orientable 4-manifold $(S^3 \underline{\times} S^1)\#(\mathbb{C}\mathbf{P}^2)^{\#5}$ on 15 vertices has a 5-dimensional second homology. It therefore carries elements with nonzero selfintersection in the second homology. Each of these two examples has a nonvanishing second Stiefel–Whitney class. It seems that no smooth tight immersion of these manifolds is known so far. For the example $SU(3)/SO(3)$ it has been conjectured in [Th] that it might admit a taut embedding. By a procedure described in [K, Sect. 6], there are tight polyhedral embeddings of connected sums of these manifolds with themselves. This construction is based on an idea of Banchoff [Ba2].

COROLLARY 3.6. *There are at least five 'sporadic' examples of tight triangulations of d-manifolds with $d \geq 3$, admitting a vertex-transitive automorphism group. The dimensions are* $4, 4, 4, 5, 8$*, the numbers of vertices are* $9, 15, 16, 13, 15$*, and the Euler characteristics are* $3, 5, 24, 0, 3$*, respectively.*

Among those sporadic examples, the tight K3 surface carries the richest topology and the most interesting intersection form on the second homology. An open problem is the following:

QUESTION 3.7. *Does there exist a* 27*-vertex triangulation of the* 16*-dimensional Cayley projective plane?*

It would necessarily be 9-neighborly and hence a tight triangulation by 2.2.

COROLLARY 3.8. *The lower bound for the number $n(d)$ of vertices of any combinatorial d-manifold (not PL homeomorphic to the d-sphere) from* 1.4 *is sharp for all dimensions $d \leq 6$ and for $d = 8$. Each of the known triangulations attaining this lower bound is a tight triangulation.*

In more detail, we have $n(2) = 6$, $n(3) = n(4) = 9$, $n(5) = 12$, $n(6) = 13$, and $n(8) = 15$. Concerning the gap $d = 7$ in this sequence, an open problem is the following:

QUESTION 3.9. *Is there a* 15*-vertex triangulation of $S^4 \times S^3$?*

If the answer turns out to be 'yes', then it follows that the bound in Theorem 1.4 is sharp for all $d \leq 8$.

PROOF OF 3.5.

Two 13-vertex triangulations of $S^3 \times S^3$

In [KöLu], two combinatorial triangulations, ${}^{6}15^{7}_{1}$ and ${}^{6}15^{7}_{2}$, of $S^3 \times S^3$ were found with 15 vertices, which are invariant under a vertex-transitive group action of $D_5 \times S_3$. (Here, the classification theorem by A.V. Žubr [Žu] for simply connected 6-manifolds was used to determine the topological type.) By using the program BISTELLAR [Lu3], which locally modifies a triangulation of a manifold by a sequence of so called bistellar flips preserving the PL homeomorphism type (cf. [BjLu]), two combinatorially distinct triangulations of $S^3 \times S^3$ were obtained with 13 vertices. Both triangulations have f-vector $f = (13, 78, 286, 715, 1014, 728, 208)$ and Altshuler-Steinberg determinant

$$det = 74571415482344619853824.$$

Nevertheless, the Altshuler-Steinberg determinants for the links of the $2 \cdot 13$ vertices of the two triangulations are 26 pairwise distinct numbers. Hence, the two triangulations are combinatorially non-equivalent and they possess no symmetry at all. By 2.2 any 4-neighborly triangulation of a 6-manifold is tight. Since $\binom{13}{4} = 715$, each of these 13-vertex triangulations is 4-neighborly and thus tight. The facets of the minimal triangulation bistellarly equivalent to the manifold ${}^{6}15^{7}_{1}$ are given in the Appendix. The facets of the minimal triangulation bistellarly equivalent to ${}^{6}15^{7}_{2}$ can be found in [Lu1].

Two 12-vertex triangulations of $S^3 \times S^2$

Again, both examples were found by applying bistellar flips. One of the examples is bistellarly equivalent to a product triangulation of $S^3 \times S^2$ with $5 \cdot 4 = 20$ vertices. It has f-vector $f = (12, 66, 220, 390, 336, 112)$ and Altshuler-Steinberg determinant $det = 4508595451809050112$. The list of its facets is given in the Appendix. In [Lu1], a second tight example with the same f-vector but with Altshuler-Steinberg determinant $det = 4471184572226676864$ was obtained by applying bistellar flips to a nearly neighborly centrally symmetric combinatorial 14-vertex triangulation of $S^3 \times S^2$ with a vertex-transitive D_{14}-action; see also Section 6. □

Proof for the tightness of the 12-vertex triangulations.

Since the first (as well as the second) example has no symmetry at all, the only way is to calculate by computer the homology of the span of each subset of vertices. The set $\{1, 2, 3, 4\}$ spans an empty tetrahedron, the set $\{1, 2, 5, 7, a\}$ is an empty 4-simplex. These two are, therefore, candidates for the generators of the second and third homology, respectively. The 0-tightness and the 1-tightness follow from the 3-neighborliness. This can be expressed by saying that the intersection with any half space is connected and simply connected, for any simplexwise embedding into Euclidean space. For the 2-tightness, we can observe (by computer) that the span of each subset of vertices has a second Betti number which is at most 1 and a third Betti number which is at most 1. Moreover, the corresponding second homology class can always be represented by an empty tetrahedron which is embedded, and the third homology class can be represented by an empty 4-simplex which is embedded. This property is sufficient for the tightness because any two empty tetrahedra or empty 4-simplices are homologous to one another.

The 5-dimensional example $SU(3)/SO(3)$

Regard the 13 vertices $0, 1, \ldots, 12$ as the elements of the field $\mathbb{F}_{13}$ and define a 5-dimensional simplicial complex M^5 as the union of the orbits of the simplices

$$012347_{39},\ 012348_{39},\ 012368_{39},\ 012458_{13},\ 012479_{13},\ 01268\,11_{13}$$

under the group $F_{13,3}$ generated by the permutations $x \mapsto x+1(13)$ and $x \mapsto 3x(13)$. This group is of order 39, the lengths of the orbits are 39,39,39,13,13,13, respectively. The f-vector of this triangulation is $f = (13, 78, 286, 533, 468, 156)$. Since $\binom{13}{3} = 286$, the triangulation is 3-neighborly. Hence, the underlying space is simply connected. As for the above homology calculations, we use the computer program HOMOLOGY by F. Heckenbach [He] to compute the integral homology vector, $(\mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z})$. Because there is 2-torsion in the homology, the only possibility for tightness is the tightness with respect to $\mathbb{Z}_2$. (Recall that the $\mathbb{Z}_2$-homology vector is $(\mathbb{Z}_2, 0, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}_2)$.)

