Triangulated Manifolds with Few Vertices and Vertex-Transitive Group Actions

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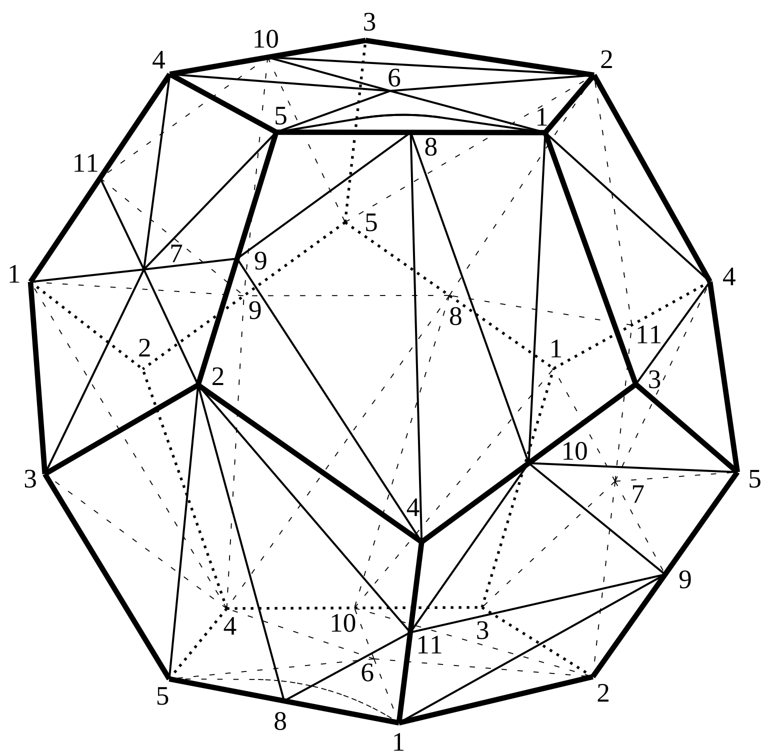
Dissertation



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Contents

Introduction

"Manifold Gardens"

In 1993, then an M.Sc. student at King's College London, I attended a conference on Quantum Probability at Nottingham. Every day on the way to the conference building, I had to walk through a little road with the beautiful name "Manifold Gardens", and I wondered whether people knew of what mathematical flavour their street name is.

Thereafter, manifolds more and more attracted my attention. For my Diploma Thesis, that I wrote under the supervision of Martin Mathieu at the University of Tübingen, I studied deformation quantizations of Poisson manifolds. In particular, I wrote the thesis on work by Marc A. Rieffel, whom I later visited in autumn 1995 at Berkeley. One particular talk that I attended during my stay, I would like to mention here. It was on the solution of the "Double Bubble Conjecture" by Hass, Hutchings, and Schlafly [69], that the least area surface enclosing two equal volumes is a double (soap) bubble. Their proof of the conjecture was based, in its main part, on computer calculations, and arose my interest in Discrete Mathematics.

In spring 1996, I became a Ph.D. student and member of the research group "Discrete Geometry" of Günter M. Ziegler at the Technical University of Berlin. Manifolds were forgotten for a while, and I mainly concentrated my work on topological aspects of the so called "Evasiveness Conjecture"; see Chapter 6. The origin of this conjecture lies in graph theory, and its essence is that *monotone* graph properties are of highest complexity. The conjecture was reformulated in the language of (finite abstract) simplicial complexes by Kahn, Saks, and Sturtevant [80], and proved by them for graph properties on prime power numbers of nodes by applying a fixed point theorem of Oliver [129]. In fact, the conjecture is of strong topological nature. It says that vertex-transitive group actions of the symmetric group S_n do not exist on (non-trivial) *non-evasive* (to be defined in Chapter 6) simplicial complexes on $\binom{n}{2}$ vertices. "Non-evasive" as a topological property implies for a simplicial complex that it is collapsible and thus contractible, and in particular, $\mathbb{Z}$-acyclic. We will show in Chapter 6 that there are, apart from a simplex, no $\mathbb{Z}$-acyclic simplicial complexes with a vertex-transitive group action in dimensions 2 and 3. In dimension 11, an example of a $\mathbb{Z}$-acyclic simplicial complex on 60 vertices with a vertex-transitive action of the alternating group A_5 was constructed by Oliver [80]. We will construct further such examples in Chapter 7, the smallest of dimension 5 on 30 vertices. For all of these complexes we will see that they are homotopy equivalent to the 2-skeleton of a cell decomposition of the Poincaré homology 3-sphere, a well known manifold!

In November 1997, I visited Anders Björner at KTH Stockholm to work on a project on the g-conjecture for simplicial spheres, which proposes a characterization of the number of faces of simplicial spheres. There have been numerous attempts to prove the conjecture, but no one ever tried seriously to find possible counterexamples. Following an idea by Björner and Ziegler, Anders Björner and I attempted to use *bistellar flips* to locally modify the triangulation of a sphere in order to search for counterexamples (cf. Chapter 1). In particular, we wanted to examine non-PL spheres. One way to construct non-PL spheres is to suspend at least twice some homology sphere. The best accessible homology sphere is the Poincaré homology 3-sphere in its description as the spherical dodecahedron space of Threlfall and Seifert [157]. A triangulation of the Poincaré homology 3-sphere was obtained by hand, and a computer program, BISTELLAR [113], was written that performs bistellar flips. We then let the program run on (suspensions of) the Poincaré sphere. Although no counterexamples to the g-conjecture for simplicial spheres appeared, we found a 16-vertex triangulation of the Poincaré homology 3-sphere. Previously, a triangulation of the Poincaré sphere was constructed by Brehm [91, p. 55] with 17 vertices, and nothing better had been known. Thus our program proved (in this and in other cases; see below) to be successful to search for small (and minimal) triangulations of manifolds.

One application for the program BISTELLAR is that it can be used as a heuristic to test whether a given (pure) simplicial complex is bistellarly equivalent to the boundary of a simplex, the minimal triangulation of a sphere. Simplicial spheres with this property are called *combinatorial spheres*. A triangulated manifold is a *combinatorial manifold* if it has a combinatorial sphere as the link for every of its vertices (cf. Chapter 1). Therefore, the program gives us a useful tool at hand to recognize combinatorial manifolds, which we applied at many places throughout this thesis.

It was only a small step to combine the methods from the search for $\mathbb{Z}$-acyclic vertex-transitive simplicial complexes with the task to construct small and minimal triangulations of manifolds by using the program BISTELLAR. In joint work with Ekkehard Köhler, a program, MANIFOLD_VT [115], was written, which effectively generates candidates for combinatorial manifolds that have a vertex-transitive group action on few vertices; see Chapter 2. Using BISTELLAR, we then verified for a candidate that it is indeed a combinatorial manifold.

Besides the main programs, BISTELLAR and MANIFOLD_VT, that are discussed in this thesis, a large "tool kit" for the "computer aided design of manifolds" was developed that includes various lemmas, "tricks", and additional computer programs. Let us briefly mention some applications. In Chapter 2, we will construct combinatorial manifolds with few vertices and combinatorial manifolds with few vertices that have a vertex-transitive automorphism group. In Chapter 3, we discuss combinatorial pseudomanifolds with a vertex-transitive automorphism group on few vertices. Chapter 4 is devoted to nearly neighborly centrally symmetric spheres and 'products of spheres' with a vertex-transitive cyclic group action on few vertices. Vertex-transitive neighborly maps and regular simplicial maps with few vertices will be in the focus of Chapter 5.

The individual chapters are in preparation for publication as distinct papers, Chapter 1 (joint work with A. Björner) [23], Chapter 2 (joint work with E. Köhler) [87], Chapter 3 [108], Chapter 4 [109], Chapter 5 [110], Chapter 6 [111], and Chapter 7 [112].

Some of the main findings of this thesis are listed in the following:

- a 16-vertex triangulation of the Poincaré homology 3-sphere (p. 17);
- non-PL d-spheres with $d+13$ vertices for $d \geq 5$ (p. 19);
- non-PL triangulations of triangulable d-manifolds with few additional vertices for $d \geq 5$ (p. 20);
- complete enumeration of all combinatorial manifolds on $n \leq 13$ vertices that have a vertex-transitive automorphism group (p. 29);
- complete enumeration of all combinatorial pseudomanifolds on $n \leq 13$ vertices that have a vertex-transitive automorphism group (p. 72);
- complete enumeration of all neighborly maps on $n \leq 22$ vertices that have a vertex-transitive automorphism group (p. 92);
- complete enumeration of all regular simplicial maps on $n \leq 22$ vertices (p. 96);
- 13 new triangulations of topologically distinct 3-manifolds on $n \leq 15$ vertices (p. 34);
- minimal triangulations of $S^2 \times S^2$ with 11 vertices (p. 87);
- minimal triangulations of $S^3 \times S^2$ with 12 vertices (p. 86);
- minimal triangulations of $S^3 \times S^3$ with 13 vertices (p. 87);
- minimal triangulations of $(S^2 \times S^2)\#(S^2 \times S^2)$ with 12 vertices (p. 39);
- a minimal triangulation of $\mathbb{R}\mathbf{P}^4$ with 16 vertices (p. 77);
- a triangulation of $\mathbb{C}\mathbf{P}^2\# - \mathbb{C}\mathbf{P}^2$ with 12 vertices that has an involution, which interchanges the two parts of the connected sum (p. 75);
- a 3-neighborly vertex-transitive triangulation with 13 vertices of a simply connected 5-manifold that has homology $H_* = (\mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z})$ (p. 41);
- a family of nearly neighborly centrally symmetric d-spheres S^d_{2d+4}, d odd, with $2d+4$ vertices (p. 80);
- examples of nearly neighborly centrally symmetric d-spheres with vertex-transitive cyclic group action for $d = 3, 5, 7$ (p. 81);
- a proof of the Evasiveness Conjecture for 2- and 3-dimensional complexes (p. 107);
- an infinite class of counterexamples to the Generalized Aanderaa-Rosenberg Conjecture (p. 108);
- a 5-dimensional example of a non-contractible $\mathbb{Z}$-acyclic simplicial complex with a vertex-transitive A_5-action (p. 121).

I wish the reader to have a good time in “Manifold Gardens”!

Chapter 1

Simplicial Manifolds, Bistellar Flips and a 16-Vertex Triangulation of the Poincaré Homology 3-Sphere

We present a computer program based on bistellar operations that provides a useful tool for the construction of simplicial manifolds with few vertices. As an example, we obtain a 16-vertex triangulation of the Poincaré homology 3-sphere; we construct an infinite series of non-PL d-dimensional spheres with $d+13$ vertices for $d \geq 5$; and we show that if a d-manifold admits any triangulation on n vertices, then it admits a non-combinatorial triangulation on $n+12$ vertices ($d \geq 5$).

1.1 Experimental Topology

In the early days of topology, manifolds were often studied via triangulations. The combinatorial structure makes the computation of various invariants possible, and theorems can be proved based on the assumption of a suitable triangulation. See e.g. [103], [127], and [155] for accounts of some main lines in the historical development. Since the manifolds themselves, and not their combinatorial structure, are the real objects of interest in topology, there was a growing desire to get away from triangulations. In the 1930's and 40's algebraic tools gradually replaced the combinatorial ones, and to the extent that from this time on there still was an interest in decomposing a manifold, the more economical CW complexes gained popularity.

While triangulations always remained of interest to discrete geometers and geometric and PL topologists, the emergence of computers has subtly changed the general situation. It is now possible (at least in principle) to study compact manifolds and compute their invariants on a machine. But a fundamental question naturally arises: *How do you present the manifold to a computer?* It is clear that some finite combinatorial encoding must be used. A decomposition as a CW complex may be elegant and also economical in terms of the number of cells, but it is in general difficult to explain the attaching maps to a computer. One needs something like a regular CW complex, where the attaching maps are determined by the combinatorics of inclusion of closed cells. However, the conceptually easiest presentation is as a simplicial complex, say, given as the list of its facets (maximal faces). Such an encoding is clear and simple, as long as it is not too large. Thus, the matter of the *size* of a triangulation has taken on practical significance. It is of interest to say something about the number of vertices, or the total number of faces, of a triangulation, and also to explicitly construct minimal or otherwise optimal triangulations.

For earlier work on the topic of minimal triangulations we refer to [10], [11], [18], [42], [44], [90], [91], [95], and [158]; and for algorithmic approaches to recognition problems for manifolds to the papers [119], [128], and [156].

The work reported in this chapter grew out of a desire to have a *computer tool for experimentation with triangulations*. We had three purposes in mind:

1. *To be able to start with some triangulation of a manifold and let the computer search for smaller triangulations.*
2. *To be able to determine, via a heuristic, the homeomorphism type of a manifold and, in particular, to recognize (combinatorial) spheres.*
3. *To be able to search for counterexamples to conjectures, where such examples might be hard to find due to their size or complexity.*

Since to determine the homeomorphism type of a manifold is a delicate and much studied matter, the second point needs immediate clarification. What we have in mind is a procedure for heuristically comparing a given test manifold with reference manifolds having similar invariants from a library of standard manifolds on few vertices, with no guarantee for success. In future work the combinatorial ideas of this paper can hopefully be expanded and combined with algorithms for computing topological invariants (not only homology, but also fundamental group, characteristic classes, intersection forms, multiplicative structure of cohomology, ...) to create a truly versatile tool for manipulation and identification of manifolds.

A computer program, BISTELLAR [113], was written which repeatedly modifies a triangulation by local so called "bistellar operations". Such operations for dimensions 2 and 3 are illustrated in Figures 1.1 and 1.2; we defer the formal definition to Section 1.2. The program accepts as input a simplicial manifold M (or any pure simplicial complex) presented via the list of its facets. It then searches through other triangulations of M via bistellar moves, using randomness controlled by a "simulated annealing" type strategy, to be explained in Section 1.3.

The program has turned out to be quite useful for the first two purposes. For reasons that will be explained later (searching for counterexamples to the "g-conjecture for spheres"), we needed non-PL triangulations of the d-sphere ($d \geq 5$) of manageable size. As a stepping stone in the construction we gave BISTELLAR the task to compute a small triangulation of what Rolfsen [139, p. 308] calls "the ubiquitous Poincaré homology sphere". As reported in Section 1.5, the program produced a triangulation on 16 vertices which seems to be the smallest known triangulation of this manifold. It follows from work of Walkup [158] that any triangulation must have at least 11 vertices. Thus, it is at the moment impossible to say where between 11 and 16 the truth about the optimal number of vertices lies. However, after having run our program over millions of triangulations, we are prepared to believe that 16 vertices might in fact be best possible for this manifold.

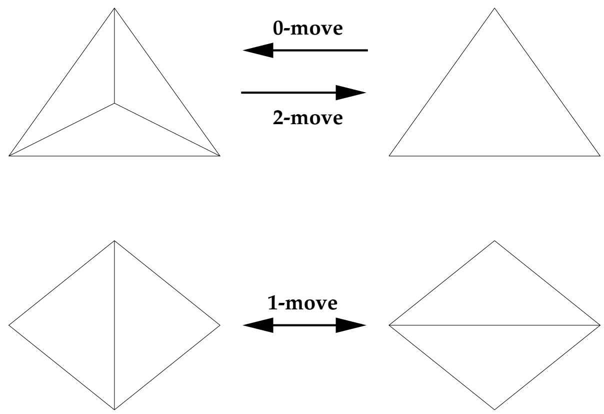


Figure 1.1: Bistellar moves for $d = 2$.

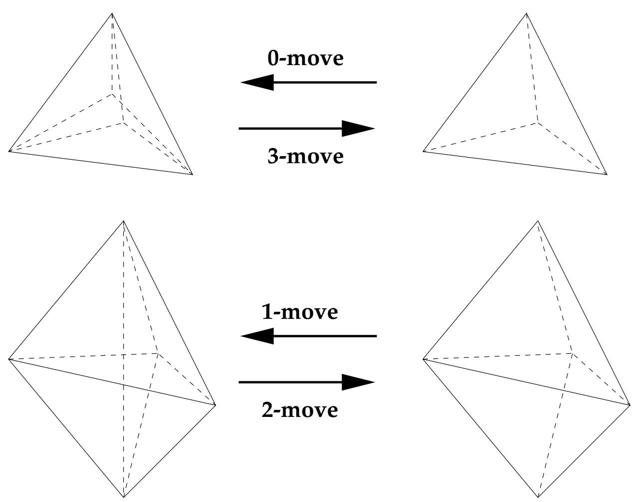


Figure 1.2: Bistellar moves for $d = 3$.

The 16-vertex triangulation of the Poincaré space is the starting point for a proof that there exist non-PL triangulations of the d-sphere on $d + 13$ vertices for all $d \geq 5$. This is in turn used to show that if an arbitrary d-manifold admits some triangulation on n vertices, then it admits a non-PL triangulation on $n + 12$ vertices ($d \geq 5$). Also, the $(d + 13)$-vertex non-PL spheres complement earlier theorems of Barnette and Gannon [18] and Brehm and Kühnel [42]; see Section 1.6.

The search for minimal triangulations using our program has been continued and has led to several new results. They will be presented elsewhere (see Chapter 2, Chapter 3, and Chapter 4), but let us summarize the main findings.

Combinatorial triangulations were found for

- $S^2 \times S^2$ on 11 vertices,
- $S^3 \times S^2$ on 12 vertices,

- $S^3 \times S^3$ on 13 vertices,
- $(S^2 \times S^2)\#(S^2 \times S^2)$ on 12 vertices,
- $\mathbb{RP}^4$ on 16 vertices.

In all these cases, the theoretically minimal numbers of vertices for combinatorial triangulations of these manifolds are achieved.

The triangulations of $S^3 \times S^2$ on 12 and of $S^3 \times S^3$ on 13 vertices are of particular interest, since they attain the minimal numbers of vertices that any (non-spherical) combinatorial 5- or 6-manifold can have. They therefore establish that the Brehm-Kühnel [42] lower bound for the number of vertices of combinatorial d-manifolds is sharp in dimensions 5 and 6. For a statement of this bound see Theorem 1.8 and the sentence following it.

An extended version of the program, BISTELLAR_EQUIVALENT [114], was used to determine the homeomorphism type of a large number of manifolds, e.g. of all triangulated 3-manifolds that have a vertex-transitive automorphism group on $n \leq 15$ vertices (cf. Chapter 2). The idea behind this is to first construct reference triangulations of interesting manifolds with few vertices. If then a test object has the same homology as a particular reference manifold (this can be checked with the computer program HOMOLOGY by Heckenbach [71]), it was possible in many cases to find a *bistellar equivalence* between the two manifolds, and thus to show that they are PL homeomorphic. For this we first searched for a small triangulation of the test object, and then applied further bistellar flips until, eventually, we were able to show that the modified test object is combinatorially isomorphic to the reference manifold.

Naturally, this works particularly well for manifolds with a unique minimal triangulation, such as PL d-spheres that can be minimally triangulated as the boundary complex of the $(d+1)$-dimensional simplex. Therefore the program can be used, at least as a heuristic, to determine whether a given simplicial complex is a combinatorial manifold (i.e., whether all vertex links are PL spheres). Other manifolds that have a unique minimal triangulation are e.g. the twisted sphere product (or 3-dimensional Klein bottle) $S^2 \underline{\times} S^1$ (cf. [10], [11], [158]) and the complex projective plane $\mathbb{CP}^2$ [95], in both cases on 9 vertices.

The program has not yet achieved any success for the third purpose, that of finding counterexamples. At the end of Section 1.2 we report on some experiments of this kind.

The chapter is structured as follows. In the next section we review some definitions and some general facts about triangulations of manifolds, bistellar flips, and the counting of faces. Section 1.3 presents the program. In Section 1.4 we discuss the Poincaré homology 3-sphere and construct some highly symmetric triangulations for input into BISTELLAR. Section 1.5 presents the 16-vertex triangulation that was found. In Section 1.6 we derive via multiple suspensions the non-PL d-spheres on few vertices, and discuss how their existence relates to the existing theoretical bounds for such objects. In the brief Section 1.7, finally, we construct a highly symmetric triangulation of $\mathbb{RP}^3$ using the same general technique as in Section 1.4.

1.2 Review of Definitions and Background

We collect here some definitions and discuss a bit more the background to this and the consecutive chapters, including some general facts concerning triangulations of manifolds. For the general notions of topology we refer to Stillwell [155] and for PL topology to Glaser [65], Hudson [74], and Rourke and Sanderson [140].

All manifolds in this and the following chapters are compact, connected, and closed. Since PL concepts play such a role here, we recall the following definitions. A *PL sphere* is a simplicial complex which is piecewise linearly homeomorphic to the boundary of a simplex. A *combinatorial manifold* (or *PL manifold*) is a triangulation of a topological manifold such that the link at every vertex is a PL sphere.

For $d \neq 4$, a triangulation of the d-sphere is PL in the first sense if and only if it is a PL manifold in the second sense. For $d \leq 3$ this follows from the work of Moise [126] and for $d \geq 5$ from the work of Kirby and Siebenmann [83]; namely, there is a unique PL structure for spheres in these dimensions. For $d = 4$ this question is not fully understood: Is a combinatorial manifold homeomorphic to the 4-sphere necessarily a PL sphere? Since in dimension 4 the category of PL manifolds is equivalent to the smooth category, the question is equivalent to: Does there exist an "exotic" 4-sphere? (We are grateful to M. Kreck for clarifying this distinction.)

It was shown by Rado (in 1924) that all 2-manifolds and by Moise (in 1952) that all 3-manifolds can be triangulated (cf. [126], [127], [132], [155]). Since the link of a vertex in a triangulated 2-manifold is a polygon and the link of a vertex in a triangulated 3-manifold is a 2-sphere (and all 2-spheres are PL), 2- and 3-dimensional manifolds are always PL.

The situation is much more subtle in dimension 4. Freedman constructed in 1982 a non-differentiable analogue of the complex projective plane (see [62], [63, Sect. 8.3 and 10.1]), and this *fake* $\mathbb{C}\mathbf{P}^2$ provides an example of a 4-manifold that cannot be triangulated as a combinatorial manifold. By combining work of Casson with that of Freedman (see [2, p. xvi]) one obtains examples of topological 4-manifolds *that cannot be triangulated at all.* For expositions of these triangulation questions and related matters see e.g. [83, Annex 2 and 3], [103], [104], [118], [127], and [155].

In 1963 Milnor (cf. Lashof [104]) listed seven problems that he thought of as the toughest and most important problems in geometric topology. Among them is the question whether every topological manifold can be triangulated, now known to have a negative answer. Also on the list is the double suspension problem that asks whether the double suspension of a homology 3-sphere is a topological sphere. This problem was settled by Edwards [60] in 1974 for the double suspension of the Mazur homology 3-sphere which he proved is a topological 5-sphere (see [56, Ch. 12]). The theorem has later been generalized:

Theorem 1.1 (Cannon [48]) *The double suspension $\mathcal{S}^2 H^d$ of any d-dimensional homology sphere H^d is homeomorphic to S^{d+2}.*

It follows that $\mathcal{S}^2 H^d$, although homeomorphic to S^{d+2}, has a non-PL structure, since H^d appears as the link of some 1-simplex in $\mathcal{S}^2 H^d$. This fact will be of importance in Section 1.6.

We now specialize the discussion to the concepts and tools that will be needed in this chapter.

