Some Results Related to the Evasiveness Conjecture

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Abstract

The Evasiveness Conjecture for graph properties has natural generalizations to simplicial complexes and to set systems. In this paper we show that the Evasiveness Conjecture for simplicial complexes holds in dimension 2 and 3. We also present an infinite class of counterexamples to the Generalized Aanderaa-Rosenberg Conjecture (the Evasiveness Conjecture for set systems). The smallest member of this class is the only previously known counterexample by Illies.

1 Introduction

Let $\mathcal{P}$ be any *graph property*, that is, a property of graphs which is invariant under graph-isomorphisms, on a fixed set of nodes V of size $n := |V|$, and let E denote the set of all edges on V, with $m := |E| = \binom{n}{2}$. We identify $\mathcal{P}$ with the set system

$$\mathcal{F}_{\mathcal{P}} := \{A \subseteq E : \text{Graph } (V, A) \text{ has property } \mathcal{P}\} \subseteq 2^{E},$$

and for an unknown graph $\mathcal{G} = (V, A)$ on V we consider the *decision problem* whether $\mathcal{G}$ has the property $\mathcal{P}$ or not. In order to find out if the edge set A of $\mathcal{G}$ belongs to $\mathcal{F}_{\mathcal{P}}$, we ask questions of the type "Is $e \in A$?", and an oracle answers (correctly) YES or NO.

The number of elements of E that we will have to test in the worst case, if we proceed according to some optimal strategy, is called the *argument complexity* $c(\mathcal{F}_{\mathcal{P}})$ of $\mathcal{P}$. Then $0 \leq c(\mathcal{F}_{\mathcal{P}}) \leq m$, and $\mathcal{P}$ is *trivial* if $c(\mathcal{F}_{\mathcal{P}}) = 0$ and *non-trivial* if $c(\mathcal{F}_{\mathcal{P}}) > 0$. $\mathcal{P}$ is called *evasive* if $c(\mathcal{F}_{\mathcal{P}}) = m$ and *non-evasive* otherwise. For general set systems $\mathcal{F} \subseteq 2^{E}$, these terms are defined analogously. A graph property is *monotone* if it is preserved under deletion of edges.

In the early seventies Richard Karp proposed the following remarkable conjecture.

Evasiveness Conjecture for Graph Properties: *Every non-trivial monotone graph property $\mathcal{P}$ is evasive.*

Extensive work has been done on determining the argument complexity of particular graph properties (see e.g. [1], [2], [4, Ch. VIII], [20], and references contained therein).

The first successful approach to Karp's Conjecture was carried through by Kahn, Saks, and Sturtevant [10] in 1984. Using methods from algebraic topology, in particular, a fixed point theorem by Oliver [15], they were able to settle the case when

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n is a prime power (and the case $n = 6$). For this, they restated KARP's Conjecture in the language of simplicial complexes: If $\mathcal{P}$ is a monotone graph property, then the corresponding set system $\mathcal{F}_{\mathcal{P}}$ is a (finite abstract) simplicial complex with the vertex set E. We call $\mathcal{F}_{\mathcal{P}}$ the *graph complex* associated with $\mathcal{P}$. Invariance under permutation of the nodes of V (what one naturally requires for $\mathcal{P}$ to be a graph property) gives rise to an induced action of the symmetric group S_n on the edge set E, and thus on the simplicial complex $\mathcal{F}_{\mathcal{P}}$. Clearly, the action of S_n is transitive on E.

By allowing the symmetry group to be any finite group G, one obtains the following more general situation.

Evasiveness Conjecture for Simplicial Complexes [10]: *If $\mathcal{F}$ is a non-evasive vertex-homogeneous simplicial complex* (VHSC) *on the vertex set $E = \{1, \ldots, m\}$ with vertex-transitive action by some group G, then it is the standard $(m-1)$-simplex Δ_{m-1}.*

To be "non-evasive" is in fact a rather strong *topological* requirement. The following sequence of implications holds for finite simplicial complexes (cf. [10]; for an exposition of topological methods in combinatorics see [3]):

$$\text{non-evasive} \Rightarrow \text{collapsible} \Rightarrow \text{contractible} \Rightarrow \mathbb{Z}\text{-acyclic} \Rightarrow \mathbb{Q}\text{-acyclic} \Rightarrow \tilde{\chi} = 0$$

and leads to further generalizations of the above conjectures (cf. Figure 1). ($\tilde{\chi}$ denotes the reduced Euler characteristic of a simplicial complex.)

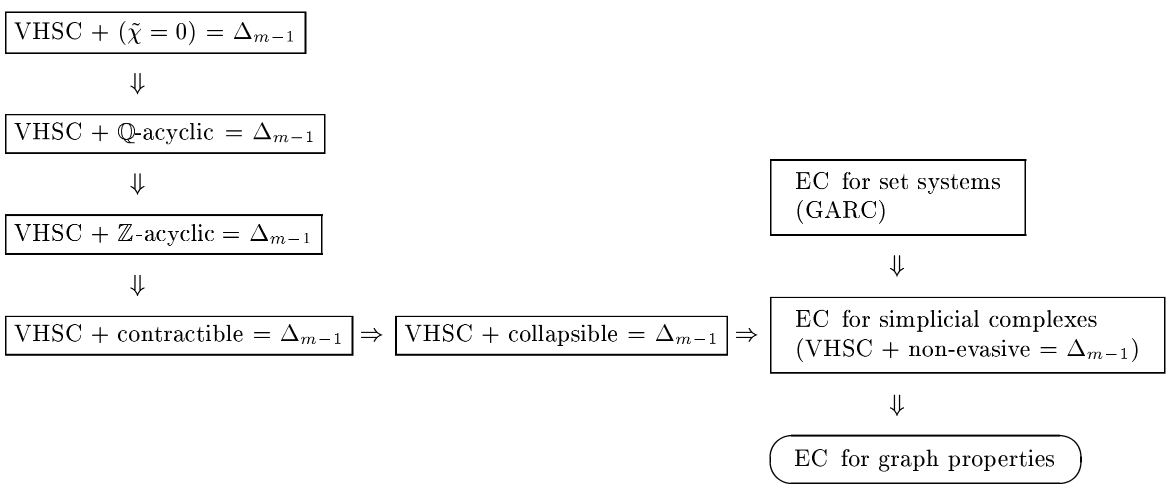


Figure 1: Generalizations of the Evasiveness Conjecture (EC) for graph properties.