Step 1. M^5 is a combinatorial manifold.

By definition this means that each vertex link (there is only one type by the vertex-transitive automorphism group) is a combinatorial 4-sphere. In fact, by using the computer program BISTELLAR it is possible to find a bistellar equivalence of the vertex link with the boundary of the 5-simplex. A verification without the use of a computer can be given as follows: At first, write down the list of all 72 top-dimensional simplices of the link of 0 in M^5. It is invariant under the automorphism $\gamma = (139)(265)(4\,\mathit{12}\,\mathit{10})(78\,\mathit{11})$. The decomposition into the sets $\{1,3,4,9,10,12\}$ and $\{2,5,6,7,8,11\}$ is γ-invariant. Secondly, consider the span of each of these two subsets in the vertex link. This leads to two subcomplexes in that link, each of which can be seen to be collapsible to a point. By standard results in PL topology this implies that the vertex link is a 4-sphere provided that it is a 4-manifold at all. Now we have to repeat this procedure for the 3-dimensional links in the 4-dimensional link. By the symmetry group it is sufficient to consider only the links of the edges 01, 02, 04, 07. These have 11 vertices, and it is possible without a computer to show that they all are combinatorial 3-spheres.

STEP 2. The topology of M^5.

By the work of D. Barden [Bar] there is a classification of simply connected 5-manifolds, in the smooth category as well as in the PL category. It turns out that there is only one candidate for M^5 with the required homology vector, denoted by X_{-1} in [Bar]. In fact, this manifold is the well-known simply connected homogeneous 5-manifold $SU(3)/SO(3)$; cf. the classification of compact homogeneous manifolds of low dimension by V.V. Gorbatsevich [G] as well as the exposition in [Kl].

One description of $SU(3)/SO(3)$ is as the union of two copies of the non-trivial 3-disc bundle over S^2, glued together by an appropriate homeomorphism which sends one generator of the second homology to its negative, thus realizing the 2-torsion. The boundary of each of these 3-disc bundles over S^2 is the 4-manifold $\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$, the total space of the nontrivial S^2-bundle over S^2. Unfortunately, by the enumeration in [KöLu], there is no 13-vertex triangulation of this 4-manifold admitting a vertex-transitive automorphism group. Hence the decomposition of this 5-manifold into the two distinct 3-disc bundles cannot be seen in a simple combinatorial way. However, there exist two vertex-transitive combinatorial 5-pseudomanifolds for which each vertex link is a combinatorial $\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$ on 12 vertices [Lu2].

STEP 3. The tightness of M^5.

By 2.3 it suffices to show the 2-tightness. The 1-tightness follows from the 3-neighborliness. It remains to show that for any subset S of vertices the span $\langle S\rangle$ injects into M at the second homology level, i.e., $H_2(\langle S\rangle;\mathbb{Z}_2) \to H_2(M;\mathbb{Z}_2)$ is injective. The 4-element subsets split into two classes: the tetrahedra of the triangulation and the *empty tetrahedra*. Under the action of the automorphism group there are only the following types of empty tetrahedra: 123 11, 1246, 124 10, 124 12, 1278, 129 11. When regarded as homology 2-cycles, the empty tetrahedra are candidates for representing the nonvanishing homology H_2 in M^5. The tightness would be violated if any of these 2-cycles would be homologous to zero in M^5. At least, any two of them are homologous to one another. We illustrate this in more detail for 1246 and 124 12. In fact, the second homology of the span of $\{1,2,4,6,12\}$

in M^5 is isomorphic to $\mathbb{Z}$. The three tetrahedra 126 12, 146 12, 246 12 are contained in M^5, thus the span of $\{1,2,4,6,12\}$ <u>is</u> a homology between the two remaining empty tetrahedra. Similarly, any two empty tetrahedra are homologous to one another. Note also that the automorphism of order 13 necessarily has to preserve the non-zero second homology class. By a computer check, for the span of any subset of vertices the rank of the second homology is either 0 or 1, and the nonzero element can always be represented by an empty tetrahedron. The 2-tightness follows. This implies the tightness by 2.3.

<u>The 4-dimensional example $(S^3 \underline{\times} S^1) \# (\mathbb{C}\mathbf{P}^2)^{\# 5}$</u>

Let the group action of $\mathbb{Z}_5 \times S_3$ on 15 elements be generated by the permutations

$$(123456789\mathit{10}\,\mathit{11}\,\mathit{12}\,\mathit{13}\,\mathit{14}\,\mathit{15}) \quad \text{and} \quad (1\mathit{11})(27)(4\mathit{14})(5\mathit{10})(8\mathit{13}).$$

Define the simplicial complex M^4 on 15 vertices as the union of the orbits

$$1236\,12_{30},\ 1237\,13_{30},\ 1238\,11_{30},\ 1236\,13_{15},\ 1237\,11_{15},\ 1238\,12_{15},\ 1368\,11_{15}.$$

With respect to the action of the transitive subgroup $\mathbb{Z}_{15}$ of $\mathbb{Z}_5 \times S_3$, the three orbits of size 30 split into the following six ones:

$$1236\,12_{15},\ 1267\,13_{15};\ 1237\,13_{15},\ 125\,11\,12_{15};\ 1238\,11_{15},\ 1368\,12_{15}.$$

The complex has the f-vector $f = (15, 105, 320, 375, 150)$, and its integral homology turns out to be $(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^5, \mathbb{Z}_2, 0)$. Again, for the tightness we have to use the field $\mathbb{Z}_2$. The $\mathbb{Z}_2$-homology vector is $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2^5, \mathbb{Z}_2, \mathbb{Z}_2)$ or simply $(1, 1, 5, 1, 1)$.