Definition 1.2 [130] *Let M be a simplicial d-manifold (or any pure d-dimensional simplicial complex). If A is a $(d-i)$-face of M, $0 \leq i \leq d$, such that $\mathrm{link}_M(A)$ is the boundary $Bd(B)$ of an i-simplex B that is not a face of M, then the operation Φ_A on M defined by*

$$\Phi_A(M) := (M \backslash (A * Bd(B))) \cup (Bd(A) * B)$$

is called a ***bistellar i-move****.*

Alternatively, we say *bistellar operations* or *bistellar flips* for bistellar moves. Bistellar i-moves with $i > \lfloor \frac{d}{2} \rfloor$ are also called *reverse $(d-i)$-moves*. Note that a 0-move adds a new vertex to a triangulation, while a reverse 0-move deletes a vertex; see Figures 1.1 and 1.2. Two pure simplicial complexes are *bistellarly equivalent* if there exists a finite sequence of bistellar operations leading from one triangulation to the other (and vice versa).

It is easy to see that bistellar equivalence implies being PL homeomorphic, for any simplicial manifolds. For combinatorial triangulations the converse is also true.

Theorem 1.3 (PACHNER [130, Thm. 1]) *Two combinatorial manifolds are bistellarly equivalent if and only if they are PL homeomorphic.*

Define the *bistellar flip graph* of a triangulable manifold M to have as nodes the triangulations of M (or, more precisely, their isomorphism classes up to relabeling the vertices), and an edge between two nodes if one triangulation can be obtained via a single bistellar flip from the other (and vice versa). If the dimension of M is at most 3, then this graph is connected, as shown by the work of Moise [126] together with Theorem 1.3. We will see in Section 1.6 that if $d \geq 5$ then this graph has infinitely many connected components. Of course, the manifolds within each connected component of the bistellar flip graph are pairwise PL homeomorphic. If M can be triangulated as a combinatorial manifold, then by PACHNER's theorem the (infinite) space of all combinatorial triangulations of M is divided into equivalence classes of pairwise PL homeomorphic triangulations which coincide with connected components of the bistellar flip graph. For a discussion of PACHNER's theorem in a topological environment see [107].

We now consider counting faces of *all* dimensions, not just vertices (dimension zero). For more details and references to this area see the survey [20], and for triangulations of spheres and polytopes [154].

Let f_i be the number of i-dimensional faces of a triangulated d-manifold M (with $f_{-1} = 1$), and define numbers h_i by

$$\sum_{i=0}^{d+1} h_i \, x^{d+1-i} = \sum_{i=0}^{d+1} f_{i-1} (x-1)^{d+1-i}. \tag{1.1}$$

The sequence $(f_0, \ldots, f_d)$ is called the *f-vector* of M, and $(h_0, \ldots, h_{d+1})$ its *h-vector*. The corresponding *g-vector* $(g_0, \ldots, g_{\lfloor (d+1)/2 \rfloor})$ is defined by $g_0 = 1$ and $g_i = h_i - h_{i-1}$, for $i \geq 1$.

It was shown by KLEE [85] for any triangulated manifold M that the face numbers $(f_0, \ldots, f_{\lfloor (d-1)/2 \rfloor})$ determine the remaining numbers $(f_{\lfloor (d+1)/2 \rfloor}, \ldots, f_d)$ via linear relations. From (1.1) we see that this means that $(h_0, \ldots, h_{\lfloor (d+1)/2 \rfloor})$, and thus also

$(g_0, \ldots, g_{\lfloor (d+1)/2 \rfloor})$, determine the complete f-vector. In other words, the g-vector of a triangulated manifold contains complete information about its f-vector.

The relevance of this for our program is the following.

Theorem 1.4 (Pachner [130, p. 83]) *If M' is obtained from M by a bistellar k-move, $0 \leq k \leq \lfloor (d-1)/2 \rfloor$, then*

$$\begin{aligned} g_{k+1}(M') &= g_{k+1}(M) + 1 \\ g_i(M') &= g_i(M) \quad \textit{for all} \quad i \neq k+1. \end{aligned}$$

Furthermore, if d is even and $k = \frac{d}{2}$, then $g_i(M') = g_i(M)$ for all i.

This means that it is very easy to follow and control the successive f-vectors during a sequence of bistellar flips. In our program we compute and store the initial g-vector, which is then updated with a $+1$ (or -1) in position $k+1$ for each k-move (or reverse k-move). (Remark: In the case of odd-dimensional manifolds the result implies that the bistellar flip graph is bipartite – it can be colored by the sum (mod 2) of the entries of the g-vector. In even dimensions, $\frac{d}{2}$-moves do not change the g-vector and sometimes even lead to a combinatorially isomorphic triangulation of a manifold, that is, the bistellar flip graph may have loops.)

The linear relations of Klee take on a particularly attractive form if M triangulates a sphere (the Dehn-Sommerville relations):

$$h_i = h_{d+1-i}. \tag{1.2}$$

If furthermore M is *polytopal* (i.e., combinatorially isomorphic to the boundary complex of a simplicial convex polytope), then by a theorem of Stanley [153]

$$(g_0, \ldots, g_{\lfloor (d+1)/2 \rfloor}) \text{ is an M-sequence.} \tag{1.3}$$

This combinatorial condition is defined as follows, showing that it can easily be tested by machine. For integers $k, n \geq 1$ there is a unique way of writing

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}$$

so that $a_k > a_{k-1} > \cdots > a_i \geq i \geq 1$. Then define

$$\partial^k(n) = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-1}.$$

Also let $\partial^k(0) = 0$. A sequence $(n_0, n_1, \ldots)$ of nonnegative integers is called an *M-sequence* (M for Macaulay) if

$$n_0 = 1 \text{ and } \partial^k(n_k) \leq n_{k-1}, \quad \text{for all} \quad k \geq 2.$$

Note that a nontrivial consequence of (1.3) is that $g_i \geq 0$ for polytopal spheres. The *"g-theorem"* states that the conditions (1.2) and (1.3) together characterize the f-vectors of polytopal spheres. The sufficiency of these conditions was proved by Billera and Lee [21].

The conjecture to which we wanted BISTELLAR to search for counterexamples is the so called *"g-conjecture for spheres"* which states that condition (1.3) is valid for *all* triangulated spheres, not just polytopal ones. If correct, this would imply a characterization of the f-vectors of spheres.

The g-conjecture can be deduced from known results for all d-spheres up to dimension 4, but is open for $d \geq 5$. Attempts during the last 20 years to prove it have so far been without success. It therefore seemed to us that the possibility of its falsity should be considered and tested.

In order to look for counterexamples we started with non-PL triangulations of the 5- and 6-sphere and let the bistellar flip program search through thousands of triangulations. This purpose is what originally made us look for small triangulations of the Poincaré 3-sphere and its suspensions; see Section 1.6 for a description of the spheres we used to start the computer search. The bistellar flip program guarantees by Theorem 1.3 that all triangulations visited during the search are non-PL, and, in particular, that they are not polytopal. At each step the g-vector is updated, as described in Theorem 1.4, and tested for being an M-sequence. The parameters for the program can be set to put priority on creating a g-vector that is not an M-sequence (if possible), e.g. a g-vector with some negative entry.

So, what was the result? No counterexamples to the g-conjecture were found. Although no conclusions can be drawn, let us hope that this is an indication that the conjecture is correct.

1.3 The Bistellar Flip Program

The computer program that will now be presented performs walks on the bistellar flip graph of triangulations of a manifold M. By necessity we must restrict attention to some connected component of this graph. For a particular triangulation of M from this component (the input) we want to perform bistellar modifications with the objective to obtain "small" (hopefully even minimal), or otherwise sought-after, triangulations of M (within the component). As an objective function that we want to optimize we could take for example the total number of faces of a triangulation. Nevertheless, the sum G of the entries of the g-vector seems to be a more appropriate objective function, since any up-move (i.e., i-move with $0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$) increases G by one and any down-move (reverse up-move) decreases G by one, so that we have good control over G. (If d is even, then $\frac{d}{2}$-moves do not change G.) In addition to the goal of minimizing the objective function G we perform moves according to *priority rules*. Reverse 0-moves are given the highest priority as they delete a vertex, then come reverse 1-moves, reverse 2-moves, etc. If no further reverse moves are available, this might be due to the fact that we have achieved a global minimum for G within our component of triangulations. But we can as well have gotten stuck in some local minimum.

A concept that is very useful in such situations is *simulated annealing* [84]. In a continuous version of simulated annealing (see e.g. [136]) one wants to find a global minimum $x_* \in \mathbb{R}^n$ for a real valued objective function $f : \mathbb{R}^n \to \mathbb{R}$, i.e., $x_* \in \mathbb{R}^n$ such that $f(x_*) \leq f(x)$ for all $x \in \mathbb{R}^n$. Starting at some initial point y one moves to a randomly picked neighboring point y' if $\Delta f = f(y') - f(y) \leq 0$. If $\Delta f > 0$, then we move "uphill" to y' with probability $\exp(-\Delta f/\beta)$ or otherwise stay put at y. In the next step a new neighboring point y'' of y' (or of y if we have not moved) is chosen at random and so on. The *cooling parameter* $\beta > 0$ describes how likely it is to move "uphill" and is usually decreased with time (the number of steps).

We now describe an appropriate simulated-annealing-type strategy for bistellar flips. As soon as we are trapped in a 'local' minimum, we perform an up-move. (Up-moves are also performed according to priority rules, such as "perform a $(k+1)$-move before a k-move".) Sometimes, this already paves the way for further reverse moves that lead away from the local minimum. But we might also fall back into the same local minimum in the following *round*. After a certain number of up-moves has become necessary (we call this the *relaxation parameter*) we start "heating up" the function G, i.e., for a number of steps given by the *heating parameter* we perform only up-moves (as long as this is possible), with the exception that we usually do not perform 0-moves, since this would blow up the size of the complex too quickly. Then we let the system relax until we have to heat up again. If there is more than one option for moves of a certain priority, we pick one of these options randomly and then execute the move.

An Implementation of the Bistellar Flip Program

We start with some triangulation of a d-manifold, represented by the list of its facets, and determine all its faces and compute its f- and g-vector. Next, we check for every $(d-i)$-face of the triangulation whether it is contained in precisely $i+1$ facets. The collection of these faces (together with their respective links) form the *raw options* for bistellar i-moves. If we want to consider *proper options* for i-moves, then we include only those raw options for i-moves for which in addition the links satisfy the condition of being the boundary of an i-simplex that is *not* a face of the triangulation. This last condition is easy to check.

When we determine the raw options at the beginning, we have to check for all f_i i-faces how often they are included in one of the f_d facets. This amounts to $f_i \cdot f_d$ operations. Nevertheless, in the following rounds we do not have to recompute the raw options from scratch, since with any bistellar flip we simply cut out a ball locally and replace it by another ball. All raw options for faces in the interior of the ball that we remove have to be deleted and raw options for the faces in the interior of the new ball have to be included. Raw options for faces on the common boundary of the balls might also change. But altogether, there is only a constant number of faces involved in updating the raw options. Finally, to find out which of the raw options of a given priority are proper options, we have to test the condition on links mentioned above.

We wrote the program BISTELLAR in GAP [144], as all operations for sets and lists that we need are available in this computer algebra package. For dimension 3, the listing of the main part of the program is as follows. Complete information about BISTELLAR is best obtained by downloading the program
(`http://www.math.TU-Berlin.de/diskregeom/stellar/`).

```
## initial settings ##

InputFacets;
Compute_RawOptions;
Compute_f_and_g_vector;
g_min:=g;

## parameters ##

rounds:=1;
```

```
relaxation:=0;
heating:=0;

while rounds <= 50000 do

      ## strategy for options ##

      options:=[];

      if heating > 0 then
         Include_MoveOptions(1);
         if options = [] then
            Include_ReverseMoveOptions(1);
            heating:=0;
         fi;
         heating:=heating-1;
      else
         Include_ReverseMoveOptions(0);
         if options = [] then
            Include_ReverseMoveOptions(1);
            if options = [] then
               Include_MoveOptions(1);
               if options = [] then
                  Include_MoveOptions(0);
               fi;
               relaxation:=relaxation+1;
               if relaxation = 10 then
                  heating:=15;
                  relaxation:=0;
               fi;
            fi;
         fi;
      fi;

      ## perform Move or ReverseMove ##

      ChooseOptionAtRandom;
      ExecuteOption;
      Update_RawOptions;
      Update_f_and_g_vector;
      Print(rounds,"  ",g,"\n");
      if g < g_min then
         g_min:=g;
         Print("f-vector = ",f,"\n");
         Print(facets,"\n");
      fi;

```

```
        rounds:=rounds+1;

od;
```

In higher dimensions, the strategy for the options can easily be adapted, although it takes time and experiments to figure out reasonable parameters for heating and relaxation. (This is a common problem with simulated annealing algorithms.)

1.4 The Ubiquitous Poincaré Homology 3-Sphere

The original example by Poincaré of a non-simply-connected manifold with the same homology as the ordinary 3-sphere appeared in [131]. It was constructed by him from two solid double tori identified along their boundary surfaces of genus 2. For this and other constructions of this space see [139, pp. 244–250 and 308–311], [155, pp. 263–266], or [160, p. 245]. This manifold, whose existence prompted the still open 3-dimensional Poincaré conjecture, has had an enormous influence on the subsequent development of topology. It is discussed in many places in the literature; in addition to the already mentioned sources see also e.g. [57], [82], [86], and [157]. We want to particularly mention the paper [82], where eight different constructions of this space are given and proved to be equivalent. Also, several of the given references discuss the fact that the fundamental group of the Poincaré homology 3-sphere is the "binary icosahedral group" of order 120.

Triangulations of the Poincaré homology 3-sphere on 17 and 18 vertices were constructed by Brehm. This is mentioned in the proof of Proposition 3.28 of [91, p. 55], but no details are given. The first task for our bistellar flip program was to try to improve on this.

In order to have a starting triangulation for the program at hand, we first construct a "small" triangulation of the Poincaré homology 3-sphere. For this, we consider the description of the Poincaré sphere as the *spherical dodecahedron space* which is the cell decomposition of the solid dodecahedron where opposite pentagons on the boundary are identified by a coherent twist of $\pi/5$ radians; see Threlfall and Seifert [157] or Weber and Seifert [160].

We triangulate the boundary of the dodecahedron by introducing a midpoint for every pair of identified opposite pentagons (see Figure 1.3). Into the interior of the dodecahedron we place an icosahedron in such a way that every vertex of the icosahedron corresponds to a copy of a midpoint of a pentagon. For every vertex of the icosahedron we form the cone over the respective pentagon. For every edge of the icosahedron we include the tetrahedron that is determined by this edge and the edge that separates the two corresponding neighboring pentagons. Similarly, for any triangle on the boundary of the icosahedron we take the tetrahedron that is made up by the triangle and the intersection-vertex of the three corresponding neighboring pentagons. Finally, we triangulate the interior of the icosahedron by introducing a center point and we take the cone over the boundary of the icosahedron with respect to the center point. The resulting triangulation of the Poincaré homology 3-sphere has $5+6+12+1=24$ vertices and is invariant under the 60-element group A_5 of rotations of the icosahedron and the dodecahedron.

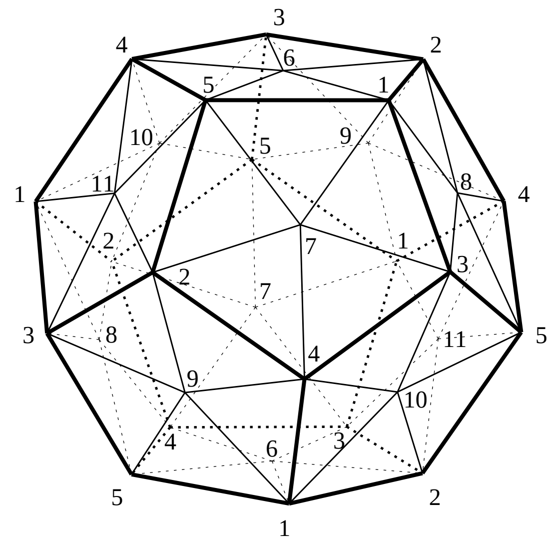


Figure 1.3: A_5-invariant triangulation of the Poincaré 3-sphere.

Instead of an icosahedron, we could also place a bipyramid over a pentagon into the interior of the dodecahedron. In this case, the north and south pole of the bipyramid are joined to the dark shaded subcomplexes of Figure 1.4. Then take one vertex of the equatorial pentagon of the bipyramid and let it correspond to the light shaded subcomplex of Figure 1.4. By rotations of the cyclic group $\mathbb{Z}_5$ we obtain four additional equatorial subcomplexes, and the seven subcomplexes that we have described cover the boundary of the dodecahedron. Now, triangulate the space between the bipyramid and

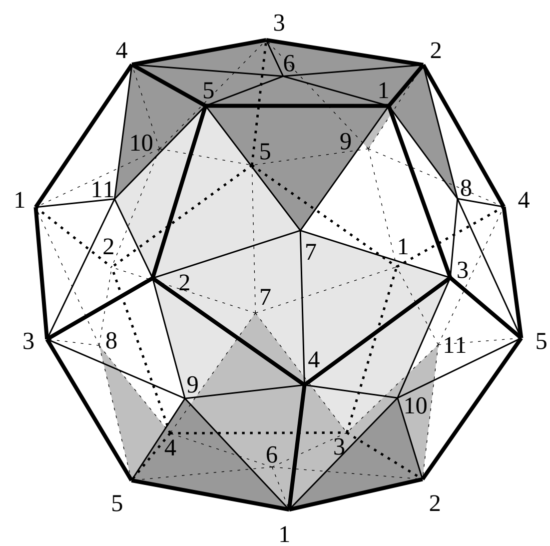


Figure 1.4: $\mathbb{Z}_5$-invariant triangulation of the Poincaré 3-sphere.

the (identified) boundary of the dodecahedron similarly as before. For the interior of the bipyramid we introduce an edge connecting north and south pole and then slice the bipyramid like an orange. This provides us with a $\mathbb{Z}_5$-invariant 18-vertex triangulation of the Poincaré sphere. As was mentioned, such a triangulation was previously found by Brehm. By some modification of the identified boundary it is not too difficult to obtain non-symmetric 17-vertex triangulations, but we were unable to reach 16 vertices by hand.

1.5 A Non-Symmetric Triangulation Σ^3_{16} on 16 Vertices

We applied the bistellar flip program to both the above 18-vertex and the 24-vertex triangulation. After some running time we obtained a 16-vertex triangulation.

Theorem 1.5 *There exists a triangulation (without any symmetries) of the Poincaré homology* 3*-sphere on* 16 *vertices with f-vector* $f = (16, 106, 180, 90)$.

Proof: The list of facets

1249	124 15	126 14	126 15	129 14	134 12
134 15	137 10	137 12	13 10 15	149 12	156 13
156 14	158 11	158 13	15 11 14	16 13 15	178 10
178 11	17 11 12	18 10 13	19 11 12	19 11 14	1 10 13 15
235 10	235 11	237 10	237 13	23 11 13	249 13
24 11 13	24 11 15	258 11	258 12	25 10 12	26 10 12
26 10 14	26 12 15	279 13	279 14	27 10 14	28 11 15
28 12 15	345 14	345 15	34 12 14	35 10 15	35 11 14
37 12 13	3 11 13 14	3 12 13 14	4567	456 14	457 15
467 11	46 10 11	46 10 14	47 11 15	489 12	489 13
48 10 13	48 10 14	48 12 14	4 10 11 13	567 13	579 13
579 15	589 12	589 13	59 10 12	59 10 15	67 11 12
67 12 13	6 10 11 12	6 12 13 15	78 10 14	78 11 15	78 14 15
79 14 15	8 12 14 15	9 10 11 12	9 10 11 16	9 10 15 16	9 11 14 16
9 14 15 16	10 11 13 16	10 13 15 16	11 13 14 16	12 13 14 15	13 14 15 16

determines a 3-dimensional (pure) simplicial complex Σ^3_{16} on 16 vertices with f-vector $f = (16, 106, 180, 90)$. Since this simplicial complex was obtained via bistellar flips starting from a triangulation of the Poincaré sphere, it is PL homeomorphic to this space.

Alternatively, we can assemble the 90 tetrahedra in the interior of the dodecahedron. Once again, we obtain a triangulation of the solid dodecahedron where opposite pentagons on the boundary are identified by a coherent twist of $\pi/5$ radians. In Figure 1.5 we depict the corresponding triangulation of the boundary with the respective identifications. The vertices 1–11 lie on the boundary of the dodecahedron whereas the vertices 12–16 lie in the interior.

If a combinatorial manifold has a (combinatorial) symmetry, then the links of the vertices that are mapped onto each other must be combinatorially equivalent. For Σ^3_{16} the links of the vertices $\{3, 6\}$, $\{10, 13, 14\}$, and $\{2, 4, 5, 7, 12\}$ are pairwise combinatorially equivalent within each group, and there are no other such equivalences. Thus, the automorphism group of Σ^3_{16} is a subgroup of $S_2 \times S_3 \times S_5$. Nevertheless, none of these 1440 permutations, apart from the identity, is in fact a symmetry, and therefore Σ^3_{16} has trivial automorphism group. □

What about a 15-vertex triangulation of the Poincaré homology 3-sphere? It follows from work of Walkup [158, Theorem 4] that at least 11 vertices are needed. (We are grateful to R. Forman for pointing this out to us.) We let our bistellar flip program run

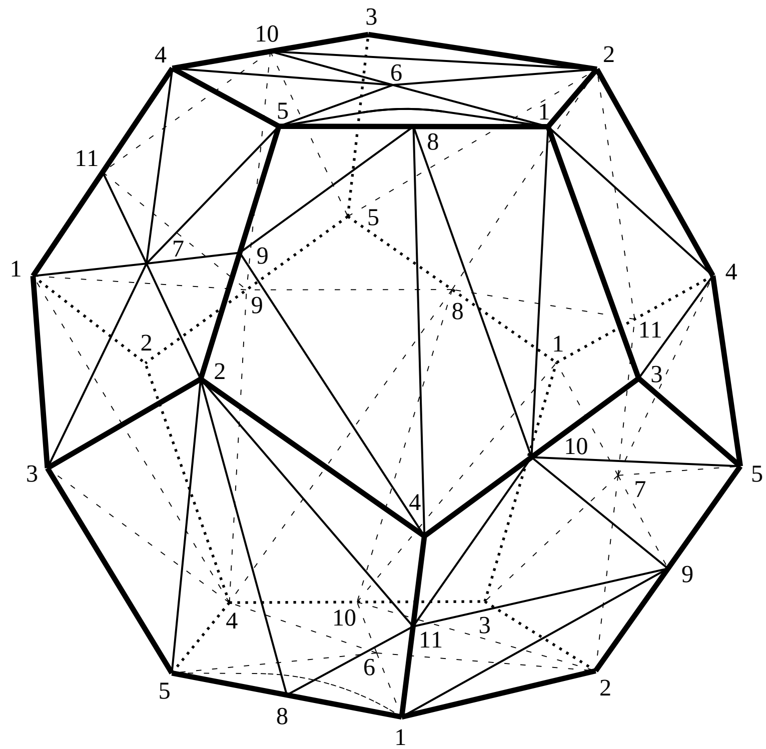


Figure 1.5: 16-vertex triangulation of the Poincaré 3-sphere.

for up to 10^6 moves with changing relaxation and heating parameters. From time to time the triangulation Σ^3_{16} appeared or other triangulations on 16 vertices with larger f-vectors, but never any smaller triangulation or any non-equivalent triangulation with the same f-vector.