Instead of relaxing the condition "non-evasive" one can alternatively remove the monotonicity. The resulting Evasiveness Conjecture for set systems is known as the

Generalized Aanderaa-Rosenberg Conjecture (GARC) [16] *Let $\mathcal{F} \subseteq 2^E$ be a set system with induced transitive symmetry group $G \subseteq S_E$. If $\emptyset \in \mathcal{F}$, but $E \notin \mathcal{F}$, then $\mathcal{F}$ is evasive.*

Albeit the latter conjecture as well as the Conjecture $[\text{VHSC} + (\tilde{\chi}=0) = \Delta_{m-1}]$ were proved by RIVEST and VUILLEMIN [16] for sets E of prime power cardinality, $m = q^s$, Illies [8] provided a counterexample to GARC for $m = 12$, and there is an abundance of counterexamples to the Conjecture [VHSC + $(\tilde{\chi} = 0) = \Delta_{m-1}$] for $m \neq q^s$.

In the next section we will review some fixed point theorems and their applications. In Section 3 we show that there is, apart from the simplex, no $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex with $m = 6$, 10, or 12 vertices. This fact implies the non-existence of 2- and 3-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes, different from a simplex.

Furthermore, we construct in Section 4 an *infinite class* of counterexamples to the Generalized Aanderaa-Rosenberg Conjecture for $m = u(u+1)$, $u \geq 3$ odd, with the Illies example as the smallest member of the class.

2 Fixed Point Theorems and Group Actions

Recall that if the vertex-transitive action of a (finite) group G on a (finite) simplicial complex K (with m vertices) has a fixed point, then K is a simplex. This can be seen geometrically by regarding K as a subcomplex of the $(m-1)$-dimensional simplex Δ_{m-1} with vertices $e_1, \ldots, e_m$. Any point x of K has a unique representation $x = \sum_{i=1}^m \lambda_i e_i$, with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. The group G then acts by permuting the coordinates, $gx = \sum_{i=1}^m \lambda_i e_{g(i)}$, $g \in G$. If G is transitive, then for every i, j there is some $g \in G$ such that $e_j = e_{g(i)}$. If, in addition, the action of G has a fixed point y, then $gy = y$ for every group element g, and therefore $\lambda_1 = \ldots = \lambda_m = \frac{1}{m}$. But $y = \frac{1}{m}\sum_{i=1}^m e_i$ is a point of K if and only if K is a simplex.

This simple fact, in combination with fixed point theorems from algebraic topology, provides an important tool for the study of group actions on simplicial complexes. It was shown by Smith [19] that if a p-group P, i.e., a group with prime power order $|P| = p^t$, acts on a $\mathbb{Z}_p$-acyclic complex, then the fixed point set for this action is $\mathbb{Z}_p$-acyclic as well. In particular, the fixed point set is not empty – hence, there are no vertex-transitive group actions of a p-group on a $\mathbb{Z}_p$-acyclic simplicial complex (that is not a simplex).

The theorem by Smith has been generalized by Oliver.

Theorem 1 (Oliver [15]) *Let G be a finite group with subsequent normal subgroups $P \lhd Q \lhd G$ such that*

(i) *P is a p-group,*

(ii) *G/Q is a q-group, and*

(iii) *Q/P is cyclic.*

If G acts on a $\mathbb{Z}_p$-acyclic complex K, then the Euler characteristic $\chi(K^G)$ of the fixed point set K^G is equivalent to $1 \pmod q$.

We say that a group G is of *Oliver (p,q)-type* if it has properties (i), (ii) and (iii) of Theorem 1. A group G of Oliver (p,q)-type with $q = 1$ is called a group of *Oliver $(p,1)$-type*. If a group G of Oliver (p,q)-type acts vertex-transitively on a $\mathbb{Z}_p$-acyclic simplicial complex K, then K is a simplex. In fact, if H is an Oliver (p,q)-type vertex-transitive subgroup of some group G, which acts on a $\mathbb{Z}_p$-acyclic complex K, then K is a simplex.

Theorem 2 (Kahn, Saks, and Sturtevant [10]) *Let $\mathcal{F}_{\mathcal{P}_n}$ be the graph complex associated with some (non-trivial) graph property $\mathcal{P}_n$ on $n = p^t$ nodes, with p prime. Then $\mathcal{F}_{\mathcal{P}_n}$ is not $\mathbb{Z}_p$-acyclic.*

dim	# vertices
1	-
2	6
3	6, 12
4	10, 12, 15, 20, 30, 60
5	10, 12, 15, 20, 30, 60
6	10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 60, 70, 84, 105, 140, 210, 420
7	10, 12, 14, 15, 20, 21, 24, 28, 30, 35, 40, 42, 56, 60, 70, 84, 105, 120, 140, 168, 210, 280, 420, 840
8	12, 14, 15, 18, 20, 21, 24, 28, 30, 35, 36, 40, 42, 45, 56, 60, 63, 70, 72, 84, 90, 105, 120, 126, 140, 168, 180, 210, 252, 280, 315, 360, 420, 504, 630, 840, 1260, 2520

Table 1: Possible numbers of vertices for low-dimensional vertex-homogeneous simplicial complexes with $\tilde{\chi} = 0$.