STEP 1. M^4 is a combinatorial manifold.

As above for M^5, the computer found that the vertex link in M^4 is bistellarly equivalent with the boundary of the 4-simplex. Without a computer one can proceed as follows: the link of the vertex 15 contains 50 tetrahedra with vertices $1, 2, 3, \ldots, 14$. It turns out that in the link the span of each of the sets $\{1,4,7,10,13\}$ and $\{2,5,8,10,14\}$ is a triangulated disc with three triangles, the remaining set $\{3,6,9,12\}$ spans three edges 3 9, 9 6, 6 12. Furthermore, the span of each of the two sets $\{1,4,7,10,13,6,12\}$ and $\{2,5,8,11,14,3,9\}$ is collapsible to a point. This proves that the link of 15 is a 3-sphere, provided that it is a 3-manifold. But that is the simple check that each link is a combinatorial 2-sphere.

STEP 2. The topology of M^4.

A natural candidate with the topology of M^4 is the connected sum of the nontrivial S^3-bundle over S^1 and five copies of $\mathbb{C}\mathbf{P}^2$. The 1-handle can be seen in the decomposition of M^4 into the span of the set $\{1,2,3,6,7,8,11,12,13\}$ and its complement $\{4,5,9,10,14,15\}$. The latter one spans the combinatorial join of the boundaries of $\{4,9,14\}$ and $\{5,10,15\}$, together with these two triangles themselves. This is homotopy equivalent to the wedge product of S^3 with two copies of S^2. The span of the former one is homotopy equivalent to the join of S^3 with three copies of S^2. Halfway between these two simply connected subcomplexes there is a 3-manifold given by all the slices through all the 4-simplices. This must be disconnected by the theorem of Seifert and van Kampen. A connection between two points in the 'upper' and 'lower' part on the two sides of the middle level defines the generator of the first homology (and of the fundamental group as well). In particular, the first homology with integer coefficients and the fundamental group are isomorphic to one another. By the classification of Z. Wang [Wan] of closed non-orientable 4-manifolds with infinite cyclic fundamental group M^4 is homeomorphic to $(S^3 \underline{\times} S^1) \# (\mathbb{C}\mathbf{P}^2)^{\#5}$. (We are grateful to M. Kreck for bringing this reference to our attention.)

STEP 3. The tightness of M^4.

By 2.3 it suffices to prove the 1-tightness for the field with 2 elements as domain of coefficients. The 0-tightness is trivially satisfied because the triangulation of M^4 is 2-neighborly. The 1-tightness is more subtle, since $H_1(M^4)$ is isomorphic to the integers. Thus we have to show that the span of each subset of vertices injects at the first homology level into M^4. By a computer check the rank (mod 2) of the homology of the span of each subset of vertices is not greater than the rank of the homology of M^4. In each case, the generating element in the first homology can be represented by an empty triangle. If such an empty triangle, spanned by a subset S, would be homologous to zero (and, by Step 2, a contractible loop) in the entire manifold, then the span of the complement of S would carry an additional sixth generator in the second homology. This is not the case by the computer check. The tightness follows. $\square$

4. Tight triangulations admitting vertex-transitive group actions

All possible transitive group actions on ≤ 22 elements have been classified (see the references in the GAP-manual [Sch]). Based on this classification, a computer aided enumeration of all combinatorial manifolds with $n \leq 15$ vertices and a vertex-transitive group action was performed (with the exception of action of groups of small order on $14 \leq n \leq 15$ vertices) and the resulting examples were listed in [KöLu], where ${}^d n^{\,i}_{k}$ denotes the k-th example of a combinatorial d-manifold listed for the i-th vertex-transitive group action on n vertices. For almost all of these examples it is easy to see that they are not tight, as the spanned subcomplex of some subset of vertices does not inject in homology, which can be tested by computer. All remaining complexes turned out to be tight.

THEOREM 4.1. *The numbers of combinatorially distinct tight triangulations of d-manifolds (others than the d-sphere) admitting a vertex-transitive automorphism group are as follows:*

— $d = 2$, $6 \leq n \leq 22$: 1081 *types (see* Section 3 *above, compare* [ABS], [Lu1]*)*
— $d = 3$, $9 \leq n \leq 15$: ${}^3 9^{\,3}_{2}$ *(1-handle; compare* [ASt], [K]*)*
— $d = 4$, $9 \leq n \leq 13$: $\mathbb{C}\mathbf{P}^2_9 = {}^4 9^{\,13}_{1}$ *(*[KBa], [KL1]*) and* ${}^4 11^{\,2}_{1}$ *(1-handle,* [K]*)*
— $d = 4$, $14 \leq n \leq 15$, *and a group of order* $\geq 2n$: ${}^4 15^{\,4}_{1}$ *(*$(S^3 \underline{\times} S^1) \# (\mathbb{C}\mathbf{P}^2)^{\#5}$*, new)*

— $d = 5$, $12 \le n \le 13$: ${}^{5}13^{2}_{7}$ *(1-handle, [K]) and* ${}^{5}13^{3}_{2}$ *($SU(3)/SO(3)$, new)*
— $d = 5$, $n = 14$, *and a group of order* $\ge 2n$: *none*
— $d = 5$, $n = 15$, *and a group of order* $\ge 3n$: *none*
— $d = 6$, $n = 13$: *none*
— $d = 6$, $n = 14$, *and a group of order* $\ge 2n$: *none*
— $d = 6$, $n = 15$, *and a group of order* $\ge 3n$: *none*
— $d = 7$, $n = 15$, *and a group of order* $\ge 3n$: *none*
— $d = 8$, $n = 15$: $M^8_{15} = {}^{8}15^{5}_{1}$ *([BrK2]).*

Altogether, this gives six types in dimension $d \ge 3$ with $n \le 15$ vertices. In addition, there is an example with $d = 4$ and $n = 16$, a triangulated K3 surface [CK]. This is the only 16-vertex triangulation of any 4-manifold, which is invariant under a primitive group action on the 16 vertices. For non-primitive actions on 16 vertices, the large number of 1954 distinct vertex-transitive group actions on 16 elements and the fact that many of the corresponding groups are of small order makes a complete enumeration for 16 vertices practically impossible.