Conjecture 1.6 *The triangulation Σ^3_{16} of the Poincaré homology 3-sphere has the component-wise minimal f-vector $f = (16, 106, 180, 90)$ for a triangulation of this manifold and is the unique triangulation with this f-vector.*

The boundary of the identified dodecahedron is a $\mathbb{Z}$-acyclic space with the same fundamental group as the Poincaré homology 3-sphere [36, p. 57]. In particular, this 2-dimensional space is not contractible. What is the minimal number of vertices of a simplicial complex that is $\mathbb{Z}$-acyclic but not contractible?

By taking the restriction of Σ^3_{16} to the boundary of the identified dodecahedron we obtain a triangulation on 11 vertices. The bistellar flip program brought this number down to 10. The corresponding f-vector is $f = (10, 40, 31)$. Subsequently another triangulation on 10 vertices with $f = (10, 40, 31)$, shown in Figure 1.6, was found by hand. Here is the list of its facets:

124	125	136	138	13 10	148	149	157
15 10	167	169	235	237	238	246	24 10
267	268	28 10	356	359	379	37 10	456
457	458	479	47 10	589	58 10	689.	

We do not know if 10 vertices is best possible for a complex with these properties.

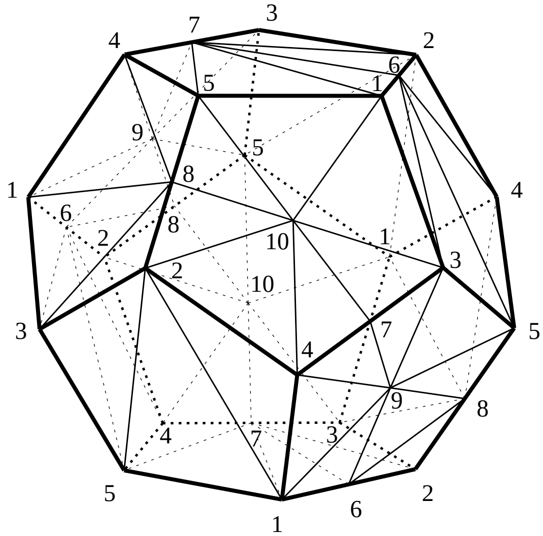


Figure 1.6: $\mathbb{Z}$-acyclic non-contractible complex on 10 vertices.

Remark: Taking instead the restriction of Σ^3_{24} (described in Section 1.4; see Figure 1.3) to the boundary of the identified dodecahedron we obtain a triangulation on 11 vertices, on which A_5 acts transitively on facets and without stationary points. Its nerve complex provides an 11-dimensional A_5-invariant vertex-transitive $\mathbb{Z}$-acyclic simplicial complex on 30 vertices (see Chapter 7).

1.6 A Series of Non-PL d-Spheres on d+13 Vertices for $d \geq 5$

It follows from Theorem 1.1 that if we suspend Σ^3_{16} twice, then we obtain a non-PL 5-sphere. If we suspend further, we obtain non-PL spheres of higher dimensions.

Theorem 1.7 *Let $d \geq 5$. Then there are non-PL triangulations of the d-dimensional sphere on $d + 13$ vertices.*

Proof: Let us first show that for $d \geq 5$ there exist particularly simple non-PL triangulations of the d-dimensional sphere on $d + 14$ vertices. For this, we suspend Σ^3_{16} $(d-3)$-times, i.e., we form $(d-3)$-times the join product of Σ^3_{16} with S^0. By the associativity of the join product with respect to the PL-structure (cf. [140, 2.24(1)]),

$$((\cdots((\Sigma^3_{16} * S^0) * S^0) * \cdots * S^0) * S^0) = \Sigma^3_{16} * (S^0 * S^0 * \cdots * S^0 * S^0) = \Sigma^3_{16} * S^{d-4}.$$

If we take for S^{d-4} the boundary complex of the $(d-3)$-simplex, then the latter simplicial complex has $16 + (d-2)$ vertices. Note also that it has $90 \cdot (d-2)$ facets, and that the list of its facets is easily compiled by concatenation from the list in Section 1.5 of the 90 facets of Σ^3_{16} with the list of all $(d-3)$-subsets of a $(d-2)$-set.

An improvement of the number of vertices by one can be obtained if we use *Datta's trick* to construct one-point suspensions of triangulated manifolds M. The Datta construction is as follows. Suspend M by using two vertices w_1 and w_2. Then pick a vertex v of M and replace the collection of facets that contain v by the facets that we obtain from the $(d-1)$-facets of the link of v by adding as an extra vertex either w_1 if w_2 is already contained in the respective $(d-1)$-facet, or otherwise w_2 if w_1 is already contained. The reverse procedure to this operation is called *starring a vertex in "an edge"* in an article by Bagchi and Datta [15, Def. 9]. The two authors remark in that paper that this generalized bistellar operation does not change the PL homeomorphism type of the suspension if M is a manifold (or a pseudomanifold). (We thank W. Kühnel for pointing out Datta's trick to us.) If we take $(d-3)$-times the one point Datta suspension of Σ^3_{16}, then we obtain a non-PL d-sphere with $d+13$ vertices. □

Theorem 1.7 complements the following two results, which show that triangulated manifolds with "few" vertices must be PL spheres.

Theorem 1.8 *Let M be a triangulated d-manifold on n vertices.*

(a) (Barnette and Gannon [18]) *If $n < d+6$ and $d \geq 5$, then M is a PL sphere.*

(b) (Brehm and Kühnel [42]) *If $n < 3\lceil \frac{d}{2} \rceil + 3$ and M is combinatorial, then M is a PL sphere.*

Brehm and Kühnel [42] also show that if $n = 3\frac{d}{2}+3$, then M is either a PL d-sphere or a "manifold like a projective plane" (the latter case can occur only for $d=2$, 4, 8, or 16). The following consequence of Theorem 1.7 shows that the assumption "combinatorial" can not be removed from the Brehm-Kühnel theorem.

Corollary 1.9 *There exist non-PL d-spheres with $n \leq 3\frac{d}{2}+3$ vertices for $d \geq 19$.*

Question 1.10 *Are there non-PL d-spheres for $d \geq 5$ with less than $d+13$ vertices?*

We tried on this question with BISTELLAR for $d=5$. Starting with the (ordinary) double suspension with 20 vertices of the 16-vertex triangulation of the Poincaré homology 3-sphere, we were able to get down to 18 vertices, but not further. The f-vector of the smallest non-PL 5-sphere that we found is $f = (18, 139, 503, 904, 783, 261)$.

We next show that for $d \geq 5$ there exists to any triangulation of a d-manifold M a non-PL triangulation of M with few additional vertices.

Theorem 1.11 *Let M be a topological d-manifold, $d \geq 5$, that can be triangulated with n vertices. Then there are non-PL triangulations of M with $n+12$ vertices.*

Proof: Let M be a simplicial d-manifold with n vertices and $d \geq 5$. If the triangulation of M is non-PL, then nothing has to be done. So assume that M is combinatorial. Let (by Theorem 1.7) Σ^d be a simplicial non-PL sphere on $d+13$ vertices. Then there exists a vertex v of Σ^d for which the corresponding link is not a combinatorial sphere. Choose a facet of Σ^d that is not contained in the star of v and delete this facet from Σ^d. Also delete some facet from M and glue the remaining complexes together along the boundaries of the deleted simplices. The resulting manifold is the connected sum $\Sigma^d \# M$. Topologically, $\Sigma^d \# M$ is homeomorphic to M, but on the PL level it provides a non-PL triangulation of M, since $\mathrm{link}_{\Sigma^d}(v) = \mathrm{link}_M(v)$. Let us count the vertices of $\Sigma^d \# M$. The complexes M and Σ^d contribute n and $d+13$ vertices respectively. By the identification of the boundaries of the two d-simplices, we loose $d+1$ vertices. Thus, $\Sigma^d \# M$ has $n + (d+13) - (d+1) = n+12$ vertices. □

Finally, we prove the result on connected components of the bistellar flip graph referred to in Section 1.2.

Theorem 1.12 *Let M be a triangulable manifold of dimension $d \geq 5$. Then there are infinitely many connected components of the bistellar flip graph of M.*

Proof: Let H be any homology 3-sphere with non-trivial fundamental group $\pi_1(H)$, e.g. let H be the Poincaré homology 3-sphere. We construct in three steps infinitely many triangulations of M that cannot pairwise be reached from one another by bistellar flips.

First, we form k-fold connected sums of H. These connected sums are again homology spheres, nevertheless they are pairwise non-homeomorphic for different values of k. This is due to the fact that the fundamental group of a connected sum $M\#N$ of two manifolds M and N, with (non-trivial) fundamental groups $\pi_1(M)$ and $\pi_1(N)$ respectively, is the free product $\pi_1(M) * \pi_1(N)$. Thus the connected sums $H^{\#k}$ and $H^{\#l}$ have distinct fundamental groups if $k \neq l$.

In the second step, we take for $k \neq l$ the join products of the boundary complex of a $(d-3)$-simplex with $H^{\#k}$ and $H^{\#l}$. The resulting simplicial complexes, S^d_k respectively S^d_l, are non-PL spheres (as in the proof of Theorem 1.7) that have the homology spheres $H^{\#k}$ and $H^{\#l}$ sitting in their respective triangulations as the links of some $(d-4)$-faces. From the combinatorics of the join construction it is easy to see that the links of $(d-4)$-faces in S^d_k are all non-homeomorphic to $H^{\#l}$, and the links of $(d-4)$-faces in S^d_l are all non-homeomorphic to $H^{\#k}$. Now, focus on a copy of $H^{\#k}$ that sits in S^d_k as the link of a $(d-4)$-face F. If we apply any bistellar flip to S^d_k, then this operation may alter but not delete this copy of $H^{\#k}$. This is so, because the definition of bistellar flips shows that the face F, or any subface of F, cannot be the pivot face of a bistellar move, and the link of F will itself be altered at most by a bistellar move and thus its homeomorphism type is preserved. The same argument used in reverse shows that the bistellar flip will not produce $H^{\#l}$ as the link of some $(d-4)$-faces in S^d_k. It thus follows that S^d_l cannot be reached from S^d_k via bistellar flips, and vice versa.

Finally, we will use the infinite number of examples of pairwise non-bistellarly equivalent triangulations of d-spheres S^d_k to obtain an infinite number of pairwise non-bistellarly equivalent triangulations of M. For this, let Φ be the set of those spheres S^d_k such that $H^{\#k}$ is not homeomorphic to the link of any of the $(d-4)$-faces of M. The set Φ is infinite, since there are only finitely many links in M. Then, just as in the proof of Theorem 1.11, form connected sums $S^d_k\#M$ of the spheres $S^d_k \in \Phi$ with M in a way that guarantees that $H^{\#k}$ remains as the link of some $(d-4)$-face of $S^d_k\#M$. By the same argument as in the second step, $S^d_k\#M$ and $S^d_l\#M$ cannot be reached from one another via bistellar flips. □

1.7 An A_5-Invariant Triangulation of $\mathbb{RP}^3$ with 29 Vertices

The idea of coherent twists on the dodecahedron can be used to create other interesting 3-manifolds besides the spherical dodecahedron space. For instance, Weber and Seifert [160] constructed a *hyperbolic dodecahedron space*, a manifold with homology $H_* = (\mathbb{Z}, \mathbb{Z}_5^3, 0, \mathbb{Z})$, by again identifying the boundary of the solid dodecahedron, this time with a coherent twist of $3\pi/5$ instead of $\pi/5$ radians.

If we twist by $5\pi/5$, we obtain $\mathbb{RP}^3$. Figure 1.7 gives a triangulation of the identified boundary for the latter manifold (where the identified boundary is the non-orientable surface $\mathbb{RP}^2$). As was done previously for the spherical dodecahedron space, we place an icosahedron with additional center point into the interior of the dodecahedron. This

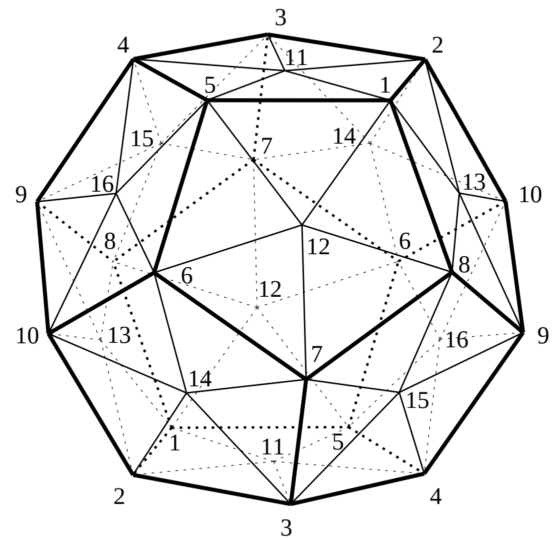


Figure 1.7: 29-vertex triangulation of $\mathbb{R}\mathbf{P}^3$.

yields an A_5-invariant triangulation of $\mathbb{R}\mathbf{P}^3$ with 29 vertices. Moreover, there is also an A_5-invariant triangulation of $\mathbb{R}\mathbf{P}^3$ on $6 + 12 + 1$ vertices that is defined by placing an icosahedron with center point into the interior of an outer icosahedron with identifications on the boundary by reflection at the origin. For a vertex-minimal triangulation of $\mathbb{R}\mathbf{P}^3$ on 11 vertices see [44], [158], and Chapter 2.

Chapter 2

Combinatorial Manifolds with Transitive Automorphism Group on Few Vertices

In this chapter, we present an enumeration algorithm for combinatorial d-manifolds with n vertices that are invariant under a given vertex-transitive group action. This algorithm is, in part, based on an earlier algorithm by Kühnel and Lassmann [98].

With a GAP-implementation, MANIFOLD_VT [115], of this algorithm, we determine, up to combinatorial equivalence, all combinatorial manifolds with a vertex-transitive automorphism group on $n \leq 13$ vertices. With the exception of actions of groups G of small order, $|G| \leq 2n$, in dimensions $4 \leq d \leq 8$, we extend this result to $n \leq 15$ vertices. For the enumeration we use a library of the computer algebra package GAP [144], which provides all equivalence classes of transitive permutation groups of degree $n \leq 22$.

Furthermore, we study connected sums and direct products, and we obtain many new examples of combinatorial manifolds with few vertices – some of them provably with the minimal number of vertices.

2.1 Outline of the Enumeration Algorithm

In the following, we describe the algorithm for the enumeration of vertex-transitive simplicial/combinatorial manifolds. We will combine the theoretical outline of the algorithm with some insight from its application in the case of few vertices.

The algorithm consists of nine steps. In the Steps 1–3, we fix the initial settings: the number of vertices n, the dimension d, and the group action n^i. In Step 4, we generate all possible candidates for vertex-transitive combinatorial d-manifolds for the group action n^i on n vertices. In Steps 5–8, we examine these candidates, i.e., we test if they are indeed manifolds and classify them up to combinatorial equivalence. In Step 9, we determine the topological type of most of these manifolds.

1. *Fix the number of vertices n.*

2. *Fix the dimension d for the manifolds.*

3. *Fix the group action.* For every n, there is a finite number of transitive permutation groups of degree n. These groups were classified for $n \leq 15$ (see [46], [47], [51], [75],

[121], [122], [141], and Table 2.1 for the number of distinct actions that occur).

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
#	5	5	16	7	50	34	45	8	301	9	63	104	1954	10	983	8	1117	164	59

Table 2.1: The number of inequivalent transitive group actions on $n \leq 22$ vertices.

We treat the group actions in decreasing group order. In fact, at the beginning we could examine the action of the symmetric group S_n or of the alternating group A_n on n vertices, but both groups are transitive on unordered $(d+1)$-sets for all d, and the only manifold on n vertices that is invariant under one of these two actions is the d-sphere S^d if $n = d+2$. Hence, we start with the next smallest group in the list of group actions for the respective n. Every time that our algorithm produces a new candidate for a combinatorial manifold, we check whether we have found this object, up to combinatorial equivalence, earlier. If not, then the automorphism group of the new object is the current group that we are working with. (The order of the automorphism group cannot be larger, since all examples have a vertex-transitive group action and we have tested actions of larger groups before.)

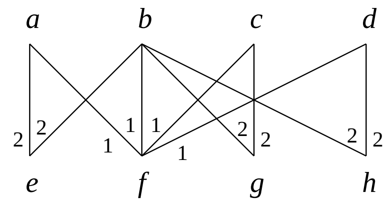
For the following, let n^i denote the i-th group action on n vertices from the GAP library.

4. *Main part: We determine all pure d-dimensional simplicial complexes on n vertices with a vertex-transitive action of group n^i that have the pseudomanifold property.*

   We first examine the induced actions of the group n^i on the set of $(d+1)$-tuples and on the set of d-tuples. In order to build a pure simplicial complex M, i.e., a candidate for a combinatorial manifold, with a vertex-transitive n^i-action, we have to form a union of orbits of $(d+1)$-tuples. For groups of small order, the enumeration of all these combinations is a hopeless task, even if n is comparably small.

   To reduce the problem, we require all candidates M to have the *pseudomanifold property*: Every $(d-1)$-dimensional face of the pure simplicial complex M must be included in *precisely two* d-dimensional facets. By transitivity, we say that an orbit of $(d-1)$-dimensional faces is included t times in an orbit of d-dimensional facets if each $(d-1)$-dimensional member of the first orbit is included in t elements of the latter orbit. If there is a $(d-1)$-dimensional orbit that is included 3 or more times in a d-dimensional orbit, then this d-dimensional orbit cannot be used for M without violating the pseudomanifold property. In a preprocessing step we sort out all these orbits. It then can happen that there are some $(d-1)$-dimensional orbits that are not included (or included only once) in any (in some) of the remaining d-dimensional orbits. We sort out these $(d-1)$-orbits (and the d-orbits containing these $(d-1)$-orbits) as well, etc.

   We associate a weighted bipartite graph with the remaining d- and $(d-1)$-orbits as nodes and an edge of weight t between two nodes whenever a $(d-1)$-orbit is included t-times in a d-orbit. For a simplicial complex with the pseudomanifold property, only those combinations of d-orbits are allowed for which the total weight of every $(d-1)$-orbit is exactly two or zero. To find such combinations efficiently, we set up the adjacency matrix for the weighted bipartite graph. Let us for example consider the action of the dihedral group D_7 on $n = 7$ vertices and let $d = 3$. There are four 3-dimensional orbits, a, b, c, d, and as well four 2-dimensional orbits, e, f, g, h, with associated weighted graph



As matrix of this graph we have

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} e & f & g & h \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}$$

We order the rows of the matrix lexicographically so that all rows with non-zero entry in the first column come first, then those rows with non-zero entry in the second column, and so on. In terms of the associated matrix, the problem of finding pure simplicial complexes with the pseudomanifold property translates to finding all combinations of row vectors such that their vector sum has entries 0 or 2 only. For this, we proceed block-wise. We first form all valid combinations (no entry larger than 2) of rows from the first block with a resulting sum entry of 2 in the first column. Next, we form all valid combinations of rows from the second block with 0 in the first column and 2 in the second column of their sum. Also we build all possible combinations of resulting vectors from the first block with vectors from the second block such that the new resulting vectors have either 0 or 2 in their second entry. We then proceed to the next block and column. As soon as a sum vector has entries 0 or 2 only we store it in a result list and do not form further combinations with such a vector. So we do not enumerate all vertex-transitive pure simplicial complexes with the pseudomanifold property this way; however, if the sum of two *closed* vectors (0 and 2 entries only) is again a closed vector, then the corresponding simplicial complex is not strongly connected (i.e., there is a pair of facets that cannot be joined by a path which moves from facet to facet only across $(d-1)$-faces) and therefore cannot be a connected manifold. For the above matrix we eventually get three valid combinations, $a + c$, $a + d$, and $c + d$. Finally, we decode this information and build the simplicial complexes corresponding to the resulting combinations of orbits.

5. *Various combinatorial tests:* Pure simplicial complexes with the pseudomanifold property can be seen as the most general form of pseudomanifolds as they comprise proper combinatorial manifolds, combinatorial pseudomanifolds, as well as Eulerian manifolds (see Chapter 3). For every simplicial complex that we found in Step 4, we perform simple tests to filter out complexes that are not manifolds. First we test if the candidate is connected. In order that a simplicial complex is a combinatorial manifold the link of any proper face has to be a combinatorial sphere. Thus, for

the link of one vertex v_0 (transitivity!) we check that this link is connected, has the Euler characteristic of a sphere, and has the pseudomanifold-property. Moreover, we perform this test for the link of every edge containing v_0 if $d \geq 3$ and for the link of every triangle containing v_0 if $d \geq 4$.

Remark: In order to enumerate all combinatorial pseudomanifolds or all Eulerian manifolds with a vertex-transitive action, we only need to alter this preselection slightly; see Chapter 3.

6. *Combinatorial equivalence:* We now determine new candidates up to combinatorial equivalence. Since testing equivalence is expensive, we first compute two combinatorial invariants, the f-vector and the *Altshuler-Steinberg determinant* [9], of a candidate, i.e., the determinant $\det(AA^T)$ of its vertex-facet incidence matrix A. Clearly, $\det(AA^T)$ is invariant under relabeling vertices or facets.

   One possibility for a combinatorial equivalence between two vertex-transitive complexes is that they are mapped onto each other by an outer automorphism of the acting group. As many group actions have the cyclic group, $\mathbb{Z}_n$, generated by the cycle $(123\ldots n)$, as a transitively acting subgroup, we restrict our attention to *multiplications* $k \mapsto (m \cdot k) \bmod n$ with $m \in \{1,2,3,\ldots,(n-1)\}$ and $\gcd(m,n) = 1$.

   In the above example, the generating simplices of the orbits a, b, c, and d are 1234_7, 1235_{14}, 1245_7, and 1246_7 respectively, the lower index indicating the size of the corresponding orbit. The union of orbits $a+c$ is mapped to $a+d$ by a multiplication with 2, and to $c+d$ by a multiplication with 3. Thus there is, up to combinatorial equivalence, a unique combinatorial 3-manifold with 7 vertices and vertex-transitive D_7-action. This manifold turns out, as expected, to be the boundary complex $BdC_4(7)$ of the cyclic 4-polytope $C_4(7)$ with 7 vertices.

   If the f-vectors and Altshuler-Steinberg determinants of two complexes are equal but the complexes are not multiplication isomorphic, we take a simplex of one complex and test all possible ways it can be mapped to the generating simplices of the orbits of the other complex. Note that by strong connectivity a combinatorial isomorphism between two simplicial/combinatorial manifolds is already determined by its restriction to one simplex.

7. *Compute homology:* We use the C-program HOMOLOGY by Heckenbach [71] to compute the homology of every candidate that is left over after the previous tests and also the homology of its vertex-link. The vertex-link must have the homology of a sphere and the homology of the manifold must obey Poincaré duality (with respect to $\mathbb{Z}_2$-coefficients). This provides us with a further simple test on our candidates. (In the range treated in the following, all our candidates that fulfilled this test and the previous tests were indeed manifolds. Nevertheless, in higher dimensions and on more vertices one possibly should also check the links of tetrahedra and higher dimensional simplices for a good preselection.)