Proof: Let $G = \mathit{Aff}(GF(p^t)) < S_n$ be the group of affine transformations of $GF(p^t)$. The group G is 2-transitive on $\{1 \ldots n\}$ and therefore transitive on the edge set E. Furthermore, G is of Oliver $(p, 1)$-type (choose $Q := G$ and $P := \{x \mapsto x{+}b : b \in GF(p^t)\}$). Hence, G is a vertex-transitive Oliver $(p, 1)$-type subgroup of the symmetric group S_n with induced action on all graph complexes $\mathcal{F}_{\mathcal{P}_n}$. But then either $\mathcal{F}_{\mathcal{P}_n}$ is a simplex, and thus $\mathcal{P}_n$ is trivial, or $\mathcal{F}_{\mathcal{P}_n}$ is not $\mathbb{Z}_p$-acyclic by Theorem 1. $\square$

If a graph complex is not $\mathbb{Z}_p$-acyclic, then it cannot be non-evasive.

Corollary 3 (Kahn, Saks, and Sturtevant [10]) *The Evasiveness Conjecture for graph properties holds for every prime power number of nodes.*

3 The Evasiveness Conjecture in Dimension 2 and 3

We will show in the following that (non-trivial) non-evasive vertex-homogeneous simplicial complexes do not exist in dimension 2 and 3.

Proposition 4 (Björner) *Let E be a finite set of cardinality $m = |E| = q_1^{\alpha_1} \cdots q_r^{\alpha_r}$ (primepower-decomposition) and $M = \max\{q_1^{\alpha_1}, \ldots, q_r^{\alpha_r}\}$. If $K \subseteq 2^E$ is a vertex-homogeneous simplicial complex on the vertex set E with reduced Euler characteristic $\tilde{\chi}(K) = 0$, then $\dim(K) \geq M - 1$.*

Proof: By the transitivity of the group action, every element of E is contained the same number of times, s, in the k-sets of every orbit $\mathcal{O}$ of $(k-1)$-dimensional faces, i.e., $k \cdot |\mathcal{O}| = s \cdot |E|$. For $M = q_i^{\alpha_i}$ this implies that $q_i \mid |\mathcal{O}|$ if $k < M$. Now, if $\dim(K) < M - 1$, then, with the exception of the orbit of the empty set, which has size 1, the size of every orbit of $(k-1)$-faces of K is divisible by q_i. Hence, $\tilde{\chi}(K) \equiv -1 \bmod q_i$, and it follows that $\tilde{\chi}(K) \neq 0$. $\square$

Corollary 5 (Rivest and Vuillemin [16]) *Conjecture* $[\mathrm{VHSC} + (\tilde{\chi} = 0) = \Delta_{m-1}]$ *holds if $m = q^s$ is a prime power.*

It follows from Proposition 4 that for every d-dimensional vertex-homogeneous simplicial complex K with reduced Euler characteristic $\tilde{\chi} = 0$ one has $M \leq d+1$. In particular, the cardinality m of the vertex set E of K can only attain finitely many different values. Furthermore, $m > d+2$. This follows from the fact that for $m = d+2$ there is, by transitivity, only one orbit of d-faces. But the boundary complex of a simplex is a sphere with $\tilde{\chi} \neq 0$. Table 1 displays the vertex-numbers that are possible for $d \leq 8$.

As a result of the above, we get a lower bound for the dimension of graph complexes $\mathcal{F}_{\mathcal{P}_n}$ with reduced Euler characteristic $\tilde{\chi} = 0$ for graph properties $\mathcal{P}_n$ on n nodes. See Table 2 for small $n \neq p^t$.

# nodes n	# vertices $m = \binom{n}{2}$	dim $\geq$	# nodes n	# vertices $m = \binom{n}{2}$	dim $\geq$
6	15	4	26	325	24
10	45	8	28	378	26
12	66	10	30	435	28
14	91	12	33	528	15
15	105	6	34	561	16
18	153	16	35	595	16
20	190	18	36	630	8
21	210	6	38	703	36
22	231	10	39	741	18
24	276	22	40	780	12

Table 2: Lower bounds for the dimension of graph complexes ($\neq$ simplex) with $\tilde{\chi} = 0$.

The 'smallest example' of a graph property $\mathcal{P}$ with $\tilde{\chi}(\mathcal{F}_{\mathcal{P}}) = 0$ can be found on 6 nodes. It is the property $\mathcal{P}_{\mathrm{A}}$ of being a subgraph of any of the first four graphs of Figure 2. The next listed three respective two graphs describe examples of higher-dimensional graph complexes on 6 nodes with $\tilde{\chi}(\mathcal{F}_{\mathcal{P}}) = 0$. All three examples have nontrivial re-

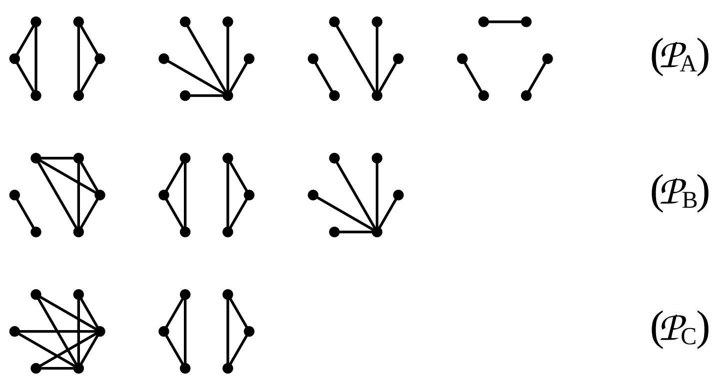


Figure 2: Three examples of graph properties with $\tilde{\chi}(\mathcal{F}_{\mathcal{P}}) = 0$.