In general, examples of combinatorial manifolds with a primitive group action seem to be rare if n is not a prime number.

Theorem 4.2. *There are altogether exactly* 68 *combinatorial manifolds with a primitive group action on prime numbers of vertices* $n \le 15$. *For non-prime numbers* $n \le 15$ *the only examples of combinatorial manifolds with a primitive group action are the boundary* $(n-2)$*-spheres of* $(n-1)$*-simplices on* n *vertices and, in addition, two examples of dimension two on* 6 *respectively* 10 *vertices, with an* A_5*-action in both cases. For non-prime numbers* $16 \le n \le 22$ *there are exactly two examples of dimension* $d = 2$ *on* 21 *vertices with an action of* $PGL(2,7)$ *and one example of dimension* $d = 4$ *on* 16 *vertices with an action of* $AGL(1,16)$. *For* $d = 3$ *no example exists for* $n = 16, 18, 20, 21, 22$, *and for* $d = 5$ *there is no example for* $n = 16, 18, 20, 22$. *If* $d = 5$ *and* $n = 21$, *then possible examples can only exist for the primitive action of* $PGL(2,7)$.

The enumeration of combinatorial manifolds with a primitive group action follows for $n \le 15$ from the results in [KöLu]. The two-dimensional example ${}^{2}6^{12}_{1}$ on 6 vertices is the unique regular, neighborly and thus tight triangulation of the projective plane. The two-dimensional example ${}^{2}10^{7}_{1}$ on 10 vertices is a neighborly triangulation of the non-orientable surface of genus 7. Both two-dimensional examples with 21 vertices are regular [Lu1, ${}^{2}_{reg}21^{20}_{1}$, ${}^{2}_{reg}21^{20}_{2}$], but they are not neighborly and thus not tight. The four-dimensional example on 16 vertices is the mentioned tight triangulation of the K3 surface [CK].

5. Tight triangulations of pseudomanifolds

Manifolds with isolated singularities occur in many branches of mathematics. In all these cases, the crucial point is the structure of the neighborhood of the singularities. We define a *combinatorial d-pseudomanifold* to be a simplicial complex such that the link of each vertex is a combinatorial $(d-1)$-manifold. If the tightness condition in 1.2 is satisfied, we can similarly talk about tight triangulations of pseudomanifolds. In fact, there are very natural examples of tight triangulations of 3-dimensional pseudomanifolds. If one applies 2.1 to each vertex link, then the following proposition is easily obtained, just by summing up over all vertices.

THEOREM 5.1. *For any 3-pseudomanifold M with $H_1(M) = 0$ the following conditions are equivalent:*

(i) *The triangulation is tight.*
(ii) *The triangulation is 3-neighborly.*
(iii) $n\binom{n-2}{2} = \binom{4}{2}\chi(M)$.

The 3-neighborliness implies that the underlying space is simply connected. Moreover, any 3-neighborly 3-pseudomanifold defines a block design $S_2(3,4;n)$, just by regarding the tetrahedra as blocks.

Classical examples

1. 70 years ago, A. Emch [Em] gave an interesting example of a 3-neighborly 8-vertex triangulation which is a tight triangulation of a 3-pseudomanifold. Each of its vertex links is the unique tight 7-vertex triangulation of the torus and is invariant under the transitive action of the affine group $AGL(1,7)$. The 8-vertex triangulation itself is invariant under the group $PGL(2,7)$, the 8-th vertex can be interpreted as ∞ of a projective line over the field $\mathbb{Z}_7$.

2. The existence of highly symmetric block designs $S_2(3,4;13)$ is mentioned in the CRC handbook of combinatorial designs VI.1.4. These designs can be interpreted as tight triangulations of 3-pseudomanifolds provided that the link of each vertex is a closed surface (and not just a surface with singularities).

3. For primes $n \equiv 5(12)$, further examples were constructed by E. Köhler in [Köh], which are invariant under the affine group, acting 2-transitively on vertices. For the tightness of the examples compare [K, Sect. 7].

THEOREM 5.2. [Lu2]: *There are precisely five distinct tight triangulations of 3-pseudomanifolds with $n \leq 15$ vertices, which are 3-neighborly (thus simply connected) and have a vertex-transitive automorphism group.*

This follows from the enumeration of all vertex-homogeneous 3-pseudomanifolds with $n \leq 15$ vertices in [Lu2]. The five examples include Emch's system with 8 vertices, three distinct 13-vertex triangulations with orientable and non-orientable vertex link (necessarily a 12-vertex surface with $\chi = -10$), and a particular 14-vertex example. The groups are $PGL(2,7)$, $F_{13,4}$ and D_7, with generators

$$(1234568),\ (132645),\ (16)(23)(45)(78);$$
$$(123456789\,\mathit{10}\,\mathit{11}\,\mathit{12}\,\mathit{13}),\ (15\,\mathit{12}\,8)(2\,\mathit{10}\,\mathit{11}\,3)(4796);$$
$$(13579\,\mathit{11}\,\mathit{13})(2468\,\mathit{10}\,\mathit{12}\,\mathit{14}),\ (1\,\mathit{12})(2\,\mathit{11})(3\,\mathit{10})(49)(58)(67)(\mathit{13}\,\mathit{14}),$$

respectively.