8. *Bistellar flips:* We use the program BISTELLAR [113] (see Chapter 1) to check if the link of a candidate is bistellarly equivalent to the boundary of a simplex. With this heuristic, it has been possible to establish for all our remaining candidates that they are indeed combinatorial manifolds and thus to complete the enumeration. (In fact, we have enumerated *all simplicial* manifolds with a vertex-transitive automorphism group on few vertices! All these manifolds turned out to be combinatorial.)

9. *Topological type:* For almost all combinatorial manifolds that we found in the previous steps it has in addition been possible to determine their homeomorphism type. This was done in most cases with the program BISTELLAR_EQUIVALENT [114], which we used to establish a bistellar equivalence between the test manifold and some reference manifold. As a reference manifold we take a known small or minimal triangulation of a manifold that has the same homology as the test object.

   Figure 2.1 displays an easy application of bistellar flips. We explicitly give a bistellar equivalence between a 9-vertex triangulation of the 2-torus and the unique 7-vertex triangulation of the torus.

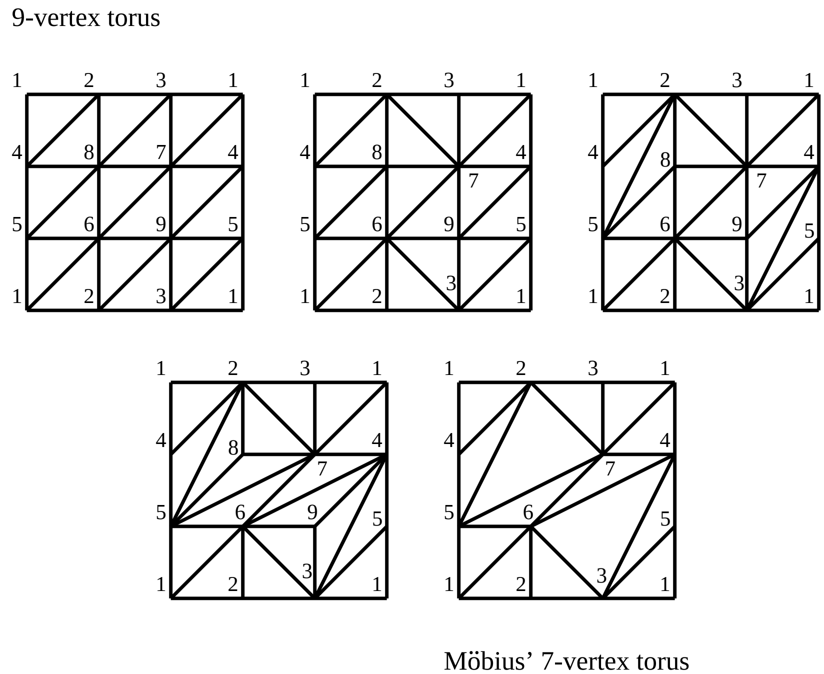


Figure 2.1: Bistellar flips on the torus to reduce the number of vertices.

Remark: Our GAP-program MANIFOLD_VT [115] is an implementation of the Steps 1–6 above. For actions of groups of small order on $n = 14$, 15 vertices, we used a C-implementation of Step 4 to enumerate all valid combinations of row vectors for the respective matrices.

2.2 Notations and Conventions

We name every vertex-transitive combinatorial manifold (up to combinatorial equivalence) that we found in the previous section with a unique symbol ${}^{d}n^{i}_{k}$ denoting the k-th example of a combinatorial manifold of dimension d with n vertices listed for the i-th transitive permutation group n^i of degree n. This group acts transitively on the vertices of ${}^{d}n^{i}_{k}$, and this action induces an action ${}^{d}n^{i}$ on the $(d+1)$-tuples of the ground set $\{1, \ldots, n\}$.

We list the combinatorial manifolds with vertex-transitive group action that were found in the Tables 2.13–2.22, together with additional information such as their topological types and where they possibly appeared previously in the literature – as far as we know. All examples can be rebuilt from their orbit representatives listed in the tables by using a GAP-program such as the following:

```
gap> G:=TransitiveGroup(7,4);
gap> facets:=[ ];
gap> UniteSet(facets,Orbit(G,[1,2,4],OnSets));
gap> Print(facets,"\n");
[ [ 1, 2, 4 ], [ 1, 2, 6 ], [ 1, 3, 4 ], [ 1, 3, 7 ], [ 1, 5, 6 ],
  [ 1, 5, 7 ], [ 2, 3, 5 ], [ 2, 3, 7 ], [ 2, 4, 5 ], [ 2, 6, 7 ],
  [ 3, 4, 6 ], [ 3, 5, 6 ], [ 4, 5, 7 ], [ 4, 6, 7 ] ]
```

(The given example is the 7-vertex torus ${}^{2}7^{4}_{1}$.)

If we are not able to determine the topological type of a manifold, then we denote it by $\sim M$ if it has the same homology vector as some particular manifold M. For more information on simplicial spheres which are *nearly neighborly centrally symmetric* (*nncs*) see Chapter 4.

For every transitive permutation group that occurs as the automorphism group of one of the combinatorial manifolds, we list the generators of the group in Table 2.23. Note that there exist finite groups that have more than one representation as a transitive permutation group on n vertices. Not all of the group actions have systematic names, thus we use the GAP-terminology to denote such groups: for example, $t8n15(32)$ is the transitive permutation group on 8 vertices with number 15 (we have added the size of the group in brackets).

2.3 Enumeration Results

As mentioned before in Chapter 1, Brehm and Kühnel [42] obtained a remarkable lower bound on the number of vertices for non-spherical combinatorial manifolds.

Theorem 2.1 (Brehm and Kühnel [42]) *Let M be a d-dimensional combinatorial manifold with n vertices. If $n < 3\lceil \frac{d}{2} \rceil + 3$, then M is a PL-sphere, i.e., M is PL-homeomorphic to the boundary of the $(d+1)$-simplex Δ_{d+1}. If $n = 3\lceil \frac{d}{2} \rceil + 3$, then either M is a PL-sphere or $d = 2$, 4, 8, 16 and M is a 'manifold like a projective plane'. In the latter case, $M = \mathbb{R}\mathbf{P}^2_6$ for $d = 2$ and $M = \mathbb{C}\mathbf{P}^2_9$ for $d = 4$.*

No similar lower bound is known for non-PL triangulations. Nevertheless, all simplicial manifolds with a vertex-transitive automorphism group that we found with MANIFOLD_VT are combinatorial manifolds.

Theorem 2.2 *There are at least 461 simplicial manifolds of dimension $2 \leq d \leq 13$ on $n \leq 15$ vertices that have a vertex-transitive automorphism group. All these 461 manifolds are combinatorial manifolds, 205 of them are spheres and 255 are non-spherical, as listed in Table 2.2.*

15	0 **17**	5 **29**	≥ 2 $\geq$ **31**	≥ 4 $\geq$ **1**	≥ 0 $\geq$ **2**	≥ 4 $\geq$ **0**	≥ 0 **1**	5	0	3	0	1
14	0 **13**	20 **35**	≥ 0 $\geq$ **7**	≥ 16 $\geq$ **9**	≥ 3 $\geq$ **0**	≥ 6	23	1	1	1	1	
13	0 **4**	6 **9**	0 **5**	17 **2**	0 **0**	6	0	1	0	1		
12	1 **30**	6 **33**	1 **7**	27 **0**	0	4	1	1	1			
11	0 **1**	3 **3**	0 **1**	3	0	1	0	1				
10	0 **3**	6 **4**	4 **0**	1	1	1	1					
9	0 **3**	1 **1**	0 **1**	2	0	1						
8	0 **1**	2	0	1	1							
7	0 **1**	1	0	1								
6	1 **1**	1	1									
5	0	1										
4	1											
n/d	2	3	4	5	6	7	8	9	10	11	12	13

Table 2.2: Combinatorial manifolds with vertex-transitive automorphism group; number of spheres and number of non-spheres (bold).

Corollary 2.3 *There are precisely* 220 *combinatorial manifolds,* 110 *spheres and* 110 *non-spherical manifolds, of dimension* $2 \leq d \leq 11$ *on* $n \leq 13$ *vertices that have a vertex-transitive automorphism group. The* 34 *homeomorphism types of these manifolds are:* S^2, $\mathbf{T}^2$, $\mathbf{T}^2\#\mathbf{T}^2$, $(\mathbf{T}^2)^{\#3}$, $(\mathbf{T}^2)^{\#4}$, $(\mathbf{T}^2)^{\#5}$, $(\mathbf{T}^2)^{\#6}$, $\mathbb{R}\mathbf{P}^2$, $\mathbb{R}\mathbf{P}^2\#\mathbb{R}\mathbf{P}^2$, $(\mathbb{R}\mathbf{P}^2)^{\#4}$, $(\mathbb{R}\mathbf{P}^2)^{\#5}$, $(\mathbb{R}\mathbf{P}^2)^{\#7}$, $(\mathbb{R}\mathbf{P}^2)^{\#8}$, $(\mathbb{R}\mathbf{P}^2)^{\#15}$; S^3, $S^2 \underline{\times} S^1$, $S^2 \times S^1$, $(S^2 \times S^1)^{\#2}$, $\mathbb{R}\mathbf{P}^3$; S^4, $\mathbb{C}\mathbf{P}^2$, $S^3 \times S^1$, $S^3 \underline{\times} S^1$, $S^2 \times S^2$, $(S^2 \times S^2)^{\#2}$; S^5, $S^4 \underline{\times} S^1$, X_{-1}; S^6; S^7; S^8; S^9; S^{10}; S^{11}.

Corollary 2.4 *There are exactly* 77 *combinatorial* 2*-manifolds on* $n \leq 15$ *vertices with a vertex-transitive automorphism group;* 3 *of these are spheres and* 74 *are non-spherical,* 42 *are orientable and* 35 *are non-orientable. In particular, there are* 18 *different topological types:* S^2, $\mathbf{T}^2$, $(\mathbf{T}^2)^{\#2}$, $(\mathbf{T}^2)^{\#3}$, $(\mathbf{T}^2)^{\#4}$, $(\mathbf{T}^2)^{\#5}$, $(\mathbf{T}^2)^{\#6}$, $(\mathbf{T}^2)^{\#8}$, $\mathbb{R}\mathbf{P}^2$, $(\mathbb{R}\mathbf{P}^2)^{\#2}$, $(\mathbb{R}\mathbf{P}^2)^{\#4}$, $(\mathbb{R}\mathbf{P}^2)^{\#5}$, $(\mathbb{R}\mathbf{P}^2)^{\#7}$, $(\mathbb{R}\mathbf{P}^2)^{\#8}$, $(\mathbb{R}\mathbf{P}^2)^{\#12}$, $(\mathbb{R}\mathbf{P}^2)^{\#15}$, $(\mathbb{R}\mathbf{P}^2)^{\#16}$, $(\mathbb{R}\mathbf{P}^2)^{\#17}$.

Corollary 2.5 *There are exactly* 166 *combinatorial* 3*-manifolds on* $n \leq 15$ *vertices that have a vertex-transitive automorphism group;* 52 *of these are spheres and* 114 *are non-spherical. The manifolds are of one of eight different topological types:* S^3, $S^2 \underline{\times} S^1$, $S^2 \times S^1$, $(S^2 \times S^1)^{\#2}$, $\mathbb{R}\mathbf{P}^3$, $L(3,1)$, S^3/Q, *and* $\mathbf{T}^3$.

Remark: For dimensions $4 \leq d \leq 8$ and $n = 14, 15$ vertices, we processed almost all transitive group actions, with the exception of the actions of type ${}^{4}14^{1}$, ${}^{4}14^{2}$, ${}^{5}14^{1}$, ${}^{5}14^{2}$, ${}^{6}14^{1}$, ${}^{6}14^{2}$, ${}^{7}14^{1}$, and ${}^{7}14^{2}$ of groups of order $|G| = 14$ on 14 vertices and actions of type ${}^{4}15^{1}$, ${}^{5}15^{1}$, ${}^{5}15^{2}$, ${}^{5}15^{3}$, ${}^{5}15^{4}$, ${}^{6}15^{1}$, ${}^{6}15^{2}$, ${}^{6}15^{3}$, ${}^{6}15^{4}$, ${}^{7}15^{1}$, ${}^{7}15^{2}$, ${}^{7}15^{3}$, ${}^{7}15^{4}$, and ${}^{8}15^{1}$ of groups of order $|G| \leq 30$ on 15 vertices.

2.4 Combinatorial Manifolds with Few Vertices

2.4.1 Join Products

The *join product* of two simplicial complexes K_1 and K_2 (with disjoint vertex sets) is the simplicial complex $K_1 * K_2$ that has as facets the concatenations of facets from K_1 with facets from K_2. The join product of two (simplicial) spheres S^k and S^l is again a sphere, $S^k * S^l \cong S^{k+l+1}$. If S^d is a triangulated d-sphere with n vertices and vertex-transitive automorphism group G, then the multiple join product $(S^d)^{*r}$ is a triangulated $(dr+r-1)$-sphere on rn vertices with the wreath product $G\,wr\,S_r$ as its vertex-transitive automorphism group. We denote the join product of a number of k-gons by $k * \cdots * k$.

2.4.2 Polytopal Spheres

A d-dimensional *polytopal sphere* is the boundary complex of a $(d+1)$-dimensional polytope P. According to Mani [117], any combinatorial d-sphere with few vertices, $n \leq d+4$, is polytopal. Furthermore, a theorem by Barnette and Gannon [18] states that every simplicial manifold with $n \leq d+5$ vertices for $d = 3$ and $d \geq 5$, and with $n \leq d+4$ vertices for $d = 4$, is a combinatorial sphere (cf. Chapter 1). Therefore, all triangulated d-manifolds with $n \leq d+4$ vertices are polytopal spheres. On the other hand, there are already exactly two non-polytopal spheres in dimension $d = 3$ with $n = 8$ vertices, the Brückner-Grünbaum sphere [68] and the Barnette sphere [17]; and Kalai [81] showed that, asymptotically, there are by far more non-polytopal than polytopal spheres.

Perles [66, p. 120] proved that, in addition to the polytopality of spheres with few vertices, every combinatorial automorphism of a polytopal d-sphere with $n \leq d+4$ vertices is realizable by an affine transformation of E^d. Also every combinatorial automorphism of a 2-sphere is realizable, by Mani [116]. In contrast, Bokowski, Ewald, and Kleinschmidt [28] found a polytopal 3-sphere with 10 vertices that has a combinatorial automorphism which cannot be realized.

Cyclic Polytopes and Crosspolytopes

We use the notation of [164] and denote by $C_d(n)$ the d-dimensional cyclic polytope and by C_d^{Δ} the d-dimensional crosspolytope. The corresponding boundary complexes we denote by $BdC_d(n)$ and BdC_d^{Δ}, respectively.

The combinatorial and geometric automorphism group of the crosspolytope C_d^{Δ} is $\mathbb{Z}_2\,wr\,S_d$, which can be seen by looking at the standard description of C_d^{Δ}: Take the convex hull of the unit vectors and their negatives, $C_d^{\Delta} = \text{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$, with boundary sphere $BdC_d^{\Delta} = (S^0)^{*d}$.

For cyclic d-polytopes, we get a vertex-transitive automorphism group if and only if the dimension is even or if the polytope is a simplex. In the latter case $(n = d+1)$, the symmetry group is S_{d+1}. So let d be even. If $n \geq d+3$, then the automorphism group of $C_d(n)$ is the dihedral group D_n (cf. [52, Theorem 5.1]). For $n = d+2$, the cyclic polytope $C_d(d+2)$ has as symmetry group the wreath product $S^2_{(d+2)/2}\,wr\,\mathbb{Z}_2$. This is due to the fact that $BdC_d(d+2)$ is the join product, $BdC_d(d+2) = Bd\Delta_{d/2} * Bd\Delta_{d/2}$, of the boundary of the $\frac{d}{2}$-simplex $\Delta_{d/2}$ with itself, which can be read off from Gale's evenness condition (cf. [164, p. 14]): A d-subset $S \subseteq \{1, 2, \ldots, n\}$ is a facet of the cyclic polytope $C_d(n)$ if and only if for any pair $i < j$ of vertices that do not belong to S the number of elements $k \in S$ between i and j is even. If $n = d+2$, the symmetry group

$S^2_{(d+2)/2}\, wr\, \mathbb{Z}_2$ of $C_d(d+2)$ is generated by the transposition $(12)(34)\dots(d+1\ d+2)$ together with all permutations of the sets $\{1,3,\dots,d+1\}$ and $\{2,4,\dots,d+2\}$.

By the *upper bound theorem* of McMullen [120] for polytopal spheres and its generalization to simplicial spheres by Stanley [152], the cyclic polytope $C_d(n)$ attains the maximal number of i-faces any simplicial d-sphere with n vertices can have (cf. [164]). However, if d is odd, the cyclic polytope $C_d(n)$ does not have, in general, the symmetry group with the largest order among all simplicial spheres with the same f-vector.

Lemma 2.6 *The combinatorial spheres* ${}^5 13^5_2$, ${}^5 14^4_5$, *and* ${}^7 13^4_2$ *have the same f-vector as the corresponding cyclic polytopes but they have a symmetry group G of order $|G| > 2n$.*

Bicyclic Polytopes

The trigonometric moment curve

$$M_{12} = \{(\cos 2\pi t, \sin 2\pi t, \cos 2\pi 2t, \sin 2\pi 2t) : 0 \le t < 1\} \subseteq \mathbf{T^2}$$

on the 2-torus $\mathbf{T^2}$ offers an alternative way for constructing cyclic 4-polytopes by taking the convex hull of n points for equally distributed values of $t = 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$. In [148], Smilansky considered generalized trigonometric moment curves

$$M_{pq} = \{(\cos 2\pi pt, \sin 2\pi pt, \cos 2\pi qt, \sin 2\pi qt) : 0 \le t < 1\} \subseteq \mathbf{T^2},$$

for positive integers p and q. The convex hulls of n equally distributed points on these curves are called *bicyclic* 4*-polytopes* by Smilansky. We denote such a polytope by $BiC(p,q;n)$ and its boundary complex by $BdBiC(p,q;n)$. (We thank W. Kühnel for pointing out Smilansky's work on bicyclic 4-polytopes to us.)

Unlike for cyclic polytopes, where we have Gale's evenness condition, there seems to be no combinatorial characterization in closed form known for the facets of general bicyclic 4-polytopes. Nevertheless, every bicyclic 4-polytope has a vertex-transitive automorphism group, and, moreover, Smilansky characterized all triples $(p,q;n)$ for which the corresponding bicyclic polytopes are simplicial. Let $\gcd(p,q) = 1$, since otherwise $M_{pq} = M_{ab}$, with $p = ka$ and $q = kb$ for positive integers a, b, and $k = \gcd(p,q)$. A bicyclic 4-polytope $BiC(p,q;n)$ with $p \neq q$ and $\gcd(p,q) = 1$ is simplicial if and only if $\gcd(p,n) \le 2$ and $\gcd(q,n) \le 2$. It follows that $\gcd(p,n) = 1$ or $\gcd(q,n) = 1$. Let $\gcd(p,n) = 1$. Then there are r and s such that $rp + sn = 1$, i.e., $rp \equiv 1 \bmod n$. If we multiply our generalized moment curve by r, we get $BiC(rp, rq; n) = BiC(1,u;n)$, with $rq = u + vn$ and $2 \le u < n$. Hence, we can always normalize $p = 1$.

Further identities for bicyclic polytopes are discussed in [30] and [145]. In particular,

- $BiC(1,q;n) = BiC(1,n-q;n)$,
- $BiC(1,2;n) = C_4(n)$
- $BiC(1,\frac{n-1}{2};n) = C_4(n)$ if n is odd,
- $BiC(1,\frac{n}{2}-1;n) = \frac{n}{2} * \frac{n}{2}$ if n is even,
- $BiC(1,q;n) = BiC(1,\frac{n+1}{q};n)$ if $\frac{n+1}{q} \in \mathbb{N}$, and
- $BiC(1,q;n) = BiC(1,\frac{n-1}{q};n)$ if $\frac{n-1}{q} \in \mathbb{N}$.

Table 2.3 lists all (non-cyclic) pairwise non-isomorphic bicyclic 4-polytopes with $n \leq 15$ vertices together with the combinatorial types of the vertex-transitive boundary spheres.

n	Bicyclic Polytope	Type
8	$BiC(1,3;8) = 4*4$	${}^{3}8^{44}_{1}$
10	$BiC(1,3;10)$	${}^{3}10^{22}_{1}$
	$BiC(1,4;10) = 5*5$	${}^{3}10^{21}_{1}$
11	$BiC(1,3;11)$	${}^{3}11^{2}_{2}$
12	$BiC(1,5;12) = 6*6$	${}^{3}12^{125}_{1}$
13	$BiC(1,3;13)$	${}^{3}13^{2}_{2}$
	$BiC(1,5;13)$	${}^{3}13^{4}_{1}$
14	$BiC(1,3;14)$	${}^{3}14^{3}_{2}$
	$BiC(1,4;14)$	${}^{3}14^{3}_{4}$
	$BiC(1,6;14) = 7*7$	${}^{3}14^{20}_{1}$
15	$BiC(1,4;15)$	${}^{3}15^{7}_{1}$

Table 2.3: Bicyclic 4-polytopes with $n \leq 15$ vertices.

The lists of facets of the bicyclic polytopes for the given values of q and $n \leq 15$ were generated with POLYMAKE [64]. In each case, we then identified the corresponding vertex-transitive 3-sphere of Table 2.14. We also determined the boundary spheres of three tricyclic 6-polytopes; see Table 2.4.

n	Tricyclic Polytope	Type
9	$TriC(1,2,5;9) = 3*3*3$	${}^{5}9^{31}_{1}$
11	$TriC(1,2,5;11)$	${}^{5}11^{2}_{2}$
12	$TriC(1,2,5;12)$	${}^{5}12^{83}_{1}$

Table 2.4: Three examples of tricyclic 6-polytopes.

2.4.3 2-Manifolds

A *map* on a surface M is a decomposition of M into polygonal regions or *countries* such that every vertex has at least degree 3 and each vertex of degree s is incident with s countries. If the decomposition is simplicial, then the map is called simplicial.

We enumerated all simplicial maps on $n \leq 15$ vertices with a vertex-transitive group action (see Table 2.13). Among them we found many classical examples, such as Möbius' unique minimal and neighborly triangulation of the 2-torus with 7 vertices [125]. The minimality of the triangulation follows from the following result.

Theorem 2.7 (Jungerman and Ringel [79], [133]) *Let M be a (compact) surface, different from the orientable surface of genus 2, the Klein bottle, and the non-orientable surface of genus 3. There exists a triangulation of M on n vertices if and only if*

$$\binom{n-3}{2} \geq 3\,(2 - \chi(M)), \tag{2.1}$$

with equality if and only if the triangulation is 2-neighborly. (For the three exceptional cases, $\binom{n-3}{2}$ has to be replaced by $\binom{n-4}{2}$, cf. [76].)

A second interesting example of a vertex-transitive simplicial map is Dyck's regular map. A map, simplicial or not, is called *regular* if it has a *flag-transitive* automorphism group, i.e., if its automorphism group is transitive on the set of all triples $\{vertex \subseteq edge \subseteq facet\}$, or *flags* for short. Regular maps can be seen as non-spherical combinatorial analogues of the Platonic solids. Besides Dyck's regular map, well known examples of flag-transitive simplicial maps with few vertices are the tetrahedron, the octahedron, the icosahedron, and $\mathbb{RP}^2_6$.