duced homology, i.e., $\tilde{H}_*(\mathcal{F}_{\mathcal{P}_{\mathrm{A}}}) = (0, 0, \mathbb{Z}^{15}, \mathbb{Z}^{15}, 0, 0)$, $\tilde{H}_*(\mathcal{F}_{\mathcal{P}_{\mathrm{B}}}) = (0, 0, \mathbb{Z}^{15}, \mathbb{Z}^{15}, 0, 0, 0)$, and $\tilde{H}_*(\mathcal{F}_{\mathcal{P}_{\mathrm{C}}}) = (0, 0, 0, \mathbb{Z}^{20} \oplus \mathbb{Z}_3, \mathbb{Z}^{20}, 0, 0, 0, 0)$, as computed with the C-program HOMOLOGY by Heckenbach [7].

Remark: Although there are many examples of graph complexes with $\tilde{\chi}(\mathcal{F}_{\mathcal{P}}) = 0$, there seems to be no example known of a nontrivial $\mathbb{Q}$-acyclic graph complex.

Note added to revised version: *Jakob Jonsson [9] has recently provided an example of an 8-dimensional non-trivial $\mathbb{Q}$-acyclic graph complex corresponding to a graph property on 6 nodes.*

We turn back to general vertex-homogeneous simplicial complexes. For every m, there is a finite number of transitive permutation groups of degree m. These groups were classified for $m \leq 15$ (see [5], [6], [13], [14], [17]), a library of the groups is contained in the computer algebra package GAP [18]. We determined for all transitive permutation groups G of degree $m = 6, 10, 12, 14, 15$ whether they are of Oliver (p, q)-type (for some p and some q) and, in addition, for the groups that are *not* of Oliver (p, q)-type if they have a transitive subgroup $H < G$ of Oliver (p, q)-type. Table 3 gives the statistics.

# Vertices m	# Transitive group actions	# Groups G not of Oliver (p,q)-type	# Groups without trans. $H < G$ of Oliver (p,q)-type
6	16	4	0
10	45	21	3
12	301	107	1
14	63	34	2
15	104	64	5

Table 3: Transitive permutation groups without subgroups of Oliver (p, q)-type.

One example of a permutation group of degree 6 is the group action of the alternating group A_5. Although A_5 is not of Oliver (p, q)-type as it is simple, on 6 vertices A_5 has A_4 as a vertex-transitive subgroup of Oliver $(2, 1)$-type.

Lemma 6 *There is, apart from the 2-simplex, no 2-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex.*

Proof: If there were a non-trivial 2-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex K, then it would have 6 vertices by Proposition 4. But every transitive permutation group of degree 6 is of Oliver (p, q)-type or has a vertex-transitive subgroup of Oliver (p, q)-type. Thus K cannot be $\mathbb{Z}$-acyclic by Theorem 1. $\square$

There exists a transitive permutation group, $[2^4]\, 3^2\!:\!4$, on 12 vertices, which is not of Oliver (p, q)-type and that has no vertex-transitive subgroup of Oliver (p, q)-type. Thus, we cannot use the above argument in the case of 3-dimensional complexes. In this case, we proceed as follows. We will first determine all simplicial complexes on 12 vertices with a vertex-transitive action of the group $[2^4]\, 3^2\!:\!4$ for which $\tilde{\chi} = 0$ and then compute their homology. It will turn out that the homology is non-trivial for each of the complexes.

Definition 7 *Let G be a transitive permutation group of the set $E = \{1, 2, \ldots, m\}$. Then G acts on the system of sets 2^E, and we denote the set of orbits for this action by $\mathrm{Orb}_G(2^E)$. We define a partial order "$<$" on $\mathrm{Orb}_G(2^E)$: For $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Orb}_G(2^E)$, $\mathcal{O}_2 < \mathcal{O}_1$ if and only if for any $B \in \mathcal{O}_2$ and $g \in G$, $gB \subseteq A$ for some $A \in \mathcal{O}_1$. The partially ordered set $(\mathrm{Orb}_G(2^E), <)$ is the **orbit poset** corresponding to the action of G on 2^E.*

It follows that simplicial complexes $K \subseteq 2^E$, which are invariant under the induced action of a transitive permutation group G on E, are in one-to-one correspondence with the ideals of the orbit poset $\mathrm{Orb}_G(2^E)$.

Since our aim is to determine vertex-homogeneous simplicial complexes with reduced Euler characteristic $\tilde{\chi} = 0$, it is natural to consider *weighted orbit posets*, where we label each orbit of k-sets by its size times the sign $(-1)^{k+1}$. Complexes with reduced Euler characteristic $\tilde{\chi} = 0$ then correspond to ideals of weighted orbit posets with $\sum_{\mathcal{O}\in I} w_{\mathcal{O}} = 0$ for the weights $w_{\mathcal{O}}$ of the orbits $\mathcal{O}$ in the ideal I.

The weighted orbit posets for the A_5-action and its transitive A_4-subaction on 6 vertices are depicted in Figure 3. The respective ideals I with $\sum_{\mathcal{O}\in I} w_{\mathcal{O}} = 0$ of these two orbit posets all correspond to the 6-vertex triangulation of the real projective plane. For the case of the A_4-action we shaded the orbit with 4 triangles in Figure 4 in grey, the other orbit of 6 triangles is in white.