In more detail the list of these five examples is the following:

n	Homology (H_0, H_1, H_2, H_3)	Comb. type	List of orbits
8	$(\mathbb{Z}, 0, \mathbb{Z}^8, \mathbb{Z})$	${}^3_p8^{43}_1$	1235_{28}
13	$(\mathbb{Z}, 0, \mathbb{Z}^{78}, \mathbb{Z})$	${}^3_p13^4_1$	1234_{26} $124\,10_{52}$ 1258_{52} 1267_{13}
	$(\mathbb{Z}, 0, \mathbb{Z}^{78}, \mathbb{Z})$	${}^3_p13^4_6$	1235_{52} $124\,12_{13}$ 1259_{52} 1267_{13} $137\,10_{13}$
	$(\mathbb{Z}, 0, \mathbb{Z}^{77} \oplus \mathbb{Z}_2, 0)$	${}^3_p13^4_3$	1234_{26} $124\,11_{52}$ 1259_{52} 1267_{13}
14	$(\mathbb{Z}, 0, \mathbb{Z}^{104} \oplus \mathbb{Z}_2, 0)$	${}^3_p14^2_{204}$	1234_7 1235_{14} 1249_{14} 1257_{14} 1269_{14} $126\,10_{14}$ 1358_{14} 1367_{14} 1379_{14} $138\,14_{14}$ $13\,10\,12_7$ $145\,14_7$ $147\,10_7$ $147\,11_{14}$ $15\,10\,14_7$ $167\,14_7$

A possible extension of the enumeration of vertex-homogeneous 3-neighborly 3-pseudomanifolds beyond 15 vertices seems to be hard. (As mentioned before, there are 1954 transitive permutation groups of degree 16, many of them of small order and thus causing a large number of distinct orbits of 3-dimensional simplices.) Nevertheless, it is promising to study particular group actions.

THEOREM 5.3. *There is exactly one* 3*-neighborly, hence tight,* 3*-pseudomanifold with a primitive group action on* $n = 16$ *vertices, and there is no such example for a primitive group action on* 18, 20, 21, *or* 22 *vertices. The unique example with* 16 *vertices is non-orientable with homology vector* $(\mathbb{Z}, 0, \mathbb{Z}^{175} \oplus \mathbb{Z}_2, 0)$. *The corresponding primitive group is* $2^4\!:\!5$ *with generators*

$$(1\,\mathit{15}\,75\,\mathit{12})(29\,\mathit{13}\,\mathit{14}\,8)(36\,\mathit{10}\,\mathit{11}\,4),\quad (1\,\mathit{16})(23)(45)(67)(89)(\mathit{10}\,\mathit{11})(\mathit{12}\,\mathit{13})(\mathit{14}\,\mathit{15}).$$

The orbit representatives are 1234_{80}, 1239_{80}, $128\,11_{20}$, $12\,12\,14_{80}$, *and* 1346_{20}.

THEOREM 5.4. *There exist precisely fourteen* 3*-neighborly tight* 3*-pseudomanifolds on* 20 *vertices that have a vertex-transitive* A_5*-action,* 13 *of them are non-orientable and one is orientable with homology vector* $(\mathbb{Z}, 0, \mathbb{Z}^{400}, \mathbb{Z})$.

The 14 tight triangulations have been found with the program MANIFOLD_VT described in [KöLu]. The orientable example has orbit representatives

$123\,11_{60}$, $123\,13_{30}$, $124\,11_{60}$, $124\,16_{30}$, $135\,14_{60}$, $135\,19_{60}$, $136\,13_{60}$, $136\,16_{30}$, 1379_5, $137\,10_{20}$, $138\,19_{60}$ $13\,11\,14_{30}$, $148\,10_{20}$, $14\,11\,19_{20}$, $14\,12\,19_{20}$, $1\,14\,16\,20_5$, with respect to the A_5-action generated by the permutations

$$(1\,6\,10\,13\,17)(2\,5\,9\,14\,18)(3\,8\,12\,15\,20)(4\,7\,11\,16\,19),$$
$$(1\,17)(2\,18)(3\,4)(5\,7)(6\,8)(9\,20)(10\,19)(11\,14)(12\,13)(15\,16).$$

REMARK. We let the program also search for 3-manifolds with transitive A_5-action on 20 vertices, and we found precisely one such object with orbit representatives $1\,3\,5\,15_{30}$, $1\,3\,5\,16_{60}$, $1\,3\,6\,16_{30}$, $1\,4\,11\,19_{20}$, $1\,4\,12\,19_{20}$. This manifold has f-vector $f = (20, 180, 320, 160)$ and the largest homology vector $(\mathbb{Z}, \mathbb{Z}^6, \mathbb{Z}^5 \oplus \mathbb{Z}_2, 0)$ that has been observed so far for a vertex-transitive combinatorial 3-manifold. It is not tight.

6. The centrally–symmetric case: Tight subcomplexes of the crosspolytope

Centrally–symmetric analogues of tight triangulations can be obtained as follows. The smallest centrally–symmetric polytope is the d-dimensional crosspolytope (the dual of the d-cube), which contains all possible edges except the d diagonals between antipodal pairs of vertices. Furthermore it contains all possible simplices except those containing a diagonal. It is therefore quite natural to consider subcomplexes of the crosspolytope. If we regard $(k-1)$-connected $2k$-manifolds M, then the tightness of M as a subset of Euclidean space E^d is equivalent to the property that M contains the complete k-dimensional skeleton of the d-dimensional crosspolytope [K, 4.1]. That is to say M is *nearly* $(k+1)$*-neighborly*, i.e., neighborly subject to the condition that only the diagonals should be avoided. A triangulation is called *centrally–symmetric* if there is a combinatorial involution σ without any fixed point on the underlying space. We can then regard each pair $x, \sigma x$ as a pair of *antipodal* vertices because there can be no edge joining them. From this it follows that any centrally–symmetric triangulation with $n = 2m$ vertices can be regarded as a subcomplex of the m-dimensional crosspolytope.