For neighborly maps with $n \leq 22$ vertices that have a vertex-transitive automorphism group and for regular simplicial maps with $n \leq 22$ vertices see Chapter 5.

2.4.4 3-Manifolds

In 1990, Kühnel [90] gave a list of six topologically distict combinatorial 3-manifolds with 15 or less vertices. These are the 3-sphere S^3, the twisted S^2-bundle over S^1, $S^2 \underline{\times} S^1$ (3-dimensional Klein bottle), the product $S^2 \times S^1$, the projective 3-space $\mathbb{RP}^3$, the 3-dimensional torus $\mathbf{T}^3$, and Cartan's hypersurface S^3/Q.

Since then, sixteen new examples have been found. Brehm and Światkowski constructed triangulations of $L(3,1)$ and $L(4,1)$ with 13 respectively 15 vertices [44], and by local modifications, Brehm found a combinatorial $L(3,1)$ on 12 vertices. Furthermore, two combinatorial manifolds with 12 vertices that have homology $H_* = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z})$ were listed in [98]. All other examples described in the following are new. In addition, we have improved the number of vertices for $L(4,1)$ to $n = 14$. Table 2.5 displays our current knowledge about the number of vertices of these manifolds.

Theorem 2.8 *There are at least* 22 *pairwise non-homeomorphic* 3*-manifolds that can be triangulated on* $n \leq 15$ *vertices.*

Small Triangulations of the Lens Spaces $L(2,1)$, $L(3,1)$, and $L(4,1)$

A minimal triangulation of the lens space $\mathbb{RP}^3 = L(2,1)$ with 11 vertices was first found by Walkup [158]. An alternative construction $S_{2\cdot 4}$ with D_8-symmetry was given by Brehm and Światkowski [44]. Both constructions lead to combinatorially isomorphic complexes with f-vector $f = (\underline{11}, 51, 80, 40)$. (We underline the k-entry of an f-vector if the triangulation is k-neighborly, that is, if $f_k = \binom{n}{k}$.)

Here, we determine the full automorphism group of the minimal triangulation of the 3-dimensional projective space. We define $\mathbb{RP}^3_{11}$ in terms of the facets

1237	123 11	1269	126 11	1279	135 10	135 11	137 10
1479	147 10	1489	148 10	1568	156 11	158 10	1689
2348	234 11	2378	246 10	246 11	248 10	2578	2579
258 10	259 10	269 10	3459	345 11	3489	359 10	3678
367 10	3689	369 10	4567	456 11	4579	467 10	5678.

For every vertex, we compute the Altshuler-Steinberg determinant of the vertex-link. Vertices 1–6 have Altshuler-Steinberg determinant 41616, the determinant of vertices 7–10 is 12096, and vertex 11 has determinant 0. Thus, the automorphism group of $\mathbb{RP}^3_{11}$ must be a subgroup of $S_6 \times S_4$. It turns out to be $2S_4$ with generators (123456)(789), (12)(36)(45)(79), and (36)(79)(8*10*), which we checked by computer.

In fact, Brehm and Światkowski [44] constructed triangulations of all lens spaces $L(p,q)$. In particular, they gave an infinite series of $D_{2\cdot(p+2)}$-symmetric trianguations $S_{2\cdot(p+2)}$ of $L(p,1)$ on $2p+7$ vertices.

Manifold	Homology	n	n_{vt}	Reference
S^3	$(\mathbb{Z},0,0,\mathbb{Z})$	$\underline{5}$	$\underline{5}$	
$S^2 \underline{\times} S^1$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z}_2,0)$	$\underline{9}$	$\underline{9}$	[10, N^9_{51}], [158]
$S^2 \times S^1$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z},\mathbb{Z})$	$\underline{10}$	$\underline{10}$	[158]
$\mathbb{R}\mathbf{P}^3 = L(2,1)$	$(\mathbb{Z},\mathbb{Z}_2,0,\mathbb{Z})$	$\underline{11}$	$\underline{12}$	[44, $S_{2\cdot 4}$], [158]; [94]
$(S^2 \times S^1)\#(S^2 \times S^1)$ $= (S^2 \times S^1)\# -(S^2 \times S^1)$	$(\mathbb{Z},\mathbb{Z}^2,\mathbb{Z}^2,\mathbb{Z})$	12	$\underline{12}$	[98, 5_{12}, 8_{12}]
$L(3,1)$	$(\mathbb{Z},\mathbb{Z}_3,0,\mathbb{Z})$	12	$\underline{14}$	[40]; [98, 3_{14}, 6_{14}, 13_{14}, 18_{14}]
$(S^2 \underline{\times} S^1)\#(S^2 \underline{\times} S^1)$ $= (S^2 \underline{\times} S^1)\#(S^2 \times S^1)$	$(\mathbb{Z},\mathbb{Z}^2,\mathbb{Z}\oplus\mathbb{Z}_2,0)$	12	?	new
$(S^2 \times S^1)^{\#3}$	$(\mathbb{Z},\mathbb{Z}^3,\mathbb{Z}^3,\mathbb{Z})$	13	?	new
$(S^2 \underline{\times} S^1)^{\#3}$	$(\mathbb{Z},\mathbb{Z}^3,\mathbb{Z}^2\oplus\mathbb{Z}_2,0)$	14	?	new
$L(4,1)$	$(\mathbb{Z},\mathbb{Z}_4,0,\mathbb{Z})$	14	?	new; [44]
$(S^2 \times S^1)\#\,\mathbb{R}\mathbf{P}^3$ $= (S^2 \times S^1)\# -\mathbb{R}\mathbf{P}^3$	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z},\mathbb{Z})$	14	?	new
$(S^2 \underline{\times} S^1)\#\,\mathbb{R}\mathbf{P}^3$	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z}_2,0)$	14	?	new
$\mathbb{R}\mathbf{P}^2 \times S^1$	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z}_2,0)$	14	17	new; [98, IV_{17}, IV_{19}]
$(S^2 \underline{\times} S^1)^{\#2}\#\,\mathbb{R}\mathbf{P}^3$	$(\mathbb{Z},\mathbb{Z}^2\oplus\mathbb{Z}_2,\mathbb{Z}\oplus\mathbb{Z}_2,0)$	14	?	new
$\mathbf{T}^3 = S^1 \times S^1 \times S^1$	$(\mathbb{Z},\mathbb{Z}^3,\mathbb{Z}^3,\mathbb{Z})$	15	$\underline{15}$	[97]
S^3/Q	$(\mathbb{Z},\mathbb{Z}_2^2,0,\mathbb{Z})$	15	$\underline{15}$	[41]
$\mathbb{R}\mathbf{P}^3\#\,\mathbb{R}\mathbf{P}^3$ $= \mathbb{R}\mathbf{P}^3\# -\mathbb{R}\mathbf{P}^3$	$(\mathbb{Z},\mathbb{Z}_2^2,0,\mathbb{Z})$	15	?	new
$(S^2 \times S^1)^{\#2}\#\,\mathbb{R}\mathbf{P}^3$	$(\mathbb{Z},\mathbb{Z}^2\oplus\mathbb{Z}_2,\mathbb{Z}^2,\mathbb{Z})$	15	?	new
$(S^2 \times S^1)^{\#4}$	$(\mathbb{Z},\mathbb{Z}^4,\mathbb{Z}^4,\mathbb{Z})$	15	?	new
$(S^2 \underline{\times} S^1)^{\#4}$	$(\mathbb{Z},\mathbb{Z}^4,\mathbb{Z}^3\oplus\mathbb{Z}_2,0)$	15	?	new
$(S^2 \times S^1)\#L(3,1)$	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_3,\mathbb{Z},\mathbb{Z})$	15	?	new
$(S^2 \underline{\times} S^1)\#L(3,1)$	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_3,\mathbb{Z}_2,0)$	15	?	new

Table 2.5: Combinatorial 3-manifolds with smallest known numbers of vertices $n \leq 15$ and n_{vt} with a vertex-transitive action (minimal if underlined).

By applying BISTELLAR flips to the triangulations $S_{2\cdot 5}$ and $S_{2\cdot 6}$ of $L(3,1)$ and $L(4,1)$ with 13 and 15 vertices, we obtained triangulations with 12 and 14 vertices respectively. As already mentioned, Brehm previously found a triangulation of $L(3,1)$ with 12 vertices by modifying $S_{2\cdot 5}$ (personal communication). We denote our example by $L(3,1)_{12}$. It has f-vector $f = (\underline{12}, \underline{66}, 108, 54)$ and facets

1234	123 10	1249	1256	1259	126 11	12 10 11	1347	1378
138 10	1479	156 12	1579	157 12	16 11 12	178 12	18 10 11	18 11 12
234 12	23 10 12	2489	248 12	2568	2589	2678	267 11	278 12
27 10 11	27 10 12	3456	345 11	3467	34 11 12	3568	3589	359 11
3678	389 10	39 10 12	39 11 12	456 10	45 10 11	4679	469 10	489 10
48 10 11	48 11 12	56 10 12	579 11	57 10 11	57 10 12	679 11	69 10 12	69 11 12.

The symmetry group of $L(3,1)_{12}$ is S_3 as a subgroup of $S_6 \times S_3 \times S_3$ with generators $(12)(36)(45)(78)(\mathit{10}\,\mathit{11})$ and $(135)(246)(789)(\mathit{10}\,\mathit{11}\,\mathit{12})$. (The Altshuler-Steinberg determinant is 134784 for vertices 1–6, 133056 for vertices 7–9, and 112320 for 10–12.)

Remark: $L(2,1)_{11}$ and $L(3,1)_{12}$ seem to be unique in the sense that there possibly are no other triangulations of these manifolds with the same f-vectors.

The triangulation of $L(4,1)$ that we found with 14 vertices is not symmetric. It has f-vector $f = (\underline{14}, 84, 140, 70)$ and facets

1238	123 14	126 13	126 14	1278	127 13	1358
1359	139 12	13 12 14	1458	1459	1478	147 13
149 11	14 11 13	16 11 12	16 11 13	16 12 14	19 11 12	238 11
23 11 14	257 10	257 13	25 10 11	25 11 12	25 12 13	269 13
269 14	278 10	28 10 11	29 11 12	29 11 14	29 12 13	3467
346 10	347 13	34 10 14	34 11 13	34 11 14	3579	357 13
358 13	3679	369 10	38 11 13	39 10 12	3 10 12 14	4568
456 10	459 14	45 10 14	4678	49 11 14	568 12	56 10 11
56 11 12	579 14	57 10 14	58 12 13	678 12	679 14	67 12 14
69 10 13	6 10 11 13	78 10 12	7 10 12 14	8 10 11 13	8 10 12 13	9 10 12 13.

Connected Sums

Let M and N be compact d-manifolds. Their *connected sum* $M\#N$ is obtained by cutting out the interior of a d-ball from each of the manifolds and then pasting together the remainders via a homeomorphism of the boundary spheres of the balls (see e.g. [73, Ch. 3]). If both manifolds are oriented, one requires that the homeomorphism reverses(!) the orientation of the boundary spheres, otherwise the connected sum is denoted by $M\#-N$. In case there exists a self homeomorphism of one of M, N that fixes some point and reverses orientation on a neighborhood of this point both types are homeomorphic.

A 3-manifold M is *prime* if $M = M_1\#M_2$ implies that either M_1 or M_2 is a 3-sphere. If M is an oriented, prime 3-manifold for which there is no orientation-reversing self homeomorphism, then $M\#M$ is not homeomorphic to $M\#-M$. Such examples exist (cf. [73]): The lens space $L(p,q)$ admits an orientation-reversing self homeomorphism if and only if $q^2 \equiv -1 \bmod p$.

Lemma 2.9 *Let M and N be d-dimensional combinatorial manifolds with m and n vertices respectively. Then there is a combinatorial triangulation of their connected sum $M\#N$ (and of $M\#-N$) on $m+n-(d+1)$ vertices.*

Proof: Suppose M and N have disjoint sets of vertices. Remove one simplex σ_M from M and one simplex σ_N from N and paste together the remainders by pairwise identifying the $d+1$ vertices of the two deleted simplices σ_M and σ_N.

In order to check that $M\#N$ is combinatorial, we have to show that the link of every vertex is a combinatorial sphere. For the vertices that lie in the interiors $M-\sigma_M$ and $N-\sigma_N$ nothing has changed. So let v be a vertex of the simplex σ_M. Its link in M is a combinatorial sphere $S^{d-1}_{(M,v)}$. Let σ_v be the maximal face of the simplex σ_M that is opposite to v. Then by [163, Theorem 3] the closure $\overline{S^{d-1}_{(M,v)}-\sigma_v}$ of the complement in $S^{d-1}_{(M,v)}$ of the ball σ_v is again a ball. This also holds for $\overline{S^{d-1}_{(N,v)}-\sigma_v}$ if we regard v as a vertex of σ_N under the identification. The two balls $\overline{S^{d-1}_{(M,v)}-\sigma_v}$ and $\overline{S^{d-1}_{(N,v)}-\sigma_v}$ are glued together along the boundary of σ_v. □

Remark: Although this lemma is very basic, in combination with bistellar flips it provides a simple and effective method to construct new combinatorial manifolds with few vertices.

Let us recall that every (compact) 3-manifold can be expressed as a connected sum of finitely many prime factors. For this and a uniqueness theorem for prime decompositions of oriented 3-manifolds see [73] and [123]. Lens spaces and 2-sphere bundles over S^1 are prime. If N is a nonorientable 3-manifold, then $N\#(S^2 \underline{\times} S^1) \cong N\#(S^2\times S^1)$.

We formed various connected sums of the prime factors $S^2 \underline{\times} S^1$, $S^2\times S^1$, $\mathbb{R}\mathbf{P}^3$, and $L(3,1)$. By lemma 2.9, the (combinatorial) connected sum $(S^2 \underline{\times} S^1)\#(S^2 \underline{\times} S^1)$ of the 9-vertex triangulation of $S^2 \underline{\times} S^1$ with itself is a combinatorial manifold with 14 vertices (that has not been observed previously). For the minimal 10-vertex triangulation of $S^2\times S^1$ we get a triangulation of $(S^2 \underline{\times} S^1)\#(S^2\times S^1)$ on 15 vertices. But this space is homeomorphic to $(S^2 \underline{\times} S^1)\#(S^2 \underline{\times} S^1)$. If we form further connected sums of the prime factors with few vertices (by $M^{\#r}$ we denote the connected sum of r copies of a factor M), then we get combinatorial manifolds with more than 15 vertices. However, by applying BISTELLAR flips to these connected sums we obtained 12 new examples of topologically distinct combinatorial manifolds with 15 or less vertices (see Table 2.5).

As mentioned, two combinatorial manifolds on 12 vertices were listed in [98, 5_{12}, 8_{12}] that have homology $H_*=(\mathbb{Z},\mathbb{Z}^2,\mathbb{Z}^2,\mathbb{Z})$. In Section 2.5, we give three further examples of combinatorial manifolds with a vertex-transitive automorphism group, ${}^312^{\,2}_{\,1}$, ${}^312^{\,2}_{\,3}$, and ${}^312^{\,13}_{\,4}$, which have the same homology. Also, there is a combinatorial manifold with this homology on 18 vertices that occurs as the vertex-link of a 2-neighborly vertex-transitive combinatorial 4-pseudomanifold with 19 vertices (see Chapter 3). All these examples are BISTELLAR_EQUIVALENT (and thus PL-homeomorphic) to a 12-vertex triangulation of $(S^2\times S^1)\#(S^2\times S^1)$. In fact, we first "minimized" the f-vector of all six examples as well as of the 16-vertex connected sum $(S^2\times S^1)\#(S^2\times S^1)$ with BISTELLAR flips. The smallest f-vector we hereby achieved is $f=(\underline{12},58,92,46)$ in each case. Then we applied further flips until two of the examples became combinatorially equivalent. During this process, we found 1182 combinatorially distinct triangulations of $(S^2\times S^1)\#(S^2\times S^1)$ with the same f-vector $f=(\underline{12},58,92,46)$, but with pairwise different Altshuler-Steinberg determinants.

Direct Products

Instead of taking connected sums of 3-manifolds, we can also form direct products of 2-manifolds with an empty triangle to obtain new combinatorial manifolds.

In general, the product $M \times N$ of a combinatorial m-manifold M with a combinatorial n-manifold N is formed by first taking the Cartesian product of every m-facet with every n-facet and then triangulating the resulting cell-complex consistently (see e.g. [106]). This leads to a product triangulation of $M \times N$ on $m \cdot n$ vertices, which is combinatorial.

After applying BISTELLAR flips, we obtained three 3-dimensional product manifolds with $n \leq 15$ vertices (see Table 2.6).

Manifold	Homology	n
$S^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$	$\underline{10}$
$\mathbb{R}\mathbf{P}^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)$	14
$\mathbf{T}^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^3, \mathbb{Z})$	15

Table 2.6: Combinatorial 3-manifolds with $n \leq 15$ vertices that are direct products.

The triangulation of $\mathbf{T}^2 \times S^1$ that we found on 15 vertices coincides with the Kühnel-Lassmann triangulation of the 3-dimensional torus [97].

Conjecture 2.10 *The 15-vertex triangulation of* Kühnel *and* Lassmann *of the 3-dimensional torus is vertex-minimal and unique with* $f = (\underline{15}, \underline{105}, 180, 90)$.

Remark: Kühnel and Lassmann enumerated in [98] all combinatorial 3-manifolds with $n \leq 15$ vertices that have a vertex-transitive cyclic group action. We determined the homeomorphism type of all these manifolds by showing for every example that it is BISTELLAR_EQUIVALENT to a reference manifold whose topological type is known. For the resulting topological types see Table 2.14. In addition to cyclic actions, Kühnel and Lassmann enumerated all 3-manifolds with a vertex-transitive dihedral action for $n \leq 19$. In their list appear two non-orientable manifolds IV_{17} and IV_{19} that are homeomorphic to each other. We established that these two manifolds are BISTELLAR_EQUIVALENT to $\mathbb{R}\mathbf{P}^2 \times S^1$.

Homology-Isomorphic Pairs of 3-Manifolds with Few Vertices

Among the 22 topologically distinct 3-manifolds, of which we now know triangulations with 15 or less vertices, there are three pairs that have the same homology:

- the triple sum $(S^2 \times S^1)^{\# 3}$ and the 3-torus $\mathbf{T}^3$,
- $\mathbb{R}\mathbf{P}^2 \times S^1$ and $(S^2 \underline{\times} S^1) \# \mathbb{R}\mathbf{P}^3$,
- and the sum $\mathbb{R}\mathbf{P}^3 \# \mathbb{R}\mathbf{P}^3$ and Cartan's hypersurface S^3/Q that is obtained as the quotient of S^3 by the 8 element quaternion group Q (see [41]).

These three pairs have distinct fundamental groups, respectively. Recall that the fundamental group $\pi_1(M)$ of a connected sum $M = M_1 \# M_2$ is isomorphic to the free product $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$. The fundamental group of $S^2 \times S^1$ is $\mathbb{Z}$, thus $(S^2 \times S^1)^{\# 3}$ has fundamental group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, compared with the fundamental group $\mathbb{Z}^3$ of the 3-torus.

$\mathbb{RP}^2 \times S^1$ and $(S^2 \underline{\times} S^1) \# \mathbb{RP}^3$ have fundamental groups $\mathbb{Z} \oplus \mathbb{Z}_2$ and $\mathbb{Z} * \mathbb{Z}_2$, respectively. S^3/Q as a finite covering has the 8-element group Q as fundamental group, in contrast to $\mathbb{RP}^3 \# \mathbb{RP}^3$ whose fundamental group $\mathbb{Z}_2 * \mathbb{Z}_2$ is infinite. In Table 2.7 we display the smallest f-vectors that we achieved for these manifolds.

Manifold	Homology	Fundamental Group	f-vector
$(S^2 \times S^1)^{\#3}$	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^3, \mathbb{Z})$	$\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$	($\underline{13}$,72,118,59)
$\mathbf{T}^3$	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^3, \mathbb{Z})$	$\mathbb{Z}^3$	($\underline{15}$,$\underline{105}$,180,90)
$(S^2 \underline{\times} S^1) \# \mathbb{RP}^3$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)$	$\mathbb{Z} * \mathbb{Z}_2$	($\underline{14}$,73,118,59)
$\mathbb{RP}^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)$	$\mathbb{Z} \oplus \mathbb{Z}_2$	($\underline{14}$,84,140,70)
$\mathbb{RP}^3 \# \mathbb{RP}^3$	$(\mathbb{Z}, \mathbb{Z}_2^2, 0, \mathbb{Z})$	$\mathbb{Z}_2 * \mathbb{Z}_2$	($\underline{15}$,86,142,71)
S^3/Q	$(\mathbb{Z}, \mathbb{Z}_2^2, 0, \mathbb{Z})$	Q	($\underline{15}$,90,150,75)

Table 2.7: Homology-isomorphic pairs of 3-manifolds with $n \leq 15$ vertices.

REMARK: In Chapter 4, we describe two non-orientable 3-manifolds on 16 vertices with homology $(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$ for which it has not been possible to find a triangulation with fewer vertices. Hence, these manifolds might be distinct from $(S^2 \underline{\times} S^1) \# (S^2 \underline{\times} S^1)$, which can be triangulated with 12 vertices.

HOMEOMORPHIC PAIRS

As a further application of BISTELLAR flips, it has been possible to show that the (combinatorial) manifolds $(S^2 \times S^1) \# (S^2 \times S^1)$ and $(S^2 \times S^1) \# -(S^2 \times S^1)$, $(S^2 \times S^1) \# \mathbb{RP}^3$ and $(S^2 \times S^1) \# -\mathbb{RP}^3$, as well as $\mathbb{RP}^3 \# \mathbb{RP}^3$ and $\mathbb{RP}^3 \# -\mathbb{RP}^3$ are BISTELLAR_EQUIVALENT and hence homeomorphic.

2.4.5 4-Manifolds

The most prominent combinatorial 4-manifold, apart from the 4-sphere, is the unique 9-vertex triangulation $\mathbb{CP}^2_9$ of the complex projective plane of KÜHNEL [95]. By the BREHM-KÜHNEL bound (Theorem 2.1, [42]), it has the minimal number of vertices that a non-spherical combinatorial 4-manifold can have.

Recently, SPARLA ([150], [151]) found a highly symmetric 12-vertex combinatorial $S^2 \times S^2$. In fact, LASSMANN and SPARLA [105] showed by a computer enumeration that there are exactly three combinatorially distinct centrally symmetric 12-vertex triangulations of $S^2 \times S^2$ that have a vertex-transitive cyclic group action. Using BISTELLAR, we were able to improve on the number of vertices and obtained several combinatorially distinct triangulations of $S^2 \times S^2$ with 11 vertices. These triangulations are minimal for $S^2 \times S^2$, since a 10-vertex triangulation would be 3-neighborly according to the following theorem, but the only 3-neighborly 4-manifolds with $n \leq 13$ vertices are $\mathbb{CP}^2_9$ and the boundary of the 5-simplex [96].

Theorem 2.11 (Kühnel [90, 4.1]) *If M is a combinatorial 4-manifold with n vertices, then*

$$\binom{n-4}{3} \geq 10\,(\chi(M) - 2), \tag{2.2}$$

with equality if and only if M is 3-neighborly.