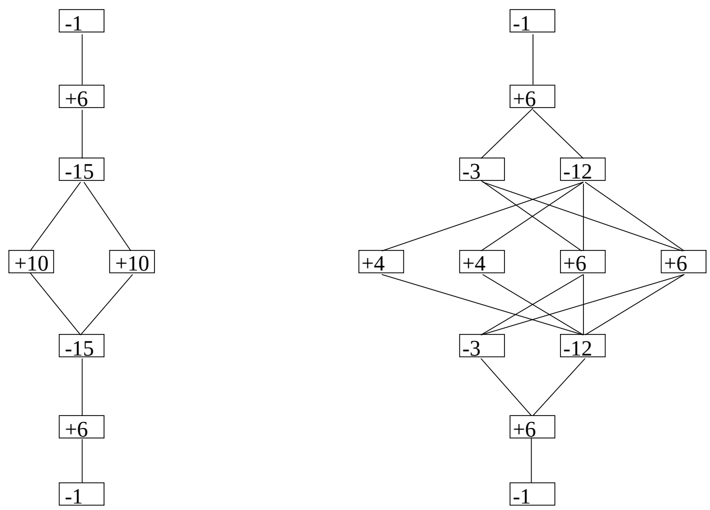


Figure 3: Weighted orbit posets of the transitive A_5- and A_4-actions on 6 vertices.

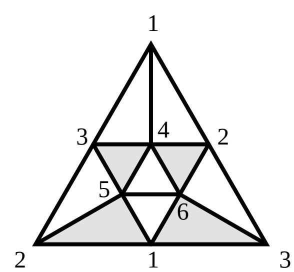


Figure 4: 6-vertex triangulation of the real projective plane.

As listed in Table 3, there are three transitive permutation groups on 10 vertices that are not of Oliver type and contain no transitive Oliver type subgroup. These groups are $A_5 < A_6 < M_{10}$ (with inclusions as transitive permutation groups). On 12 vertices there is only one such group, $[2^4]\,3^2:4$. There are two groups on 14 vertices, $PSL_2(7)$ and $PSL_2(13)$, which are not included in each other; and on 15 vertices we have five groups, with the inclusions $A_5 < S_5, A_6$; $S_5 < S_6, A_7$; and $A_6 < S_6, A_7$. If we want to generate all vertex-homogeneous simplicial complexes with $\tilde{\chi} = 0$ for these actions, it would be sufficient to do this for the actions $A_5(10)$, $[2^4]\,3^2\!:\!4$, $PSL_2(7)$, $PSL_2(13)$, and $A_5(15)$, since these are transitive subgroups of the other groups. The weighted orbit poset for $A_5(10)$ is displayed in Figure 5. Nevertheless, we only succeeded with the generation

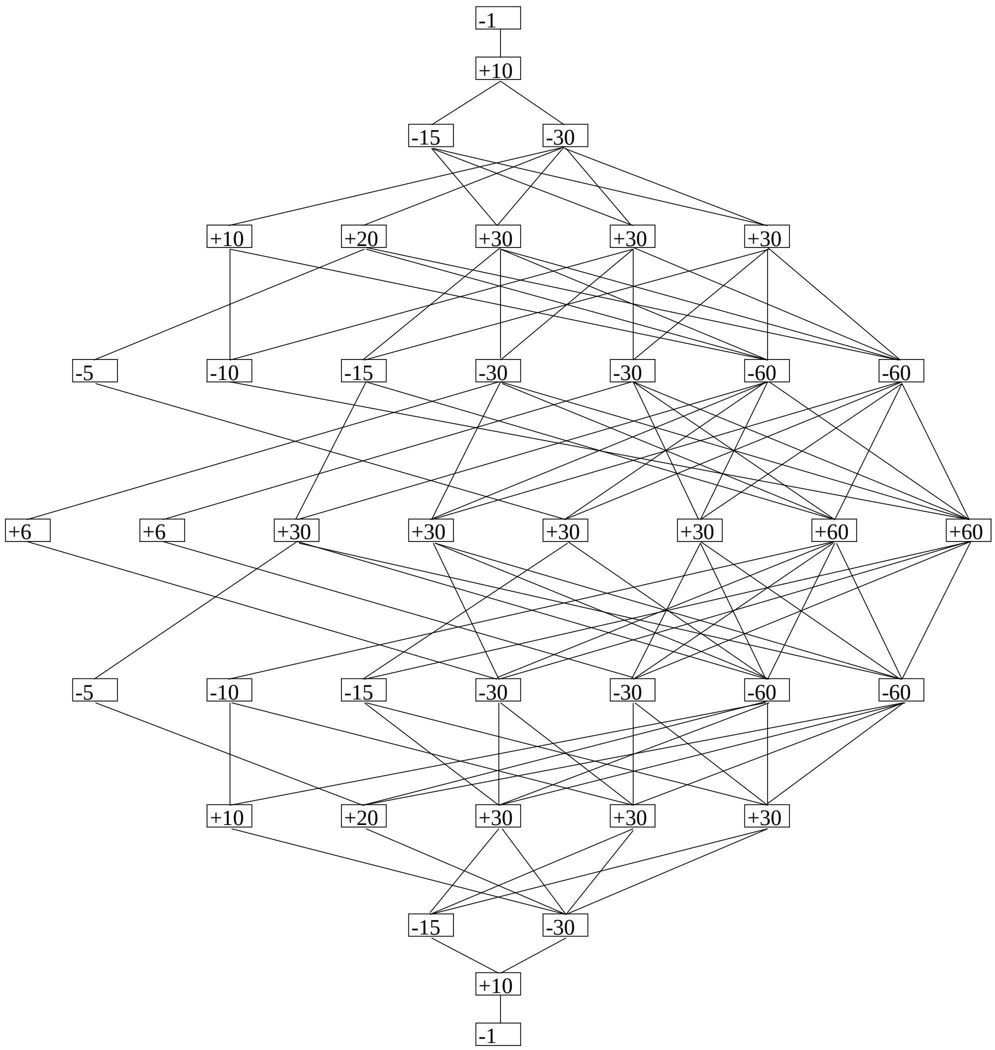


Figure 5: Weighted orbit poset of the transitive A_5-action on 10 vertices.

of all ideals (simplicial complexes) with $\sum_{\mathcal{O}\in I} w_{\mathcal{O}} = 0$ ($\tilde{\chi} = 0$) for the three actions on 10 vertices, for the action of $[2^4]\,3^2:4$ on 12 vertices, of $PSL_2(13)$ on 14 vertices, and of A_7 on 15 vertices. Table 4 lists the number of complexes that we found with a GAP-program.