PROPOSITION 6.1. *Let M be a centrally–symmetric n-vertex triangulation of a $(k-1)$-connected $2k$-manifold. Then $n = 2m$ is an even number, and the following conditions are equivalent:*

(i) *The manifold can be tightly embedded as a subcomplex of the m-dimensional crosspolytope.*
(ii) *The triangulation is nearly $(k+1)$-neighborly.*

The simplest nontrivial example is the unique centrally–symmetric 8-vertex triangulation of the 2-torus, regarded as a subcomplex of the 4-dimensional crosspolytope. This torus can be defined as the union of the $\mathbb{Z}_8$-orbits of the triangles 124 and 134 with diagonals 15, 26, 37, 48 (however, its full automorphism group is the affine group in dimension 1 over the ring $\mathbb{Z}_8$). For 2-dimensional surfaces a purely combinatorial condition, by analogy with 2.1, was obtained in [K, 2.22]. The existence of orientable triangulations of that type was previously established by M. Jungerman and G. Ringel. The higher-dimensional case was first studied by E. Sparla [Sp2]. Here an analogue of 2.2 was given, and the first non-trivial 4-dimensional example was obtained, a 12-vertex triangulation of $S^2 \times S^2$, which can be regarded as a tight subcomplex of the 6-dimensional crosspolytope. It is nearly 3-neighborly, and once again a vertex-transitive automorphism group is present. In 6.3 we present further centrally–symmetric triangulations of $S^k \times S^m$ with a vertex-transitive automorphism group, in particular, nearly $(k+1)$-neighborly triangulations of $S^k \times S^k$ for $k = 3$ and $k = 4$. By the results of [Sp2] the following conjecture is true for $k = m = 2$. It is also true for $k = m = 1$.

Conjecture 6.2. *Any centrally–symmetric combinatorial triangulation of the manifold $S^k \times S^m$ (or of any manifold with the same homology as $S^k \times S^m$) has*

$$n \geq 2(k+m)+4$$

vertices, and it is tight as a subcomplex of the $\frac{n}{2}$-dimensional crosspolytope if and only if equality is attained.

Theorem 6.3. *There exist centrally–symmetric triangulations of the sphere products*

$$\begin{array}{ccccccc} S^1 \times S^1, & S^2 \times S^1, & S^3 \times S^1, & S^4 \times S^1, & S^5 \times S^1, & S^6 \times S^1, & S^7 \times S^1, \\ & & S^2 \times S^2, & S^3 \times S^2, & & S^5 \times S^2, & \\ & & & & S^3 \times S^3, & S^4 \times S^3, & S^5 \times S^3, \\ & & & & & & S^4 \times S^4 \end{array}$$

on $n = 2(k+m)+4$ vertices, which have a vertex-transitive dihedral group action. The examples satisfy equality in the above conjecture. They are tight at least in the cases $S^k \times S^k$ and $S^k \times S^1$.

The list of examples is given in [Lu1]; compare the following tables (with the permutations $a_{2m} = (123\ldots 2m)$ as generators of the cyclic group actions and the permutations a_{2m} and $b_{2m} = (1\ 2m)(2\ 2m-1)\ldots(m\ m+1)$ as generators of the dihedral group actions). The examples of type $S^k \times S^k$ are tight because they are nearly $(k+1)$-neighborly. Similarly, the examples of type $S^k \times S^m$ are at least $(m-1)$-tight because they are nearly $(m+1)$-neighborly. The examples of type $S^k \times S^1$ are tight because they appear as the boundary of a $(k+2)$-dimensional 1-handle which is given as the orbit of the simplex $123\ldots k+3$ on $n = 2(k+3)$ vertices under the cylic group of order n. This 1-handle itself is a tight subcomplex of the $(k+3)$-dimensional crosspolytope. This can be seen along the lines of the analogous proof in the case of the tight triangulations in 3.3; compare [K, 5B]. We strongly conjecture that the other examples are tight in the crosspolytope as well, in accordance with Conjecture 6.2.

Table 1: Centrally symmetric products of spheres with $n = 2d + 4$ vertices and cyclic group action.

d	n	Top. type	f-vector	Comb. type	List of orbits	Remarks
2	8	$S^1 \times S^1$	(*24*,16)	${}^{2}_{\times}8^{\,di}_{1}$	124_{16}	[KL2, $M_1^2(8)$], [KöLu, ${}^{2}8^{\,15}_{1}$]
3	10	$S^2 \times S^1$	(*40*,60,30)	${}^{3}_{\times}10^{\,di}_{1}$	1235_{20} 1245_{10}	[Wa], [KL2, $M_2^3(10)$], [KöLu, ${}^{3}10^{\,3}_{2}$]
4	12	$S^3 \times S^1$	(*60*,120,120,48)	${}^{4}_{\times}12^{\,di}_{2}$	12346_{24} 12356_{24}	[KL2, $M_3^4(12)$], [KöLu, ${}^{4}12^{\,12}_{1}$]
		$S^2 \times S^2$	(*60*,*160*,180,72)	${}^{4}_{\times}12^{\,cy}_{1}$	12345_{12} 12356_{12} $1236\,11_{12}$ 12569_{12} $1269\,11_{12}$ $1358\,10_{12}$	[Sp1, M_1], [KöLu, ${}^{4}12^{\,11}_{1}$]
				${}^{4}_{\times}12^{\,cy}_{2}$	12345_{12} 12356_{12} $1236\,11_{12}$ $1256\,10_{12}$ $1269\,11_{12}$ $1358\,10_{12}$	[Sp1, $M = M_2$], [Sp2], [KöLu, ${}^{4}12^{\,124}_{1}$]
				${}^{4}_{\times}12^{\,di}_{1}$	12345_{12} $1235\,10_{24}$ $1236\,10_{12}$ 12459_{12} $1358\,10_{12}$	[Sp1, M_3], [KöLu, ${}^{4}12^{\,28}_{1}$]
5	14	$S^4 \times S^1$	(*84*,210,280,210,70)	${}^{5}_{\times}14^{\,di}_{1}$	123457_{28} 123467_{28} 123567_{14}	[KL2, $M_4^5(14)$] [KöLu, ${}^{5}14^{\,3}_{8}$]
		$S^3 \times S^2$	(*84*,*280*,490,420,140)	${}^{5}_{\times}14^{\,di}_{2}$	123467_{28} $12346\,12_{28}$ 123567_{14} $12357\,11_{28}$ $12457\,13_{14}$ $1246\,10\,12_{28}$	[KöLu, ${}^{5}14^{\,3}_{9}$]
6	16	$S^5 \times S^1$	(*112*,336,560,560,336,96)	${}^{6}_{\times}16^{\,di}_{2}$	1234568_{32} 1234578_{32} 1234678_{32}	[KL2, $M_5^6(16)$]
		$S^3 \times S^3$	(*112*,*448*,*1120*,1568,1120,320)	${}^{6}_{\times}16^{\,cy}_{1}$	1234567_{16} 1234578_{16} $123458\,15_{16}$ $123478\,13_{16}$ $12347\,13\,14_{16}$ $12348\,13\,15_{16}$ $1234\,13\,14\,15_{16}$ $123568\,15_{16}$ $123678\,12_{16}$ $12368\,12\,13_{16}$ $12368\,13\,15_{16}$ $12378\,12\,13_{16}$ $124578\,11_{16}$ $12457\,11\,14_{16}$ $12458\,11\,14_{16}$ $12478\,11\,13_{16}$ $1247\,11\,13\,14_{16}$ $1248\,11\,13\,15_{16}$ $1268\,11\,13\,15_{16}$ $1357\,10\,12\,14_{16}$	
				${}^{6}_{\times}16^{\,di}_{1}$	1234567_{16} 1234578_{32} $123458\,14_{16}$ $123478\,13_{32}$ $123567\,12_{16}$ $12356\,12\,15_{32}$ $123578\,12_{32}$ $12358\,12\,15_{16}$ $12378\,12\,13_{16}$ $12458\,11\,14_{16}$ $12467\,11\,13_{16}$ $12468\,11\,13_{32}$ $1247\,11\,13\,14_{32}$ $1357\,10\,12\,14_{16}$	
7	18	$S^6 \times S^1$	(*144*,504,1008, 1260,1008,504,126)	${}^{7}_{\times}18^{\,di}_{1}$	12345679_{36} 12345689_{36} 12345789_{36} 12346789_{18}	[KL2, $M_6^7(18)$]