As we did before with 3-manifolds, we form several connected sums of the combinatorial manifolds $\mathbb{C}\mathbf{P}^2_9$, $S^3\times S^1$, $S^3 \underline{\times}\, S^1$, and $S^2\times S^2$. By Lemma 2.9, we know, for example, that there are triangulations of $\mathbb{C}\mathbf{P}^2 \# \mathbb{C}\mathbf{P}^2$ and of $\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$ with $9+9-(4+1) = 13$ vertices (both manifolds are not homeomorphic to each other, and are also distinct from $S^2\times S^2$, which has the same homology), and of $\mathbb{C}\mathbf{P}^2 \# (\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2) \cong \mathbb{C}\mathbf{P}^2 \# (S^2\times S^2)$ with $9 + 11 - (4 + 1) = 15$ vertices. By applying BISTELLAR flips, we found triangulations of these manifolds with 12 respectively 13 vertices. For further connected sums see Table 2.8.

Manifold	Homology	n	n_{vt}	Reference
S^4	$(\mathbb{Z},0,0,0,\mathbb{Z})$	$\underline{6}$	$\underline{6}$	
$\mathbb{C}\mathbf{P}^2$	$(\mathbb{Z},0,\mathbb{Z},0,\mathbb{Z})$	$\underline{9}$	$\underline{9}$	[95], [96]
$S^3\times S^1$	$(\mathbb{Z},\mathbb{Z},0,\mathbb{Z},\mathbb{Z})$	$\underline{11}$	$\underline{11}$	[88]
$S^2\times S^2$	$(\mathbb{Z},0,\mathbb{Z}^2,0,\mathbb{Z})$	$\underline{11}$	$\underline{12}$	new, Ch. 4; [150], [151]
$S^3 \underline{\times}\, S^1$	$(\mathbb{Z},\mathbb{Z},0,\mathbb{Z}_2,0)$	12	$\underline{12}$	new
$\mathbb{C}\mathbf{P}^2 \# \mathbb{C}\mathbf{P}^2$	$(\mathbb{Z},0,\mathbb{Z}^2,0,\mathbb{Z})$	12	?	new
$\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2 = S^2 \underline{\times}\, S^2$	$(\mathbb{Z},0,\mathbb{Z}^2,0,\mathbb{Z})$	12	?	new, Ch. 3
$(S^2\times S^2)\#(S^2\times S^2)$	$(\mathbb{Z},0,\mathbb{Z}^4,0,\mathbb{Z})$	$\underline{12}$	$\underline{12}$	new
$\mathbb{C}\mathbf{P}^2 \# (\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2)$ $= \mathbb{C}\mathbf{P}^2 \# (S^2\times S^2)$	$(\mathbb{Z},0,\mathbb{Z}^3,0,\mathbb{Z})$	13	?	new
$\mathbb{C}\mathbf{P}^2 \# (S^2\times S^2)^{\#2}$	$(\mathbb{Z},0,\mathbb{Z}^5,0,\mathbb{Z})$	13	?	new
$(S^3\times S^1)\# \mathbb{C}\mathbf{P}^2$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z},\mathbb{Z},\mathbb{Z})$	14	?	new
$(S^3 \underline{\times}\, S^1)\# \mathbb{C}\mathbf{P}^2$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z},\mathbb{Z}_2,0)$	14	?	new
$(\mathbb{C}\mathbf{P}^2 \# \mathbb{C}\mathbf{P}^2)^{\#2}$	$(\mathbb{Z},0,\mathbb{Z}^4,0,\mathbb{Z})$	14	?	new
$(S^2\times S^2)^{\#3}$	$(\mathbb{Z},0,\mathbb{Z}^6,0,\mathbb{Z})$	14	?	new
$(S^3\times S^1)\#(S^3\times S^1)$	$(\mathbb{Z},\mathbb{Z}^2,0,\mathbb{Z}^2,\mathbb{Z})$	15	?	new
$(S^3 \underline{\times}\, S^1)\#(S^3 \underline{\times}\, S^1)$	$(\mathbb{Z},\mathbb{Z}^2,0,\mathbb{Z}\oplus\mathbb{Z}_2,0)$	15	?	new
$\sim(S^3 \underline{\times}\, S^1)\#(\mathbb{C}\mathbf{P}^2)^{\#5}$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z}^5,\mathbb{Z}_2,0)$	15	15	new
$(S^3 \underline{\times}\, S^1)\#(\mathbb{C}\mathbf{P}^2)^{\#5}$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z}^5,\mathbb{Z}_2,0)$	16	?	new
$(S^3\times S^1)\#(S^2\times S^2)$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z}^2,\mathbb{Z},\mathbb{Z})$	16	?	new
$(S^3 \underline{\times}\, S^1)\#(S^2\times S^2)$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z}^2,\mathbb{Z}_2,0)$	16	?	new
$\mathbb{R}\mathbf{P}^4$	$(\mathbb{Z},\mathbb{Z}_2,0,\mathbb{Z}_2,0)$	$\underline{16}$	?	new, Ch. 3
K3 surface	$(\mathbb{Z},0,\mathbb{Z}^{22},0,\mathbb{Z})$	$\underline{16}$	$\underline{16}$	[49]

Table 2.8: Combinatorial 4-manifolds with smallest known numbers of vertices $n \leq 16$ and n_{vt} with a vertex-transitive action (minimal if underlined).

Using MANIFOLD_VT, we found two vertex-transitive triangulations, ${}^4 12^{\,2}_{\,1}$ and ${}^4 12^{\,2}_{\,2}$, of $(S^2{\times}S^2)\#(S^2{\times}S^2)$ with 12 vertices. These triangulations are minimal by Theorem 2.11.

Corollary 2.12 *There are exactly two vertex-transitive minimal triangulations of the manifold $(S^2\times S^2)\#(S^2\times S^2)$ on 12 vertices.*

Explicit triangulation of the manifolds $\mathbb{C}\mathbf{P}^2_9$, $S^3 \times S^1$, and $S^2 \times S^2$ can be found in the literature (see [88], [95], and [151]), but no small triangulation of $S^3 \underline{\times} S^1$ has been described previously. By work of BREHM and KÜHNEL [42], non-simply connected manifolds of dimension $d \geq 3$, in particular, $S^3 \times S^1$ and $S^3 \underline{\times} S^1$, require at least $2d+3$ vertices for a combinatorial triangulation. With the exception of $\mathbb{R}\mathbf{P}^2$, which can be triangulated with 6 vertices, this lower bound also holds in dimension 2.

Theorem 2.13 (KÜHNEL [88]) *There is a combinatorial triangulation with transitive dihedral automorphism group on $2d+3$ vertices of*

- $S^{(d-1)} \times S^1$ *if d is even, and of*
- $S^{(d-1)} \underline{\times} S^1$ *if d is odd.*

Furthermore, if d is odd, then there is a combinatorial triangulation of $S^{(d-1)} \times S^1$ with dihedral group action on $2d+4$ vertices; see KÜHNEL and LASSMANN [100].

Conjecture 2.14 *The vertex-transitive combinatorial* KÜHNEL-LASSMANN *triangulation of $S^{(d-1)} \times S^1$ on $2d+4$ vertices is vertex-minimal if d is odd. If d is even, then there is a minimal triangulation of $S^{(d-1)} \underline{\times} S^1$ as well on $2d+4$ vertices.*

In dimension 2, the existence and the minimality of 8-vertex triangulations of the Klein bottle are well known. Hence, the first open case for the existence part of Conjecture 2.14 is: Does there exist a 12-vertex triangulation of $S^3 \underline{\times} S^1$?

Lemma 2.15 *There are triangulations of $S^{(d-1)} \underline{\times} S^1$ with $3d+3$ vertices.*

Proof: Let I be an interval, triangulated with 4 vertices. If we take the boundary of the d-simplex with $d+1$ vertices for the sphere $S^{(d-1)}$, then the product $S^{(d-1)} \times I$ has triangulations on $4(d+1)$ vertices. The boundary of $S^{(d-1)} \times I$ consists of two disjoint copies of the boundary of the d-simplex. By identifying the two boundary spheres (with taking care of the orientation), we obtain the wanted triangulation of $S^{(d-1)} \underline{\times} S^1$ with $3d+3$ vertices. $\square$

We used BISTELLAR to obtain 12-vertex triangulations of $S^3 \underline{\times} S^1$, and the program BISTELLAR_EQUIVALENT to show that the vertex-transitive combinatorial 4-manifolds, ${}^4 12^{54}_{1}$, ${}^4 14^{3}_{6}$, and ${}^4 14^{3}_{7}$, which we found with homology $(\mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z}_2, 0)$, are also triangulations of $S^3 \underline{\times} S^1$.

Corollary 2.16 *There exists a vertex-transitive 12-vertex triangulation of $S^3 \underline{\times} S^1$.*

From the (incomplete) list of 60 combinatorial 4-manifolds with a vertex-transitive automorphism group on $n \leq 15$ vertices that we found with MANIFOLD_VT, we were able to determine the homeomorphism type of all but one manifold, ${}^4 15^{4}_{1}$. This manifold has homology $(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^5, \mathbb{Z}_2, 0)$ and is a new example of a tight triangulation [102]. A possible reference manifold for ${}^4 15^{4}_{1}$ is $(S^3 \underline{\times} S^1) \# (\mathbb{C}\mathbf{P}^2)^{\# 5}$, which we triangulated with 16 vertices. But our attempts failed to obtain a 15-vertex triangulation of this manifold with BISTELLAR.

A further example of a tight 4-manifold is the 16-vertex triangulation of the K3-surface of CASELLA and KÜHNEL [49]. For the notion of tightness see [91].

2.4.6 5-Manifolds

Our enumeration with MANIFOLD_VT produced a 3-neighborly and thus simply connected combinatorial 5-manifold ${}^{5}13^{3}_{2}$ that has homology $H_* = (\mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z})$. According to the classification of all simply connected five-manifolds by Barden [16], there is only one simply connected five-manifold with this homology, which he denoted by X_{-1}.

Theorem 2.17 *There is a 3-neighborly triangulation of the simply connected 5-manifold X_{-1} with homology $H_* = (\mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z})$. The triangulation has a vertex-transitive action of the group $13{:}3$ and has f-vector $f = (\underline{13}, \underline{78}, \underline{286}, 533, 468, 156)$.*

Using BISTELLAR, we obtained two combinatorially distinct minimal and tight triangulations of $S^3 \times S^2$ with 12 vertices (see Chapter 4 and [102]). These are the first known examples of non-spherical combinatorial 5-manifolds that attain the Brehm-Kühnel lower bound of 12 vertices. As starting triangulations for BISTELLAR we used the centrally symmetric triangulation ${}^{5}14^{3}_{8}$ and a product triangulation of $S^3 \times S^2$ on 20 vertices. There are three further combinatorial manifolds with a vertex-transitive dihedral group action on 14 vertices , ${}^{5}14^{3}_{13}$, ${}^{5}14^{3}_{14}$, and ${}^{5}14^{3}_{15}$, which have the homology of $S^3 \times S^2$. We applied BISTELLAR flips to these triangulations as well and again obtained 3-neighborly triangulations on 12 vertices. Hence, the examples ${}^{5}14^{3}_{13}$, ${}^{5}14^{3}_{14}$, and ${}^{5}14^{3}_{15}$ are also simply connected. By the classification of Barden, there are precisely two simply connected 5-manifolds with the homology of $S^3 \times S^2$, namely $M_\infty = S^3 \times S^2$ with trivial and X_∞ with non-vanishing second Stiefel-Whitney class. Now, the centrally symmetric example ${}^{5}14^{3}_{8}$ is embedded in the 6-dimensional boundary complex BdC^{Δ}_{7} of the 7-dimensional crosspolytope C^{Δ}_{7} and therefore its Stiefel-Whitney numbers are all zero (for details see Chapter 4). But the examples ${}^{5}14^{3}_{13}$, ${}^{5}14^{3}_{14}$, and ${}^{5}14^{3}_{15}$ are 2-neighborly and thus do not embed into BdC^{Δ}_{7}. Hence, all we can say about these three examples is that they are either $S^3 \times S^2$ or X_∞.

Manifold	Homology	n	n_{vt}	Reference
S^5	$(\mathbb{Z}, 0, 0, 0, 0, \mathbb{Z})$	$\underline{7}$	$\underline{7}$	
$S^3 \times S^2$	$(\mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z})$	$\underline{12}$	$\underline{14}$	new, Ch. 4
X_{-1}	$(\mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z})$	13	$\underline{13}$	new
$S^4 \underline{\times} S^1$	$(\mathbb{Z}, \mathbb{Z}, 0, 0, \mathbb{Z}_2, 0)$	$\underline{13}$	$\underline{13}$	[88]
$S^4 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, 0, 0, \mathbb{Z}, \mathbb{Z})$	14	$\underline{14}$	[100]

Table 2.9: Combinatorial 5-manifolds with smallest known numbers of vertices $n \leq 16$ and n_{vt} with a vertex-transitive action (minimal if underlined).

2.4.7 6-Manifolds

We found two vertex-transitive combinatorial 6-manifolds, ${}^{6}15^{7}_{1}$, and ${}^{6}15^{7}_{2}$, which have the homology of $S^3 \times S^3$. An application of BISTELLAR flips to these triangulations yielded in both cases minimal and tight triangulations with 13 vertices that attain the Brehm-Kühnel lower bound in dimension 6 (see Chapter 4). Both 13-vertex triangulations are 4-neighborly, and thus the manifolds are 2-connected. It follows from work of Žubr [165] that any 2-connected 6-manifold is completely determined by its Euler characteristic (cf. Chapter 4). Hence, ${}^{6}15^{\,7}_{\,1}$ and ${}^{6}15^{\,7}_{\,2}$ are triangulations of $S^3 \times S^3$.

Manifold	n	n_{vt}	Reference
S^6	$\underline{8}$	$\underline{8}$	
$S^3 \times S^3$	$\underline{13}$	15	new
$S^5 \times S^1$	$\underline{15}$	$\underline{15}$	[88]

Table 2.10: Combinatorial 6-manifolds with smallest known numbers of vertices $n \leq 16$ and n_{vt} with a vertex-transitive action (minimal if underlined).

2.4.8 7-Manifolds

Instead of enumerating *all* symmetric combinatorial manifolds for a particular transitive group action, we can as well enumerate only those manifolds that have specific properties. In Chapter 4, we will consider vertex-transitive combinatorial manifolds that are centrally symmetric. This additional requirement leads to a restriction of the number of valid orbits, which we have to deal with, and thus enables us to enumerate manifolds with this property for larger numbers of vertices.

As examples, we obtained combinatorial 7-manifolds with 18 vertices that have the homology of $S^5 \times S^2$ respectively of $S^4 \times S^3$.

Manifold	n	n_{vt}	Reference
S^7	$\underline{9}$	$\underline{9}$	
$S^6 \mathbin{\underline{\times}} S^1$	$\underline{17}$	$\underline{17}$	[88]
$S^6 \times S^1$	18	18	[100]
$\sim S^5 \times S^2$	18	18	new, Ch. 4
$\sim S^4 \times S^3$	18	18	new, Ch. 4

Table 2.11: Combinatorial 7-manifolds with smallest known numbers of vertices $n \leq 20$ and n_{vt} with a vertex-transitive action (minimal if underlined).

2.4.9 8-Manifolds

With the exception of 8-manifolds that have the cyclic group ${}^{8}15^{\,1}$ as vertex-transitive automorphism group, we were able to enumerate all combinatorial 8-manifolds with a vertex-transitive automorphism group on $n \leq 15$ vertices (cf. Table 2.13).

According to the Brehm-Kühnel lower bound (Theorem 2.1), a non-spherical combinatorial 8-manifold has at least 15 vertices. It is a 'manifold like a projective plane' if $n = 15$. Such an example with a vertex-transitive A_5-action and two further non-transitive examples were found by Brehm and Kühnel [43]. They proved that their vertex-transitive example is unique with an A_5-action. We show, that it is, in fact, the only non-spherical vertex-transitive example on 15 vertices.

Proposition 2.18 *There is exactly one* 15*-vertex triangulation of a non-spherical vertex-transitive combinatorial* 8*-manifold.*

Proof: A non-spherical combinatorial 8-manifold M with 15 vertices is a 'manifold like a projective plane' and has Euler characteristic $\chi = 3$. Furthermore it is 5-neighborly, which, together with the Euler characteristic, completely determines the f-vector

$$f = (\underline{15}, \underline{105}, \underline{455}, \underline{1365}, \underline{3003}, 4515, 4230, 2205, 490)$$

by the Dehn-Sommerville-equations (see the discussion in [43]).

Our example ${}^{8}15^{5}_{1}$ with a vertex-transitive A_5-action coincides with the example M^8_{15} of Brehm and Kühnel. By our enumeration, there is no further M with a vertex-transitive action of a group G of order $|G| \geq 30$. The only group of order 15 with a vertex-transitive group action on 15 vertices is the cyclic group with action 15^{1}. But Brehm showed (personal communication) that a 'manifold like a projective plane' cannot have a cyclic action on 15 vertices. The cyclic action 15^{1} has 335 orbits of 9-tuples and, by complementarity, also 335 orbits of 6-tuples, 333 of size 15 and 2 of size 5 in both cases. In order to compose an 8-manifold with 490 8-simplices, both small orbits of 9-tuples of size 5 have to be present. These orbits are generated by the simplices 123678 11 12 13 and 124679 11 12 14. On the other hand, the two orbits of 6-tuples of size 5 cannot be faces of M as $f_5 = 4515$. But these two orbits are generated by the simplices 1267 11 12 and 1368 11 13, and are thus included in the above orbits. Contradiction. □

Manifold	n	n_{vt}	Reference
S^8	$\underline{10}$	$\underline{10}$	
$\sim\mathbb{HP}^2$	$\underline{15}$	$\underline{15}$	[43]
$S^7 \times S^1$	$\underline{19}$	$\underline{19}$	[88]
$\sim S^5 \times S^3$	20	20	new, Ch. 4
$\sim S^4 \times S^4$	20	20	new, Ch. 4

Table 2.12: Combinatorial 8-manifolds with smallest known numbers of vertices $n \leq 20$ and n_{vt} with a vertex-transitive action (minimal if underlined).

2.4.10 And beyond 15 vertices?

So far we have not mentioned computation time nor have we said anything about the range of parameters for which an enumeration may be continued. The two main obstacles to overcome for the enumeration of vertex-transitive combinatorial manifolds on more vertices and in higher dimensions are:

- *Computing the result vectors of the matrices in Step 4 of the enumeration algorithm.* The 'most difficult' matrix, which we were able to process, was for the cyclic action $\mathbb{Z}_{13}$ in dimension 5. The size of this matrix is 132×99, representing 132 orbits of dimension 5 and 99 orbits of dimension 4. The computation time for our C-program for Step 4 was about an hour on a CRAY J932. Due to the fact that we have chosen a straight forward implementation for computing the result vectors in order to keep the implementation time down, all combinations of row vectors are kept in the main memory at any time. For the above matrix, this amounted to 5.7 GB at the peak of the computation. An improved implementation could certainly do better on that. But on the other hand, the matrices to process grow quickly in size for actions of groups of small order, such as cyclic group actions;

- *The tests that we perform in Steps 5 and 6.* Testing equivalence in Step 6 is very expensive for higher-dimensional complexes. To avoid this in part, we could perform additional checks for the links of higher-dimensional simplices in Step 5 to filter out further complexes that are not manifolds. Or we could compute additional invariants, for example not only the Altshuler-Steinberg determinant of the complex itself but also the determinant of its vertex-link. These tests, on the other hand, require additional processing time. So it is a matter of experiments to find out good strategies. Certainly, it would be helpful to compute the homology of a complex at an early stage. But the homology program by Heckenbach that we use is in C, and to export the complexes from GAP to compute their homology and then process them further with GAP causes some effort. An alternative could be to implement the complete algorithm in C. Our experience with Step 4 is that clever implementations in C can be up to '1000 times faster' than the same code in GAP.

Although a complete enumeration of all vertex-transitive combinatorial manifolds seems to be hard for 14 vertices and out of reach for 15 vertices, we can restrict our interest to (pseudo)manifolds that have a large transitive automorphism group (Chapter 3) or that have additional properties, as being centrally symmetric (Chapter 4) or that are neighborly or regular maps (Chapter 5). In these cases, the restrictions lead to smaller matrices that are computable for a wider range of parameters and with fewer resulting complexes that have to be examined.

2.5 Tables of Manifolds

Table 2.13: 2-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Or.	Gen. g	f-vector (f_1, f_2)	Automorphism group	Type	List of orbits	Remarks
4	+	0	$(\underline{6},\underline{4})$	S_4	$^{2}4^{5}_{1}$	123_{4}	tetrahedron, regular
6	+	0	(12,8)	$[2^3]S_3 = 2wrS_3$	$^{2}6^{11}_{1}$	123_{8}	octahedron, regular
	−	1	$(\underline{15},10)$	A_5	$^{2}6^{12}_{1}$	123_{10}	$\mathbb{R}\mathbf{P}^{2}_{6}$, regular
7	+	1	$(\underline{21},14)$	$7:6$	$^{2}7^{4}_{1}$	124_{14}	Möbius' torus, [55], [88, M^2], [99], [100, M^2_1], [125]
8	+	1	(24,16)	$t8n15(32)$	$^{2}8^{15}_{1}$	123_{16}	[100, $M^2_1(8)$]
9	+	1	(27,18)	D_9	$^{2}9^{3}_{1}$	124_{18}	[100, $M^2_1(9)$]
				$\mathbb{Z}_3^2 : D_6$	$^{2}9^{18}_{1}$	136_{18}	regular, [3], [35], [54], [161]
	−	5	$(\underline{36},24)$	$S_3 \times \mathbb{Z}_3$	$^{2}9^{4}_{1}$	124_{18} 138_{6}	[8, $N(9,2)$], [53]
10	+	1	(30,20)	D_{10}	$^{2}10^{3}_{2}$	124_{20}	[100, $M^2_1(10)$]
	−	2	(30,20)	D_{10}	$^{2}10^{3}_{1}$	123_{10} 137_{10}	Klein bottle
		7	$(\underline{45},30)$	A_5	$^{2}10^{7}_{1}$	123_{30}	[8, $N(10,13)$], [34, 380, (i)], [53]
11	+	1	(33,22)	D_{11}	$^{2}11^{2}_{1}$	124_{22}	[100, $M^2_1(11)$]
12	+	0	(30,20)	$[2]A_5$	$^{2}12^{76}_{1}$	126_{20}	icosahedron, regular
		1	(36,24)	D_{12}	$^{2}12^{12}_{1}$	124_{24}	[100, $M^2_1(12)$]
					$^{2}12^{12}_{2}$	125_{24}	
				$D_4 \times S_3$	$^{2}12^{28}_{1}$	126_{24}	
				$S_4 \times S_3$	$^{2}12^{83}_{1}$	123_{24}	regular, [3], [35], [54], [161]
		2	(42,28)	$\mathbb{Z}_{12}$	$^{2}12^{1}_{3}$	123_{12} 137_{12} 159_{4}	
					$^{2}12^{1}_{10}$	126_{12} 127_{12} 159_{4}	
				D_6	$^{2}12^{3}_{3}$	124_{12} 145_{12} 159_{4}	
		3	(48,32)	A_4	$^{2}12^{4}_{6}$	124_{12} 127_{12} 138_{4} $16\,11_{4}$	
				$t12n8(24) = S_4$	$^{2}12^{8}_{1}$	124_{24} 137_{8}	
				$t12n113(192)$	$^{2}12^{113}_{1}$	124_{32}	Dyck's regular map, [24], [32], [38], [58], [59], [146], [161]
		4	(54,36)	D_6	$^{2}12^{3}_{1}$	124_{12} 137_{12} 145_{12}	
					$^{2}12^{3}_{2}$	124_{12} 137_{12} $14\,12_{12}$	
				A_4	$^{2}12^{4}_{4}$	123_{12} 125_{12} 138_{4} 159_{4} $16\,11_{4}$	
				$A_4(12) \times \mathbb{Z}_2$	$^{2}12^{6}_{1}$	123_{24} 125_{12}	
		5	(60,40)	$\mathbb{Z}_{12}$	$^{2}12^{1}_{1}$	123_{12} 136_{12} 148_{12} 159_{4}	
					$^{2}12^{1}_{2}$	123_{12} 136_{12} 149_{12} 159_{4}	
					$^{2}12^{1}_{6}$	124_{12} 126_{12} 136_{12} 159_{4}	
				$\frac{1}{2}[3:2]4$	$^{2}12^{5}_{1}$	123_{12} 124_{12} 145_{12} 159_{4}	
		6	$(\underline{66},44)$	A_4	$^{2}12^{4}_{5}$	124_{12} 127_{12} 138_{4} 159_{4} $15\,11_{12}$	[6, N^{12}_{58}]
	−	4	(42,28)	A_4	$^{2}12^{4}_{2}$	123_{12} 124_{12} 138_{4}	
		8	(54,36)	$\mathbb{Z}_{12}$	$^{2}12^{1}_{4}$	124_{12} 125_{12} 137_{12}	
					$^{2}12^{1}_{5}$	124_{12} 125_{12} 139_{12}	
					$^{2}12^{1}_{7}$	124_{12} 127_{12} $13\,10_{12}$	
					$^{2}12^{1}_{8}$	125_{12} 127_{12} 148_{12}	
					$^{2}12^{1}_{9}$	126_{12} 127_{12} 135_{12}	

Table 2.13: 2-dimensional combinatorial manifolds with vertex-transitive automorphism group.