Several of the complexes are combinatorially isomorphic, but none of the complexes with 10 or 12 vertices is $\mathbb{Z}$-acyclic, which we checked with the C-program HOMOLOGY by Heckenbach [7]. (For most of the complexes with 14 and 15 vertices, it was not possible to determine their homology with the program.)

Theorem 8 *There is, other than a simplex, no $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex on $m = 6, 10, 12$ vertices. In particular, there is no 3-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex other than the 3-simplex.*

# Vertices m	Weighted orbit poset	# Complexes with $\tilde{\chi} = 0$	# $\mathbb{Z}$-acyclic complexes
10	A_5	112	0
	A_6	8	0
	M_{10}	0	0
12	$[2^4]\,3^2{:}4$	336	0
14	$PSL_2\,(7)$	?	?
	$PSL_2\,(13)$	140	?
15	A_5	?	?
	S_5	?	?
	A_6	?	?
	S_6	?	?
	A_7	42	?

Table 4: Transitive permutation groups that do not have any subgroup of Oliver (p, q)-type.

Corollary 9 *The Evasiveness Conjecture holds for* 2- *and* 3-*dimensional simplicial complexes.*

REMARK: An A_5-invariant 5-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex on 30 vertices with 932 faces and f-vector $f = (1, 30, 195, 340, 255, 96, 15)$ will be presented in [11] (see also [12]). Additional examples of higher dimension and with more faces exist on 30 and 60 vertices. The first such example of dimension 11 on 60 vertices was found by OLIVER [10]. We believe that there is, apart from the 4-simplex, no $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex of dimension 4.

4 An Infinite Class of Counterexamples to the Generalized Aanderaa-Rosenberg Conjecture

We construct an infinite class of counterexamples to the Generalized Aanderaa-Rosenberg Conjecture (see p. 2) in three steps.

Let $m = u(u+1)$, for $u \geq 3$ odd. As group of symmetries we consider $G = \mathbb{Z}_m$, with action on the ground set $E = \{1, 2, \ldots, m\}$ by translation (mod m).

1. Let $\mathcal{A}_m$ be the set system of all subsets of the sets of the $\mathbb{Z}_m$-orbit

$$\begin{array}{ll}
\{1,\, u{+}1,\, 2u{+}1,\, \ldots,\, (u{-}1)u{+}1,\, uu{+}1\}, & \{1,4,7,10\}, \\
\{2,\, u{+}2,\, 2u{+}2,\, \ldots,\, (u{-}1)u{+}2,\, uu{+}2\}, & \{2,5,8,11\}, \\
\{3,\, u{+}3,\, 2u{+}3,\, \ldots,\, (u{-}1)u{+}3,\, uu{+}3\}, & \\
\cdots & \\
\{u,\, u{+}u,\, 2u{+}u,\, \ldots,\, (u{-}1)u{+}u,\, uu{+}u\}, & \{3,6,9,12\}
\end{array}$$

with u sets of $u+1$ elements each. (On the right hand side, we note ILLIES' example [8] for $u = 3$.)

2. Let $\mathcal{B}_m$ be the set system of all subsets of the sets of the $\mathbb{Z}_m$-orbit

$$\begin{array}{ll}
\{1,\, 2u{+}1,\, 4u{+}1,\, \ldots,\, (u{-}1)u{+}1\}, & \{1,7\}, \\
\{2,\, 2u{+}2,\, 4u{+}2,\, \ldots,\, (u{-}1)u{+}2\}, & \{2,8\}, \\
\{3,\, 2u{+}3,\, 4u{+}3,\, \ldots,\, (u{-}1)u{+}3\}, & \{3,9\}, \\
\cdots & \cdots \\
\{2u,\, 2u{+}2u,\, 4u{+}2u,\, \ldots,\, (u{-}1)u{+}2u\}, & \{6,12\}
\end{array}$$

with $2u$ sets of $(u+1)/2$ elements each.

3. Let $\mathcal{C}_m$ be the set system of all subsets of the sets of the $\mathbb{Z}_m$-orbit

$$\begin{array}{ll}
\{1,\, (u{+}1){+}1,\, 2(u{+}1){+}1,\, \ldots,\, (u{-}1)(u{+}1){+}1\}, & \{1,5,9\}, \\
\{2,\, (u{+}1){+}2,\, 2(u{+}1){+}2,\, \ldots,\, (u{-}1)(u{+}1){+}2\}, & \{2,6,10\}, \\
\{3,\, (u{+}1){+}3,\, 2(u{+}1){+}3,\, \ldots,\, (u{-}1)(u{+}1){+}3\}, & \{3,7,11\}, \\
\cdots & \\
\{(u{+}1),\, (u{+}1){+}(u{+}1),\, 2(u{+}1){+}(u{+}1),\, \ldots,\, (u{-}1)(u{+}1){+}(u{+}1)\}, & \{4,8,12\}
\end{array}$$

with $(u+1)$ sets of u elements each.

Proposition 10 *The set system $\mathcal{F}_m = (\mathcal{A}_m \setminus \mathcal{B}_m) \cup \mathcal{C}_m$ is non-evasive and thus provides an infinite class of counterexamples to the Generalized Aanderaa-Rosenberg Conjecture.*

Proof: Let $A \in 2^E$. We want to determine whether A is in $\mathcal{F}_m$ or not by asking questions "Is $e \in A$?". An oracle answers YES or NO. In order to show that $\mathcal{F}_m$ is non-evasive, we give a decision tree of depth $m-1$.