Table 1: Centrally symmetric products of spheres with $n = 2d + 4$ vertices and cyclic group action.

		$S^5 \times S^2$	(*144*,*672*,1764, 2772,2688,1512,378)	${}^{7}_{\times}18^{di}_{2}$	12345689 $_{36}$ 1234568 16 $_{36}$ 12345789 $_{36}$ 1234579 15 $_{36}$ 12346789 $_{18}$ 1234679 17 $_{36}$ 123468 14 16 $_{36}$ 1235679 17 $_{18}$ 123579 13 15 $_{36}$ 123579 13 17 $_{36}$ 124579 15 17 $_{18}$ 12468 12 14 16 $_{36}$	
		$S^4 \times S^3$	(*144*,*672*,*2016*, 3780,4200,2520,630)	${}^{7}_{\times}18^{di}_{3}$	1234579 15 $_{36}$ 1234579 17 $_{36}$ 1234679 14 $_{36}$ 1234679 17 $_{36}$ 123467 14 17 $_{36}$ 1234689 14 $_{36}$ 1234689 16 $_{36}$ 123479 14 15 $_{36}$ 1235679 13 $_{36}$ 1235679 17 $_{18}$ 1235689 13 $_{36}$ 1235689 16 $_{36}$ 123569 16 17 $_{36}$ 1235789 13 $_{36}$ 123589 15 16 $_{36}$ 123679 13 14 $_{36}$ 123689 13 14 $_{36}$ 123789 13 15 $_{18}$ 124589 15 16 $_{18}$	
8	20	$S^7 \times S^1$	(*180*,720,1680,2520, 2520,1680,720,160)	${}^{8}_{\times}20^{di}_{2}$	12345678 10 $_{40}$ 12345679 10 $_{40}$ 12345689 10 $_{40}$ 12345789 10 $_{40}$	[KL2, $M^8_7(20)$]
		$S^5 \times S^3$	(*180*,*960*,*3360*,7560, 10920,9840,5040,1120)	${}^{8}_{\times}20^{di}_{3}$	12345689 10 $_{40}$ 12345689 17 $_{40}$ 1234568 10 19 $_{40}$ 1234569 10 18 $_{40}$ 12345789 10 $_{40}$ 1234578 10 16 $_{40}$ 1234579 10 18 $_{40}$ 123457 10 16 18 $_{40}$ 1234678 10 19 $_{40}$ 1234679 10 15 $_{40}$ 123467 10 15 19 $_{40}$ 123469 10 15 18 $_{40}$ 123469 15 17 18 $_{40}$ 123479 10 15 18 $_{40}$ 1235679 10 18 $_{40}$ 1235689 14 17 $_{40}$ 123568 14 17 19 $_{40}$ 123569 14 17 18 $_{40}$ 123578 10 16 19 $_{40}$ 123679 10 15 18 $_{40}$ 124579 10 16 18 $_{40}$ 124579 13 16 18 $_{40}$ 12458 10 13 16 17 $_{40}$ 124679 13 15 18 $_{40}$ 12469 10 13 15 18 $_{40}$ 12478 10 13 15 16 $_{40}$ 12478 10 13 15 19 $_{40}$ 12478 10 15 16 19 $_{40}$	
		$S^4 \times S^4$	(*180*,*960*,*3360*,*8064*, 12600,12000,6300,1400)	${}^{8}_{\times}20^{di}_{1}$	123456789 $_{20}$ 12345679 10 $_{40}$ 1234567 10 18 $_{20}$ 1234569 10 17 $_{40}$ 12345789 16 $_{40}$ 1234578 16 19 $_{40}$ 1234579 10 16 $_{40}$ 123457 10 16 19 $_{20}$ 123459 10 16 17 $_{20}$ 12346789 15 $_{20}$ 1234679 10 18 $_{40}$ 1234679 15 18 $_{40}$ 123469 10 17 18 $_{40}$ 1234789 15 16 $_{40}$ 123479 10 16 18 $_{40}$ 1235679 10 14 $_{40}$ 123567 10 14 18 $_{20}$ 1235689 10 14 $_{40}$ 123568 10 14 17 $_{40}$ 12356 10 14 17 19 $_{40}$ 12356 10 14 18 19 $_{20}$ 1235789 14 16 $_{20}$ 123578 14 16 19 $_{40}$ 123579 10 14 16 $_{40}$ 12357 10 14 16 19 $_{40}$ 123589 10 14 16 $_{40}$ 123589 10 16 17 $_{40}$ 12358 10 14 16 19 $_{20}$ 123679 10 14 18 $_{40}$ 123679 14 15 18 $_{40}$ 12368 10 14 15 17 $_{40}$ 12369 10 14 15 18 $_{20}$, 123789 14 15 16 $_{20}$ 124579 10 13 16 $_{40}$ 12457 10 13 16 19 $_{20}$ 12459 10 13 16 17 $_{20}$ 12467 10 13 15 18 $_{20}$ 124689 13 15 17 $_{20}$ 12468 10 13 15 17 $_{40}$ 12469 10 15 17 18 $_{40}$ 12469 13 15 17 18 $_{40}$ 12479 10 13 15 18 $_{40}$ 13579 12 14 16 18 $_{20}$	
		....	....	${}^{8}_{\times}20^{cy}_{\dots}$	....	