				$A_4(6) \times \mathbb{Z}_2$	${}^2 12^{7}_{1}$	124_{24} 137_{12}	
				$t12n9(24) = S_4$	${}^2 12^{9}_{1}$	123_{24} 137_{12}	
		10	(60,40)	A_4	${}^2 12^{4}_{1}$	123_{12} 124_{12} 137_{12} 159_{4}	
					${}^2 12^{4}_{3}$	123_{12} 125_{12} 137_{12} $16\,11_{4}$	
				$t12n8(24) = S_4$	${}^2 12^{8}_{2}$	124_{24} $13\,10_{12}$ $15\,10_{4}$	
13	+	1	(39,26)	D_{13}	${}^2 13^{2}_{1}$	124_{26}	[100, $M^2_1(13)$]
				$13:6$	${}^2 13^{5}_{1}$	125_{26}	
	−	15	$(\underline{78},52)$	$\mathbb{Z}_{13}$	${}^2 13^{1}_{1}$	123_{13} 137_{13} 148_{13} 149_{13}	
				$13:3$	${}^2 13^{3}_{1}$	123_{39} 138_{13}	
14	+	1	(42,28)	D_{14}	${}^2 14^{3}_{2}$	124_{28}	[100, $M^2_1(14)$]
					${}^2 14^{3}_{3}$	125_{28}	
		8	(84,56)	$2[\frac{1}{2}]7:6$	${}^2 14^{4}_{1}$	123_{42} 137_{14}	
					${}^2 14^{4}_{2}$	124_{42} 137_{14}	
					${}^2 14^{4}_{3}$	124_{42} $13\,11_{14}$	
				$7:3 \times \mathbb{Z}_2$	${}^2 14^{5}_{1}$	123_{42} 137_{14}	
					${}^2 14^{5}_{2}$	124_{42} $13\,11_{14}$	
	−	2	(42,28)	D_{14}	${}^2 14^{3}_{1}$	123_{14} 139_{14}	Klein bottle
		16	(84,56)	$\mathbb{Z}_{14}$	${}^2 14^{1}_{1}$	123_{14} 136_{14} 149_{14} 159_{14}	
					${}^2 14^{1}_{2}$	123_{14} 136_{14} $14\,10_{14}$ 159_{14}	
					${}^2 14^{1}_{3}$	124_{14} 126_{14} 137_{14} 149_{14}	
					${}^2 14^{1}_{4}$	124_{14} 127_{14} $13\,12_{14}$ 159_{14}	
					${}^2 14^{1}_{5}$	124_{14} $12\,10_{14}$ $13\,12_{14}$ 159_{14}	
15	+	1	(45,30)	D_{15}	${}^2 15^{2}_{1}$	124_{30}	[100, $M^2_1(15)$]
					${}^2 15^{2}_{2}$	127_{30}	
				$D_5 \times S_3$	${}^2 15^{7}_{1}$	125_{30}	
					${}^2 15^{7}_{2}$	126_{30}	
		6	(75,50)	$[5^2:2]S_3$	${}^2 15^{18}_{1}$	123_{50}	regular, [161]
	−	7	(60,40)	A_5	${}^2 15^{5}_{1}$	125_{30} 179_{10}	
		12	(75,50)	$\mathbb{Z}_{15}$	${}^2 15^{1}_{1}$	123_{15} 136_{15} $14\,10_{15}$ $16\,11_{5}$	
					${}^2 15^{1}_{2}$	123_{15} 137_{15} $15\,10_{15}$ $16\,11_{5}$	
					${}^2 15^{1}_{3}$	123_{15} 137_{15} $15\,11_{15}$ $16\,11_{5}$	
					${}^2 15^{1}_{6}$	125_{15} 126_{15} $14\,10_{15}$ $16\,11_{5}$	
				$\mathbb{Z}_5 \times S_3$	${}^2 15^{4}_{2}$	123_{30} 138_{15} $16\,11_{5}$	
		17	(90,60)	$\mathbb{Z}_{15}$	${}^2 15^{1}_{4}$	123_{15} 138_{15} 149_{15} $14\,10_{15}$	
					${}^2 15^{1}_{5}$	123_{15} 138_{15} $14\,10_{15}$ $14\,11_{15}$	
				$D_5 \times \mathbb{Z}_3$	${}^2 15^{3}_{1}$	123_{30} 136_{30}	
					${}^2 15^{3}_{2}$	123_{30} 139_{15} 147_{15}	
				$\mathbb{Z}_5 \times S_3$	${}^2 15^{4}_{1}$	123_{30} 136_{30}	
				S_5	${}^2 15^{10}_{1}$	$12\,10_{60}$	

Table 2.14: 3-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Top. type	f-vector (f_1, f_2, f_3)	Automorphism group	Comb. type	List of orbits	Remarks
5	S^3	$(\underline{10},\underline{10},\underline{5})$	S_5	${}^3 5^{5}_{1}$	1234_{5}	$Bd\,\Delta_4$, regular
6	S^3	$(\underline{15},18,9)$	$[S_3^2]2 = S_3 wr 2$	${}^3 6^{13}_{1}$	1234_{9}	$BdC_4(6) = 3*3$, [98, I_6]
7	S^3	$(\underline{21},28,14)$	D_7	${}^3 7^{2}_{1}$	1234_{7} 1245_{7}	$BdC_4(7)$, [98, I_7]
8	S^3	$(24,32,16)$	$[2^4]S_4$	${}^3 8^{44}_{1}$	1234_{16}	$BdC_4^{\Delta} = BdBiC(1,3;8) = 4*4$, regular, nncs
		$(\underline{28},40,20)$	D_8	${}^3 8^{6}_{1}$	1234_{8} 1245_{8} 1256_{4}	$BdC_4(8)$, [98, I_8]
9	S^3	$(\underline{36},54,27)$	D_9	${}^3 9^{3}_{1}$	1234_{9} 1245_{9} 1256_{9}	$BdC_4(9)$, [98, I_9]
	$S^2 \underline{\times} S^1$	$(\underline{36},54,27)$	D_9	${}^3 9^{3}_{2}$	1235_{18} 1245_{9}	minimal, tight, [10, N^9_{51}], [88, M^3], [98, II_9], [100, M^3_2], [102], [158]
10	S^3	$(35,50,25)$	$[D_5^2]2$	${}^3 10^{21}_{1}$	1234_{25}	$BdBiC(1,4;10) = 5*5$
		$(40,60,30)$	$\frac{1}{2}[5:4]2$	${}^3 10^{4}_{1}$	1235_{20} 1245_{5} 1289_{5}	
			$S_5 \times \mathbb{Z}_2$	${}^3 10^{22}_{1}$	1234_{30}	$BdBiC(1,3;10)$, nncs, [66, p. 116]
		$(\underline{45},70,35)$	$\mathbb{Z}_{10}$	${}^3 10^{1}_{1}$	1235_{10} 1236_{10} 1246_{10} 1368_{5}	[4, N^{10}_{3574}], [98, 1_{10}]
			D_{10}	${}^3 10^{3}_{1}$	1234_{10} 1245_{10} 1256_{10} 1267_{5}	$BdC_4(10)$, [4, N^{10}_4] , [98, I_{10}]
			$\frac{1}{2}[5:4]2$	${}^3 10^{4}_{2}$	1245_{5} 1246_{20} 1267_{10}	[4, N^{10}_{425}]
	$S^2 \times S^1$	$(40,60,30)$	D_{10}	${}^3 10^{3}_{2}$	1235_{20} 1245_{10}	minimal, [100, $M^3_2(10)$], [158]
		$(\underline{45},70,35)$	$\mathbb{Z}_{10}$	${}^3 10^{1}_{2}$	1236_{10} 1237_{10} 1257_{10} 1368_{5}	minimal, [4, N^{10}_{3611}], [98, 2_{10}]
	$S^2 \underline{\times} S^1$	$(\underline{45},70,35)$	D_{10}	${}^3 10^{3}_{3}$	1236_{20} 1256_{10} 1368_{5}	[4, N^{10}_{3629}], [98, $\tilde{\mathrm{II}}_{10}$]
				${}^3 10^{3}_{4}$	1246_{20} 1249_{10} 1267_{5}	[4, N^{10}_{3631}], [98, II_{10}]
11	S^3	$(44,66,33)$	D_{11}	${}^3 11^{2}_{2}$	1234_{11} 1245_{11} 1259_{11}	$BdBiC(1,3;11)$
		$(\underline{55},88,44)$	$\mathbb{Z}_{11}$	${}^3 11^{1}_{1}$	1234_{11} 1248_{11} 1258_{11} $125\,10_{11}$	[98, 1_{11}]
			D_{11}	${}^3 11^{2}_{1}$	1234_{11} 1245_{11} 1256_{11} 1267_{11}	$BdC_4(11)$, [98, I_{11}]
	$S^2 \underline{\times} S^1$	$(44,66,33)$	D_{11}	${}^3 11^{2}_{4}$	1235_{22} 1245_{11}	[100, $M^3_2(11)$]
		$(\underline{55},88,44)$	$\mathbb{Z}_{11}$	${}^3 11^{1}_{2}$	1235_{11} 1239_{11} 1246_{11} 1256_{11}	[98, 2_{11}]
			D_{11}	${}^3 11^{2}_{3}$	1234_{11} 1248_{22} 1268_{11}	[98, II_{11}]
12	S^3	$(48,72,36)$	$[S_3^2]D_4 = D_6 wr 2$	${}^3 12^{125}_{1}$	1234_{36}	$BdBiC(1,5;12) = 6*6$
		$(60,96,48)$	$\mathbb{Z}_{12}$	${}^3 12^{1}_{1}$	1234_{12} 1246_{12} $126\,11_{12}$ $135\,10_{12}$	nncs
			D_6	${}^3 12^{3}_{4}$	1235_{12} $123\,12_{3}$ 1246_{12} $124\,11_{6}$ 1256_{6} 1368_{6} $1\,10\,11\,12_{3}$	
				${}^3 12^{3}_{5}$	1236_{12} $123\,12_{3}$ 1245_{12} $124\,11_{6}$ 1256_{6} 1368_{6} $1\,10\,11\,12_{3}$	
			$t12n13(24)$	${}^3 12^{13}_{1}$	1235_{24} 1245_{12} $124\,11_{12}$	nncs
		$(\underline{66},108,54)$	D_{12}	${}^3 12^{12}_{1}$	1234_{12} 1245_{12} 1256_{12} 1267_{12} 1278_{6}	$BdC_4(12)$, [98, I_{12}]

Table 2.14: 3-dimensional combinatorial manifolds with vertex-transitive automorphism group.

$S^2 \times S^1$	(48,72,36)	D_{12}	$^{3}12^{12}_{2}$	1235_{24} 1245_{12}	[100, $M_2^3(12)$]
	(54,84,42)	$S_3 \times \mathbb{Z}_2^2$	$^{3}12^{10}_{1}$	1234_{12} 1236_{24} 1458_{6}	
		$S_3 \times \mathbb{Z}_4$	$^{3}12^{11}_{1}$	1237_{24} 1238_{12} 1379_{6}	
		D_{12}	$^{3}12^{12}_{3}$	1237_{24} 1267_{12} 1379_{6}	
			$^{3}12^{12}_{4}$	1245_{12} $124\,10_{24}$ 1379_{6}	
		$t12n13(24)$	$^{3}12^{13}_{3}$	1237_{24} 1267_{12} 1379_{6}	
	(60,96,48)	$\mathbb{Z}_{12}$	$^{3}12^{1}_{3}$	1235_{12} $123\,10_{12}$ 1246_{12} 1256_{12}	
	($\underline{66}$,108,54)	$\mathbb{Z}_{12}$	$^{3}12^{1}_{2}$	1234_{12} 1248_{12} 1268_{12} $126\,11_{12}$ 1379_{6}	[98, 1_{12}]
			$^{3}12^{1}_{4}$	1236_{12} 1239_{12} 1256_{12} $128\,10_{12}$ 1379_{6}	[98, 2_{12}]
			$^{3}12^{1}_{8}$	1247_{12} $124\,11_{12}$ 1257_{12} $125\,11_{12}$ 1379_{6}	[98, 6_{12}]
			$^{3}12^{1}_{9}$	1248_{12} $124\,10_{12}$ 1258_{12} $125\,10_{12}$ 1379_{6}	[98, 7_{12}]
		$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	$^{3}12^{2}_{2}$	1245_{12} 1247_{12} 1258_{12} 1278_{6} $13\,10\,12_{6}$ 1458_{6}	
		$t12n8(24) = S_4$	$^{3}12^{8}_{1}$	1246_{12} 1249_{12} $127\,10_{24}$ $128\,10_{6}$	
			$^{3}12^{8}_{2}$	1249_{12} $124\,10_{24}$ 1279_{12} $128\,10_{6}$	
		$t12n13(24)$	$^{3}12^{13}_{5}$	1245_{12} $124\,10_{24}$ $125\,10_{12}$ 1379_{6}	
			$^{3}12^{13}_{6}$	1245_{12} $124\,11_{12}$ $125\,11_{24}$ 1379_{6}	
$(S^2 \times S^1)^{\#2}$	($\underline{66}$,108,54)	$\mathbb{Z}_{12}$	$^{3}12^{1}_{7}$	1247_{12} 1248_{12} 1278_{6} 1357_{12} 1369_{12}	[98, 5_{12}]
			$^{3}12^{1}_{10}$	1248_{12} $124\,10_{12}$ 1278_{6} $127\,10_{12}$ 1357_{12}	[98, 8_{12}]
		$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	$^{3}12^{2}_{1}$	1245_{12} 1247_{12} 1256_{12} $127\,10_{12}$ $13\,10\,12_{6}$	
			$^{3}12^{2}_{3}$	1245_{12} $124\,10_{12}$ 1256_{12} $139\,12_{12}$ $13\,10\,12_{6}$	
		$t12n13(24)$	$^{3}12^{13}_{4}$	1245_{12} 1248_{24} 1278_{6} 1357_{12}	
$\mathbb{R}\mathbf{P}^3$	(60,96,48)	$t12n13(24)$	$^{3}12^{13}_{2}$	1235_{24} $125\,10_{12}$ $12\,10\,11_{12}$	24-cell/$\mathbb{Z}_2$, [94]
$S^2 \underline{\times} S^1$	(54,84,42)	$D_4 \times S_3$	$^{3}12^{28}_{1}$	1258_{24} $125\,10_{12}$ 1278_{6}	
		$S_4 \times S_3$	$^{3}12^{83}_{2}$	1247_{24} $124\,11_{18}$	
	(60,96,48)	$t12n54(96)$	$^{3}12^{54}_{1}$	$124\,10_{48}$	
	($\underline{66}$,108,54)	$\mathbb{Z}_{12}$	$^{3}12^{1}_{5}$	1245_{12} 1247_{12} $125\,11_{12}$ 1278_{6} $128\,10_{12}$	[98, 3_{12}]
			$^{3}12^{1}_{6}$	1245_{12} 1248_{12} $125\,11_{12}$ 1278_{6} $127\,10_{12}$	[98, 4_{12}]
		D_6	$^{3}12^{3}_{1}$	1234_{12} 1239_{12} 1246_{12} 1346_{6} 1357_{6} $167\,12_{3}$ $1\,10\,11\,12_{3}$	
			$^{3}12^{3}_{2}$	1234_{12} $123\,10_{12}$ 1247_{6} 1256_{6} 1267_{12} $167\,12_{3}$ $1\,10\,11\,12_{3}$	
			$^{3}12^{3}_{3}$	1235_{12} $123\,10_{12}$ 1346_{6} 1357_{6} 1367_{12} $167\,12_{3}$ $1\,10\,11\,12_{3}$	
		$\frac{1}{2}[3:2]4$	$^{3}12^{5}_{1}$	1236_{12} 1237_{12} 1278_{3} 1367_{12} $145\,10_{12}$ $147\,10_{3}$	
			$^{3}12^{5}_{2}$	1236_{12} 1237_{12} 1278_{3} $136\,12_{12}$ $137\,12_{12}$ $147\,10_{3}$	
		$S_4 \times S_3$	$^{3}12^{83}_{1}$	1246_{36} $124\,11_{18}$	[98, II_{12}]

Table 2.14: 3-dimensional combinatorial manifolds with vertex-transitive automorphism group.

13	S^3	(52,78,39)	$13:4$	$^3 13^4_1$	1234_{26} $124\,12_{13}$	$BdBiC(1,5;13)$
		(65,104,52)	$\mathbb{Z}_{13}$	$^3 13^1_1$	1234_{13} 1247_{13} $127\,12_{13}$ 1369_{13}	
			D_{13}	$^3 13^2_2$	1234_{13} 1245_{13} 1256_{13} $126\,10_{13}$	$BdBiC(1,3;13)$
		($\underline{78}$,130,65)	$\mathbb{Z}_{13}$	$^3 13^1_3$	1234_{13} 1248_{13} $126\,10_{13}$ $126\,12_{13}$ $128\,10_{13}$	[98, 2_{13}]
				$^3 13^1_5$	1235_{13} 1236_{13} 1249_{13} 1269_{13} 1358_{13}	[98, 4_{13}]
			D_{13}	$^3 13^2_1$	1234_{13} 1245_{13} 1256_{13} 1267_{13} 1278_{13}	$BdC_4(13)$, [98, I_{13}]
	$S^2 \underline{\times} S^1$	(52,78,39)	D_{13}	$^3 13^2_6$	1235_{26} 1245_{13}	[100, $M^3_2(13)$]
		(65,104,52)	$\mathbb{Z}_{13}$	$^3 13^1_7$	1235_{13} $123\,11_{13}$ 1246_{13} 1256_{13}	
			D_{13}	$^3 13^2_3$	1234_{13} 1248_{26} $137\,10_{13}$	
				$^3 13^2_4$	1234_{13} 1249_{26} 1279_{13}	
		($\underline{78}$,130,65)	$\mathbb{Z}_{13}$	$^3 13^1_2$	1234_{13} 1248_{13} 1268_{13} $126\,12_{13}$ 1379_{13}	[98, 1_{13}]
				$^3 13^1_4$	1234_{13} 1249_{13} 1267_{13} $126\,12_{13}$ $127\,10_{13}$	[98, 3_{13}]
				$^3 13^1_6$	1235_{13} $123\,10_{13}$ 1246_{13} 1257_{13} 1267_{13}	[98, 5_{13}]
				$^3 13^1_8$	1236_{13} 1237_{13} 1257_{13} $1\underline{3}68_{13}$ 1379_{13}	[98, 6_{13}]
			D_{13}	$^3 13^2_5$	1234_{13} $124\,12_{13}$ $136\,10_{26}$ $137\,10_{13}$	[98, II_{13}]
14	S^3	(63,98,49)	D_{14}	$^3 14^3_4$	1234_{14} 1245_{14} $125\,12_{14}$ $148\,11_7$	$BdBiC(1,4;14)$
			$[D_7^2]2 = D_7 wr 2$	$^3 14^{20}_1$	1234_{49}	$BdBiC(1,6;14) = 7*7$
		(70,112,56)	D_{14}	$^3 14^3_2$	1234_{14} 1245_{14} 1256_{14} $126\,11_{14}$	$BdBiC(1,3;14)$
		(77,126,63)	$\mathbb{Z}_{14}$	$^3 14^1_2$	1234_{14} 1246_{14} $126\,13_{14}$ $135\,12_{14}$ $138\,10_7$	
				$^3 14^1_3$	1234_{14} 1248_{14} 1258_{14} $125\,13_{14}$ $148\,11_7$	
				$^3 14^1_{13}$	1235_{14} 1236_{14} 1246_{14} 1368_{14} $138\,10_7$	
				$^3 14^1_{28}$	1237_{14} 1238_{14} 1268_{14} 1357_{14} $138\,10_7$	
		(84,140,70)	$\mathbb{Z}_{14}$	$^3 14^1_1$	1234_{14} 1245_{14} $125\,10_{14}$ $126\,10_{14}$ $126\,12_{14}$	nncs
			D_7	$^3 14^2_1$	1234_7 1236_{14} $124\,13_7$ 1267_{14} 1278_7 1289_7 $145\,14_7$ $167\,14_7$	
				$^3 14^2_3$	1234_7 1237_{14} $124\,12_{14}$ 1256_7 $126\,11_7$ $127\,10_7$ $147\,10_7$ $15\,10\,14_7$	
		($\underline{91}$,154,77)	$\mathbb{Z}_{14}$	$^3 14^1_4$	1234_{14} 1248_{14} 1268_{14} $126\,13_{14}$ 1357_{14} $138\,10_7$	[98, 1_{14}]
				$^3 14^1_7$	1234_{14} 1248_{14} $126\,11_{14}$ $126\,13_{14}$ 1289_7 $129\,11_{14}$	[98, 7_{14}]
				$^3 14^1_8$	1234_{14} 1248_{14} $128\,13_{14}$ 1357_{14} $138\,10_7$ $138\,11_{14}$	[98, 4_{14}]
				$^3 14^1_{11}$	1234_{14} 1249_{14} 1269_{14} $126\,13_{14}$ $147\,10_{14}$ $148\,11_7$	[98, 8_{14}]
				$^3 14^1_{14}$	1235_{14} 1238_{14} 1246_{14} 1257_{14} 1268_{14} $138\,10_7$	[98, 9_{14}]
				$^3 14^1_{17}$	1236_{14} 1237_{14} 1247_{14} 1248_{14} 1258_{14} $148\,11_7$	[98, 10_{14}]
				$^3 14^1_{18}$	1236_{14} 1237_{14} 1257_{14} 1357_{14} 1368_{14} $138\,10_7$	[98, 11_{14}]
				$^3 14^1_{26}$	1237_{14} 1238_{14} 1246_{14} 1248_{14} $135\,12_{14}$ $138\,10_7$	[98, 16_{14}]
				$^3 14^1_{27}$	1237_{14} 1238_{14} 1258_{14} $125\,13_{14}$ $126\,13_{14}$ $148\,11_7$	[98, 17_{14}]
			D_{14}	$^3 14^3_1$	1234_{14} 1245_{14} 1256_{14} 1267_{14} 1278_{14} 1289_7	$BdC_4(14)$, [98, I_{14}]

Table 2.14: 3-dimensional combinatorial manifolds with vertex-transitive automorphism group.