CASE I ($A \cap \{1, 2, \ldots, m-2u\} \neq \emptyset$)

We start with elements in $\{1, 2, \ldots, m-2u\}$ and ask successively "Is $1 \in A$?", "Is $2 \in A$?", "Is $3 \in A$?", … If none of these elements is in A, we have checked that $\{1, 2, \ldots, m-2u\} \cap A = \emptyset$, and this case will be discussed later. Otherwise, the first time we get YES as an answer, say, for "Is $r \in A$?" with $r \leq m-2u$, we next test the elements in $\{r+u, r+3u, \ldots\} \subseteq \{1, 2, \ldots, m\}$, consecutively. Upon completion, we see by the construction of the set system $\mathcal{F}_m$ that, since $r \leq m-2u$, there exists at least one element, namely, $r+2u$, such that either both the sets A and $A \cup \{ r+2u \}$ lie in $\mathcal{F}_m$, or both do not. Hence, we do not have to ask for $r+2u$ in order to check whether A is in $\mathcal{F}_m$ or not. This gives a leaf of the decision tree, which we depict in Figure 6 by an oval, containing all the elements we do not have to ask for. Such a leaf has a depth of altogether at most $m-1$. Finally, if we have tested all the elements $\{ r+u, r+3u, \ldots, r+(2k+1)u \}$, with k the greatest integer such that $r+(2k+1)u \leq m$, and none of them is contained in A, then for A to lie in $\mathcal{F}_m$ it has to be in $\mathcal{C}_m$. But then there is again at least one element that we do not have to ask for, namely $r+(u+1)$. Thus, this leaf has also at most depth $m-1$.

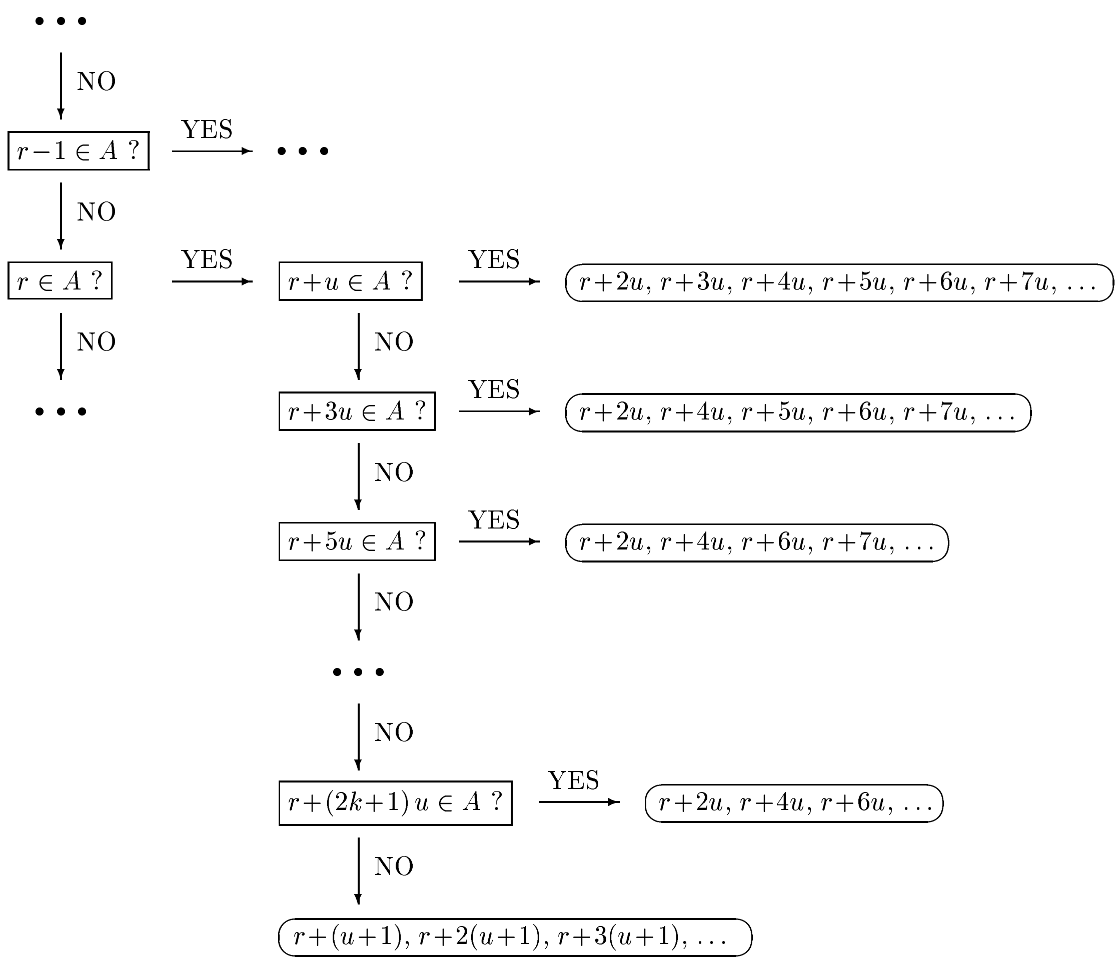


Figure 6: Decision tree for $r \leq m-2u$.