Appendix: List of facets of a 13-vertex triangulation of $S^3 \times S^3$,

$123456a$	$123456b$	$12345ac$	$12345bd$	$12345cd$	$123469a$	$123469b$	$12349ab$
$1234abc$	$1234bcd$	$123567a$	$123567b$	$12357ab$	$12358ab$	$12358ac$	$12358bd$
$12358cd$	$123679a$	$123679b$	$12379ab$	$1238abc$	$1238bcd$	$12456ad$	$12456bd$
$12457ac$	$12457ad$	$12457cd$	$124689c$	$124689d$	$12468bc$	$12468bd$	$12469ad$
$12469bc$	$12478ac$	$12478ad$	$12478cd$	$12489ac$	$12489ad$	$1248bcd$	$1249abc$
$125679b$	$125679d$	$12567ad$	$12569bd$	$12578ab$	$12578ac$	$12578bd$	$12578cd$
$12579bd$	$12679ad$	$12689bc$	$12689bd$	$1278abd$	$1279abd$	$1289abc$	$1289abd$
1345689	$134568b$	$134569a$	$134589d$	$13458bd$	$13459ad$	$1345acd$	1346789
$134678c$	$134679b$	$13467bc$	$13468bc$	$134789d$	$13478cd$	$13479ab$	$13479ad$
$1347abc$	$1347acd$	$1348bcd$	$13567ab$	$135689c$	$13568ab$	$13568ac$	$13569ad$
$13569cd$	$1356acd$	$13589cd$	$136789c$	$13679ad$	$13679cd$	$1367abc$	$1367acd$
$1368abc$	$13789cd$	$145689d$	$14568bd$	$14569ad$	$1457acd$	$146789c$	$14679bc$
$14789ac$	$14789ad$	$1479abc$	$15679bc$	$15679cd$	$1567abc$	$1567acd$	$15689bc$
$15689bd$	$1568abc$	$15789bc$	$15789bd$	$15789cd$	$1578abc$	$1789abc$	$1789abd$
2345678	$234567b$	$234568a$	2345789	$234579b$	$234589a$	$23459ac$	$23459bc$
$2345bcd$	2346789	$234679b$	$234689a$	$2349abc$	$235678a$	$235789c$	$23578ab$
$23578bd$	$23578cd$	$23579bd$	$23579cd$	$23589ac$	$2359bcd$	$236789c$	$23678ad$
$23678cd$	$23679ad$	$23679cd$	$23689ac$	$2368acd$	$2369acd$	$2378abd$	$2379abd$
$238abcd$	$239abcd$	$245678a$	$24567ad$	$24567bc$	$24567cd$	$2456bcd$	$245789c$
$24578ac$	$24579bc$	$24589ac$	$246789c$	$24678ad$	$24678cd$	$24679bc$	$24689ad$
$2468bcd$	$25679bc$	$25679cd$	$2569bcd$	$2689abc$	$2689abd$	$268abcd$	$269abcd$
$345678b$	$345689a$	$345789b$	$34589bd$	$3459acd$	$3459bcd$	$34678bd$	$34678cd$
$3467bcd$	$3468bcd$	$34789bd$	$3479abd$	$347abcd$	$349abcd$	$35678ab$	$35689ac$
$3569acd$	$35789bd$	$35789cd$	$3678abd$	$367abcd$	$368abcd$	$45678ab$	$4567abc$
$4567acd$	$45689ad$	$4568abd$	$456abcd$	$45789ab$	$45789ac$	$4579abc$	$4589abd$
$459abcd$	$4678abd$	$467abcd$	$4789abd$	$5689abc$	$5689abd$	$569abcd$	$5789abc$,

and of a 12-vertex triangulation of $S^3 \times S^2$,

123678	$12367c$	$12368c$	$12378c$	$12458a$	$12458b$	$12459b$	$12459c$
$1245ac$	$1248ab$	$1249ab$	$1249ac$	125789	$12578b$	$12579b$	$12589c$
$1258ac$	126789	$12679c$	$12689c$	$1278ab$	$1278ac$	$1279ab$	$1279ac$
$13456a$	$13456c$	$1345ac$	134679	$13467c$	$13469b$	$1346ab$	$13479a$
$1347ac$	$1349ab$	$1356ab$	$1356bc$	$13589a$	$13589c$	$1358ac$	$1359ab$
$1359bc$	136789	$13689c$	$1369bc$	$13789a$	$1378ac$	$1456ab$	$1456bc$
$1458ab$	$1459bc$	$14679c$	$1469bc$	$1479ac$	$15789a$	$1578ab$	$1579ab$
234567	$23456a$	$23457b$	$23459a$	$23459b$	234678	$23468b$	$2346ab$
$23478b$	$2349ab$	$23567b$	$2356ab$	$2359ab$	$2367bc$	$2368bc$	$2378bc$
245678	$24568a$	$24578b$	$2459ac$	$2468ab$	256789	$25679a$	$2567ab$
$25689a$	$2579ab$	$2589ac$	$2679ac$	$267abc$	$2689ac$	$268abc$	$278abc$
$34567c$	$3457bc$	$3459ac$	$3459bc$	346789	$34689b$	$34789c$	$3478bc$
$3479ac$	$3489bc$	$3567bc$	$3589ac$	$3689bc$	$3789ac$	$45678b$	$4567bc$
$4568ab$	$46789c$	$4678bc$	$4689bc$	$56789a$	$5678ab$	$6789ac$	$678abc$.

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