Periodic Foams and Simplicial Manifolds with Small Valence

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(joint work with Thom Sulanke and John M. Sullivan)

Chemist/metallurgists have found periodic foam structures in various materials, such as transition metal alloys, salt and gas hydrates [2, 3, 4]. Hereby, the Voronoi diagram for the positions of the occurring atoms forms a foam, which is a collection of CMC (constant mean curvature) surfaces that meet according to Plateau's rule (three cells around an edge, four cells meeting at a vertex). In turn, the Delaunay triangulation for the positions of the atoms in a periodic foam is built of nearly regular tetrahedra and is often called a tetrahedrally close-packed (TCP) structure.

All observed periodic foams in nature are built of only four types of cells, the dodecahedron with pentagons only and three further cells that have 2, 3, and 4 additional 6-gons as facets, respectively. Kusner and Sullivan [5] (see also [8, 9]) defined a TCP triangulation of a (closed) 3-manifold to be a triangulation s.t. each edge has valence 5 or 6 (i.e., each edge lies in 5 or 6 tetrahedra), and no triangle has two edges of valence 6.

According to Brady, McCammond, and Meier [1], every (closed, orientable) 3-manifold can be triangulated with edge valences at most 6. Sullivan asked in [10] whether every 3-manifold even has a TCP triangulation?

So far, TCP triangulations were known for the 3-torus (which give periodic foams) as well as for products of arbitrary surfaces with S^1 [9] and some spherical 3-manifolds (the boundary of the 600-cell and quotients thereof).

We add to this list a TCP triangulation of the Poincaré homology 3-sphere and of the octahedral space, respectively. Moreover, we determine all triangulated 3-manifolds with edge valences at most 5 (there are 4787 examples, $4761 \times S^3$, $22 \times \mathbb{R}\mathbf{P}^3$, $2 \times L(3,1)$, $1 \times L(4,1)$, and $1 \times S^3/Q$; cf. also [6]) as well as all proper 3-pseudomanifolds with edge valences at most 5 (there are 41 such examples, all suspensions of the projective plane $\mathbb{R}\mathbf{P}^2$).

We also show that (strongly connected) d-pseudomanifolds with codimension two faces of valence at most 4 are joins of boundaries of simplices.

(It was independently proven by Matveev and Shevchishin [7] that all triangulated 3-manifolds with edge degrees at most five are spherical and have at most 600 tetrahedra.)

References

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