Case II $(A \cap \{ 1, 2, \ldots, m-2u \} = \emptyset)$

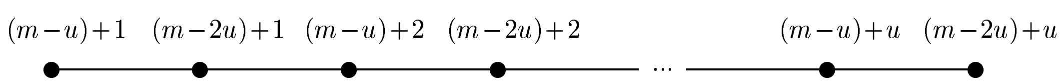
In this part, we further analyze A in the case where $A \cap \{ 1, 2, \ldots, m-2u \} = \emptyset$. For this, we can restrict $\mathcal{F}_m$ to sets that do not contain the elements $1, 2, \ldots, m-2u$. The corresponding sets in $\mathcal{A}_m \setminus \mathcal{B}_m$ are

$$
\begin{array}{ll}
\{\,(m-2u)+1,(m-u)+1\,\}, & \{\,7,10\,\}, \\
\{\,(m-2u)+2,(m-u)+2\,\}, & \{\,8,11\,\}, \\
\quad\cdots & \\
\{\,(m-2u)+u,(m-u)+u\,\}, & \{\,9,12\,\},
\end{array}
$$

and the remaining sets of $\mathcal{C}_m$ are

$$
\begin{array}{ll}
\{\,(m-2u)+1,(m-u)+1+1\,\}, & \{\,7,11\,\}, \\
\{\,(m-2u)+2,(m-u)+1+2\,\}, & \\
\quad\cdots & \\
\{\,(m-2u)+(u-1),(m-u)+1+(u-1)\,\} & \{\,8,12\,\} \\
 & \\
\{\,(m-2u)+1\,\}, & \{\,7\,\}, \\
\{\,(m-2u)+2\,\}, & \{\,8\,\}, \\
\quad\cdots & \quad\cdots \\
\{\,(m-2u)+2u\,\}, & \{\,12\,\}, \\
 & \\
\{\ \}, & \{\ \}.
\end{array}
$$

If we denote by $\overline{\mathcal{A}_m}$, $\overline{\mathcal{B}_m}$, and $\overline{\mathcal{C}_m}$ the restrictions of $\mathcal{A}_m$, $\mathcal{B}_m$, and $\mathcal{C}_m$ to the set of remaining elements $\{\,m-2u+1,\ldots,m\,\}$ respectively, then the restriction $\overline{\mathcal{F}_m} = (\overline{\mathcal{A}_m} \setminus \overline{\mathcal{B}_m}) \cup \overline{\mathcal{C}_m}$ of $\mathcal{F}_m$ is a path



Since paths are non-evasive and can be tested in # vertices – 1 steps, the depth of the leaf corresponding to the above path is again $m-1$. Altogether, it follows that $\mathcal{F}_m$ is non-evasive. $\square$

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References

[1] M. Aigner and E. Triesch. Searching for an edge in a graph. *J. Graph Theory* **12**, 45–57 (1988).

[2] E. Babson, A. Björner, S. Linusson, J. Shareshian, and V. Welker. Complexes of not i-connected graphs. *Topology* **38**, 271–299 (1999).

[3] A. Björner. Topological methods. *Handbook of Combinatorics* (R. Graham, M. Grötschel, and L. Lovász, eds.), 1819–1872. Elsevier, Amsterdam, 1995.

[4] B. Bollobás. *Extremal Graph Theory.* London Mathematical Society Monographs **11**. Academic Press, London, 1978.

[5] G. Butler. The transitive groups of degree fourteen and fifteen. *J. Symb. Comput.* **16**, 413–422 (1993).

[6] G. Butler and J. McKay. The transitive groups of degree up to eleven. *Commun. Algebra* **11**, 863–911 (1983).

[7] F. Heckenbach. *Die Möbiusfunktion und Homologien auf partiell geordneten Mengen.* Thesis for Diploma at University Erlangen-Nuremberg, 1997. Computer program HOMOLOGY. (`http://www.mi.uni-erlangen.de/~heckenb/`)

[8] N. Illies. A counterexample to the generalized Aanderaa-Rosenberg conjecture. *Inf. Process. Lett.* **7**, 154–155 (1978).

[9] J. Jonsson. A $\mathbb{Q}$-acyclic graph complex. In preparation.

[10] J. Kahn, M. Saks, and D. Sturtevant. A topological approach to evasiveness. *Combinatorica* **4**, 297–306 (1984).

[11] F. H. Lutz. Examples of $\mathbb{Z}$-acyclic and contractible vertex-homogeneous simplicial complexes. Preprint 1999.

[12] F. H. Lutz. *Triangulated Manifolds with Few Vertices and Vertex-Transitive Group Actions.* Dissertation. Shaker Verlag, Aachen, 1999.

[13] G. A. Miller. List of transitive substitution groups of degree twelve. *Quart. J. Pure Appl. Math.* **28**, 193–231 (1896). Errata: *Quart. J. Pure Appl. Math.* **29**, 249 (1897).

[14] G. A. Miller. On the transitive substitution groups of degrees thirteen and fourteen. *Quart. J. Pure Appl. Math.* **29**, 224–249 (1897).

[15] R. Oliver. Fixed-point sets of group actions on finite acyclic complexes. *Comment. Math. Helv.* **50**, 155–177 (1975).

[16] R. L. Rivest and J. Vuillemin. On recognizing graph properties from adjacency matrices. *Theor. Comput. Sci.* **3**, 371–384 (1976).

[17] G. F. Royle. The transitive groups of degree twelve. *J. Symb. Comput.* **4**, 255–268 (1987).

[18] M. Schönert et al. *GAP – Groups, Algorithms, and Programming.* Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, fifth edition, 1995. (`http://www.math.rwth-aachen.de/~GAP/WWW/`)

[19] P. A. Smith. Fixed points of periodic transformations. *Algebraic Topology* (S. Lefschetz, ed.). American Mathematical Society Colloquium Publications **27**, 350–373. American Mathematical Society, New York, 1942.

[20] A. C.-C. Yao. Monotone bipartite graph properties are evasive. *SIAM J. Comput.* **17**, 517–520 (1988).

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