$S^2 \times S^1$	(56,84,42)	D_{14}	${}^3 14^3_6$	1235_{28} 1245_{14}	[100, $M_2^3(14)$]
	(63,98,49)	$\mathbb{Z}_{14}$	${}^3 14^1_{33}$	1238_{14} 1239_{14} 1279_{14} $138\,10_7$	
	(70,112,56)	$\mathbb{Z}_{14}$	${}^3 14^1_{16}$	1235_{14} $123\,12_{14}$ 1246_{14} 1256_{14}	
		D_{14}	${}^3 14^3_8$	1237_{28} 1267_{14} 1357_{14}	
			${}^3 14^3_9$	1237_{28} 1267_{14} 1379_{14}	
			${}^3 14^3_{10}$	1237_{28} 1267_{14} $137\,11_{14}$	
	(77,126,63)	$\mathbb{Z}_{14}$	${}^3 14^1_{12}$	1234_{14} 1249_{14} 1279_{14} $127\,13_{14}$ $138\,10_7$	
			${}^3 14^1_{29}$	1237_{14} 1238_{14} 1268_{14} 1379_{14} $138\,10_7$	
	(84,140,70)	$\mathbb{Z}_{14}$	${}^3 14^1_{15}$	1235_{14} $123\,11_{14}$ 1246_{14} 1257_{14} 1267_{14}	
			${}^3 14^1_{22}$	1236_{14} $123\,11_{14}$ 1256_{14} $12\,10\,12_{14}$ 1357_{14}	
			${}^3 14^1_{23}$	1236_{14} $123\,11_{14}$ 1256_{14} $12\,10\,12_{14}$ 1379_{14}	
			${}^3 14^1_{24}$	1236_{14} $123\,11_{14}$ 1256_{14} $12\,10\,12_{14}$ $137\,11_{14}$	
	($\underline{91}$,154,77)	$\mathbb{Z}_{14}$	${}^3 14^1_5$	1234_{14} 1248_{14} 1268_{14} $126\,13_{14}$ 1379_{14} $138\,10_7$	[98, 2_{14}]
			${}^3 14^1_9$	1234_{14} 1248_{14} $128\,13_{14}$ 1379_{14} $138\,10_7$ $138\,11_{14}$	[98, 5_{14}]
			${}^3 14^1_{19}$	1236_{14} 1237_{14} 1257_{14} 1368_{14} 1379_{14} $138\,10_7$	[98, 12_{14}]
$L(3,1)$	(77,126,63)	$\mathbb{Z}_{14}$	${}^3 14^1_{30}$	1237_{14} 1238_{14} 1268_{14} $137\,11_{14}$ $138\,10_7$	
	($\underline{91}$,154,77)	$\mathbb{Z}_{14}$	${}^3 14^1_6$	1234_{14} 1248_{14} 1268_{14} $126\,13_{14}$ $137\,11_{14}$ $138\,10_7$	[98, 3_{14}]
			${}^3 14^1_{10}$	1234_{14} 1248_{14} $128\,13_{14}$ $137\,11_{14}$ $138\,10_7$ $138\,11_{14}$	[98, 6_{14}]
			${}^3 14^1_{20}$	1236_{14} 1237_{14} 1257_{14} 1368_{14} $137\,11_{14}$ $138\,10_7$	[98, 13_{14}]
			${}^3 14^1_{31}$	1237_{14} 1238_{14} $126\,12_{14}$ $128\,12_{14}$ $137\,10_{14}$ $138\,10_7$	[98, 18_{14}]
$S^2 \underline{\times} S^1$	(63,98,49)	D_{14}	${}^3 14^3_{11}$	1238_{28} 1278_{14} $138\,10_7$	
			${}^3 14^3_{13}$	1259_{28} $125\,12_{14}$ 1289_7	
	(70,112,56)	D_{14}	${}^3 14^3_5$	1234_{14} $124\,10_{28}$ $127\,10_{14}$	
	(77,126,63)	$\mathbb{Z}_{14}$	${}^3 14^1_{32}$	1237_{14} $123\,10_{14}$ 1267_{14} $129\,11_{14}$ $138\,10_7$	
			${}^3 14^1_{36}$	1246_{14} $124\,13_{14}$ $126\,13_{14}$ 1358_{14} $148\,11_7$	
		D_{14}	${}^3 14^3_7$	1236_{28} 1256_{14} 1368_{14} $138\,10_7$	
	(84,140,70)	D_7	${}^3 14^2_2$	1234_7 1237_{14} $124\,11_{14}$ 1268_{14} 1278_7 1458_7 $145\,14_7$	
			${}^3 14^2_4$	1234_7 1237_{14} $124\,12_{14}$ 1258_{14} 1278_7 1458_7 $145\,14_7$	
			${}^3 14^2_5$	1234_7 1238_{14} $124\,11_{14}$ 1256_7 1258_{14} $147\,12_7$ $167\,12_7$	
		D_{14}	${}^3 14^3_3$	1234_{14} 1245_{14} $125\,10_{28}$ $127\,10_{14}$	
	($\underline{91}$,154,77)	$\mathbb{Z}_{14}$	${}^3 14^1_{21}$	1236_{14} $123\,10_{14}$ 1256_{14} $129\,11_{14}$ $12\,10\,12_{14}$ $138\,10_7$	[98, 14_{14}]
			${}^3 14^1_{25}$	1236_{14} $123\,11_{14}$ 1257_{14} 1267_{14} 1368_{14} $138\,10_7$	[98, 15_{14}]
			${}^3 14^1_{34}$	1245_{14} 1248_{14} 1256_{14} $126\,13_{14}$ 1289_7 $129\,11_{14}$	[98, 19_{14}]
			${}^3 14^1_{35}$	1245_{14} 1249_{14} 1256_{14} $126\,13_{14}$ 1289_7 $128\,11_{14}$	[98, 20_{14}]
		D_{14}	${}^3 14^3_{12}$	1245_{14} $124\,10_{28}$ $125\,12_{14}$ $127\,10_{14}$ $148\,11_7$	[98, II_{14}]

Table 2.14: 3-dimensional combinatorial manifolds with vertex-transitive automorphism group.

15	S^3	(75,120,60)	$D_5 \times S_3$	$^3 15^{7}_{1}$	1234_{30} 1245_{30}	BdBiC(1,4;15)
		(90,150,75)	$D_5 \times \mathbb{Z}_3$	$^3 15^{3}_{1}$	1234_{30} 1248_{15} 1258_{30}	
		$(\underline{105},180,90)$	$\mathbb{Z}_{15}$	$^3 15^{1}_{3}$	1234_{15} 1248_{15} $125\,12_{15}$ $125\,14_{15}$ $128\,12_{15}$ $147\,11_{15}$	[98, 3_{15}]
				$^3 15^{1}_{13}$	1234_{15} $124\,12_{15}$ 1258_{15} $125\,14_{15}$ $128\,12_{15}$ $137\,11_{15}$	[98, 9_{15}]
			D_{15}	$^3 15^{2}_{1}$	1234_{15} 1245_{15} 1256_{15} 1267_{15} 1278_{15} 1289_{15}	$BdC_4(15)$, [98, I_{15}]
	$\mathbf{T}^3$	$(\underline{105},180,90)$	$5\!:\!4 \times S_3$	$^3 15^{11}_{1}$	1248_{30} $124\,12_{60}$	[97], [98, III_{15}], [99], [100, M^3_1]
	$\mathbb{R}\mathbf{P}^3$	(90,150,75)	$D_5 \times S_3$	$^3 15^{7}_{2}$	1234_{30} $124\,14_{30}$ $136\,13_{15}$	
		$(\underline{105},180,90)$	$\mathbb{Z}_{15}$	$^3 15^{1}_{2}$	1234_{15} 1247_{15} $127\,14_{15}$ 1369_{15} $148\,11_{15}$ $148\,12_{15}$	[98, 2_{15}]
	S^3/Q	(90,150,75)	$[3]A_5 = GL(2,4)$	$^3 15^{15}_{1}$	1235_{60} $123\,15_{15}$	[41, M^3_p]
		$(\underline{105},180,90)$	$\mathbb{Z}_{15}$	$^3 15^{1}_{10}$	1234_{15} $124\,10_{15}$ $126\,10_{15}$ $126\,14_{15}$ $138\,12_{15}$ $139\,12_{15}$	[98, 7_{15}]
				$^3 15^{1}_{12}$	1234_{15} $124\,10_{15}$ $127\,10_{15}$ $127\,14_{15}$ $148\,11_{15}$ $148\,12_{15}$	[98, 8_{15}]
	$S^2 \times S^1$	(60,90,45)	D_{15}	$^3 15^{2}_{7}$	1235_{30} 1245_{15}	[100, $M^3_2(15)$]
		(75,120,60)	$\mathbb{Z}_{15}$	$^3 15^{1}_{18}$	1235_{15} $123\,13_{15}$ 1246_{15} 1256_{15}	
			D_{15}	$^3 15^{2}_{3}$	1234_{15} 1249_{30} $138\,11_{15}$	
				$^3 15^{2}_{4}$	1234_{15} $124\,10_{30}$ $128\,10_{15}$	
		(90,150,75)	$\mathbb{Z}_{15}$	$^3 15^{1}_{6}$	1234_{15} 1248_{15} $128\,14_{15}$ 1379_{15} $139\,12_{15}$	
				$^3 15^{1}_{7}$	1234_{15} 1249_{15} 1269_{15} $126\,14_{15}$ $148\,11_{15}$	
				$^3 15^{1}_{8}$	1234_{15} 1249_{15} 1279_{15} $127\,14_{15}$ $138\,10_{15}$	
				$^3 15^{1}_{11}$	1234_{15} $124\,10_{15}$ 1278_{15} $127\,14_{15}$ $128\,11_{15}$	
				$^3 15^{1}_{17}$	1235_{15} $123\,12_{15}$ 1246_{15} 1257_{15} 1267_{15}	
				$^3 15^{1}_{21}$	1237_{15} 1238_{15} 1268_{15} 1379_{15} $138\,10_{15}$	
			D_{15}	$^3 15^{2}_{2}$	1234_{15} 1245_{15} $125\,11_{30}$ $127\,11_{15}$	
				$^3 15^{2}_{6}$	1234_{15} $124\,14_{15}$ $136\,12_{30}$ $137\,12_{15}$	
		$(\underline{105},180,90)$	$\mathbb{Z}_{15}$	$^3 15^{1}_{1}$	1234_{15} 1245_{15} $125\,10_{15}$ 1278_{15} $127\,13_{15}$ $128\,11_{15}$	[98, 1_{15}]
				$^3 15^{1}_{4}$	1234_{15} 1248_{15} 1268_{15} $126\,14_{15}$ 1379_{15} $138\,10_{15}$	[98, 4_{15}]
				$^3 15^{1}_{5}$	1234_{15} 1248_{15} $128\,14_{15}$ $136\,10_{15}$ $136\,12_{15}$ $137\,12_{15}$	[98, 5_{15}]
				$^3 15^{1}_{9}$	1234_{15} $124\,10_{15}$ $125\,10_{15}$ $125\,11_{15}$ $127\,11_{15}$ $127\,14_{15}$	[98, 6_{15}]
				$^3 15^{1}_{14}$	1234_{15} $124\,14_{15}$ $136\,10_{15}$ 1379_{15} $137\,13_{15}$ $139\,12_{15}$	[98, 10_{15}]
				$^3 15^{1}_{15}$	1235_{15} $123\,11_{15}$ 1246_{15} 1257_{15} 1268_{15} 1278_{15}	[98, 11_{15}]
				$^3 15^{1}_{16}$	1235_{15} $123\,11_{15}$ $124\,11_{15}$ $125\,10_{15}$ $135\,10_{15}$ $138\,11_{15}$	[98, 12_{15}]
				$^3 15^{1}_{19}$	1236_{15} 1237_{15} 1257_{15} 1368_{15} 1379_{15} $138\,10_{15}$	[98, 13_{15}]
				$^3 15^{1}_{20}$	1237_{15} 1238_{15} 1248_{15} 1249_{15} 1269_{15} $148\,11_{15}$	[98, 14_{15}]
				$^3 15^{1}_{22}$	1237_{15} 1238_{15} $126\,14_{15}$ $128\,14_{15}$ 1379_{15} $139\,12_{15}$	[98, 15_{15}]
			D_{15}	$^3 15^{2}_{5}$	1234_{15} $124\,14_{15}$ 1368_{15} $137\,11_{30}$ $137\,12_{15}$	[98, II_{15}]

Table 2.15: 4-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Top. type	f-vector (f_1,f_2,f_3,f_4)	Automorphism group	Comb. type	List of orbits	Remarks
6	S^4	$(\underline{15},\underline{20},\underline{15},\underline{6})$	S_6	${}^{4}6^{16}_{1}$	12345_{6}	$Bd\,\Delta_5$, regular
9	$\mathbb{C}\mathbf{P}^2$	$(\underline{36},\underline{84},90,36)$	$\mathbb{Z}_3^2{:}\mathbb{Z}_6$	${}^{4}9^{13}_{1}$	12345_{9} 12347_{27}	$\mathbb{C}\mathbf{P}^2_9$, minimal, tight, [91], [95], [96], [102]
10	S^4	(40,80,80,32)	$[2^5]S_5$	${}^{4}10^{39}_{1}$	12345_{32}	BdC_5^{Δ}, regular, nncs
		$(\underline{45},100,105,42)$	$\mathbb{Z}_5$	${}^{4}10^{1}_{1}$	12345_{10} 12356_{10} 12367_{10} 12379_{10} 13579_{2}	
			D_5	${}^{4}10^{2}_{1}$	12345_{10} 12356_{10} 12367_{10} 12379_{10} 13579_{2}	
			$\frac{1}{2}[5:4]2$	${}^{4}10^{4}_{1}$	12345_{10} 12347_{20} 12359_{10} 13579_{2}	
11	$S^3\times S^1$	$(\underline{55},110,110,44)$	D_{11}	${}^{4}11^{2}_{1}$	12346_{22} 12356_{22}	min., tight, [88, M^4], [100, M_3^4], [102]
12	S^4	(60,140,150,60)	$A_5\times\mathbb{Z}_2$	${}^{4}12^{75}_{1}$	12469_{60}	$\mathbb{R}\mathbf{P}^2_6 *_{\Delta} \mathbb{R}\mathbf{P}^2_6$, nncs, deleted join [142], [143]
	$S^3\times S^1$	(60,120,120,48)	D_{12}	${}^{4}12^{12}_{1}$	12346_{24} 12356_{24}	[100, $M_3^4(12)$]
	$S^2\times S^2$	(60,160,180,72)	$S_3\times\mathbb{Z}_4$	${}^{4}12^{11}_{1}$	12345_{24} 12356_{24} $1236\,11_{12}$ 12569_{12}	[105], [150, M_1]
			$S_3\times D_4$	${}^{4}12^{28}_{1}$	12345_{24} $1235\,10_{24}$ $1236\,10_{24}$	[105], [150, M_3]
			$[2]A_5{:}2$	${}^{4}12^{124}_{1}$	12469_{60} $1247\,11_{12}$	[150, $M=M_2$], [151]
	$(S^2\times S^2)\#(S^2\times S^2)$	$(\underline{66},204,240,96)$	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	${}^{4}12^{2}_{1}$	12346_{12} $1234\,10_{12}$ 12367_{12} $1237\,10_{12}$ 12467_{12} 12478_{12} $124\,10\,12_{12}$ $128\,10\,12_{12}$	minimal
				${}^{4}12^{2}_{2}$	12347_{12} $1234\,10_{12}$ 12357_{12} $1235\,11_{12}$ $123\,10\,11_{12}$ $1247\,11_{12}$ 12578_{12} $1258\,12_{12}$	minimal
	$S^3\mathbin{\underline{\times}}S^1$	$(\underline{66},144,150,60)$	$t12n54(96)$	${}^{4}12^{54}_{1}$	$1245\,10_{48}$ $145\,10\,12_{12}$	
13	$S^3\times S^1$	(65,130,130,52)	D_{13}	${}^{4}13^{2}_{1}$	12346_{26} 12356_{26}	[100, $M_3^4(13)$]
		$(\underline{78},182,195,78)$	$\mathbb{Z}_{13}$	${}^{4}13^{1}_{1}$	12346_{13} $1234\,11_{13}$ 12356_{13} $123\,10\,11_{13}$ 12457_{13} 12467_{13}	
			D_{13}	${}^{4}13^{2}_{2}$	12346_{26} 12357_{26} 12367_{26}	
				${}^{4}13^{2}_{3}$	12347_{26} 12367_{26} 12457_{26}	
				${}^{4}13^{2}_{4}$	12356_{26} $1235\,11_{26}$ $1246\,12_{26}$	
14	$S^3\times S^1$	(70,140,140,56)	D_{14}	${}^{4}14^{3}_{2}$	12346_{28} 12356_{28}	[100, $M_3^4(14)$]
		(77,168,175,70)	D_{14}	${}^{4}14^{3}_{5}$	12349_{28} $1238\,10_{14}$ $1249\,10_{28}$	
		(84,196,210,84)	D_{14}	${}^{4}14^{3}_{3}$	12346_{28} 12357_{28} 12367_{28}	
				${}^{4}14^{3}_{4}$	12347_{28} 12367_{28} 12457_{28}	
		$(\underline{91},224,245,98)$	D_{14}	${}^{4}14^{3}_{1}$	12345_{14} $1235\,10_{28}$ $1238\,10_{14}$ $1245\,10_{14}$ $1249\,10_{28}$	
	$S^3\mathbin{\underline{\times}}S^1$	(77,168,175,70)	D_{14}	${}^{4}14^{3}_{7}$	12378_{28} 12379_{28} $1238\,10_{14}$	
		$(\underline{91},224,245,98)$	D_{14}	${}^{4}14^{3}_{6}$	12367_{28} 12368_{28} 12379_{28} $1238\,10_{14}$	
	....	....	$\mathbb{Z}_{14}$, D_7	....	....	

Table 2.15: 4-dimensional combinatorial manifolds with vertex-transitive automorphism group.

15	S^4	(90,230,255,102)	A_5	$^4 15^5_2$	$1235\ 12_{30}$ $1235\ 13_{30}$ $123\ 12\ 15_{15}$ $123\ 13\ 15_{15}$ $12\ 10\ 11\ 13_6$ $139\ 10\ 14_6$	
		($\underline{105}$,290,330,132)	A_5	$^4 15^5_1$	$1235\ 12_{30}$ $1235\ 13_{30}$ $123\ 12\ 13_{60}$ $12\ 10\ 11\ 13_6$ $139\ 10\ 14_6$	
	$S^3 \times S^1$	(75,150,150,60)	D_{15}	$^4 15^2_1$	12346_{30} 12356_{30}	[100, $M^4_3(15)$]
		(90,210,225,90)	D_{15}	$^4 15^2_2$	12346_{30} 12357_{30} 12367_{30}	
				$^4 15^2_4$	12347_{30} 12367_{30} 12457_{30}	
				$^4 15^2_{13}$	12356_{30} $1235\ 13_{30}$ $1246\ 14_{30}$	
				$^4 15^2_{16}$	12367_{30} $1236\ 12_{30}$ $1257\ 11_{30}$	
				$^4 15^2_{22}$	12457_{30} $1247\ 13_{30}$ $1257\ 11_{30}$	
			$D_5 \times S_3$	$^4 15^7_2$	12378_{60} $1237\ 11_{30}$	
			$S_5 \times S_3$	$^4 15^{29}_2$	12458_{60} $124\ 10\ 13_{30}$	
		($\underline{105}$,270,300,120)	D_{15}	$^4 15^2_3$	12346_{30} 12357_{30} 12368_{30} 12378_{30}	
				$^4 15^2_5$	12347_{30} 12367_{30} 12458_{30} 12478_{30}	
				$^4 15^2_6$	12347_{30} 12367_{30} $1247\ 14_{30}$ $124\ 11\ 13_{30}$	
				$^4 15^2_7$	12347_{30} 12368_{30} 12378_{30} 12457_{30}	
				$^4 15^2_8$	12348_{30} 12378_{30} 12458_{30} $124\ 12\ 13_{30}$	
				$^4 15^2_9$	12348_{30} 12378_{30} 12468_{30} $1246\ 14_{30}$	
				$^4 15^2_{10}$	12348_{30} 12378_{30} $1248\ 10_{30}$ $124\ 10\ 12_{30}$	
				$^4 15^2_{11}$	12349_{30} 12378_{30} 12379_{30} $1249\ 11_{30}$	
				$^4 15^2_{12}$	12356_{30} $1235\ 13_{30}$ $1246\ 12_{30}$ $1358\ 10_{30}$	
				$^4 15^2_{14}$	12367_{30} 12368_{30} $1237\ 11_{30}$ $1268\ 12_{30}$	
				$^4 15^2_{15}$	12367_{30} $1236\ 12_{30}$ $1257\ 10_{30}$ $125\ 10\ 11_{30}$	
				$^4 15^2_{17}$	12367_{30} $1236\ 12_{30}$ $1257\ 13_{30}$ $136\ 10\ 12_{30}$	
				$^4 15^2_{18}$	12368_{30} $1236\ 12_{30}$ 12378_{30} $1257\ 11_{30}$	
				$^4 15^2_{19}$	12457_{30} 12478_{30} $1248\ 10_{30}$ $124\ 10\ 13_{30}$	
				$^4 15^2_{20}$	12457_{30} $1247\ 10_{30}$ $124\ 10\ 11_{30}$ $124\ 11\ 13_{30}$	
				$^4 15^2_{21}$	12457_{30} $1247\ 13_{30}$ $1257\ 10_{30}$ $125\ 10\ 11_{30}$	
				$^4 15^2_{23}$	12458_{30} 12478_{30} $1247\ 13_{30}$ $1257\ 11_{30}$	
				$^4 15^2_{24}$	$1245\ 10_{30}$ $1247\ 10_{30}$ $1247\ 14_{30}$ $125\ 10\ 11_{30}$	
			$\mathbb{Z}_5 \times S_3$	$^4 15^4_2$	12458_{30} 12478_{30} $1247\ 13_{30}$ $125\ 10\ 11_{30}$	
			$5:4[\frac{1}{2}]S_3$	$^4 15^6_1$	12458_{30} $1245\ 11_{60}$ $124\ 10\ 13_{30}$	
			$D_5 \times S_3$	$^4 15^7_1$	12348_{60} 12378_{60}	
			$S_5 \times S_3$	$^4 15^{29}_1$	12458_{60} $124\ 10\ 12_{60}$	
	$\sim(S^3 \underline{\times} S^1)\#(\mathbb{C}\mathbf{P}^2)^{\#5}$	($\underline{105}$,320,375,150)	$\mathbb{Z}_5 \times S_3$	$^4 15^4_1$	$1236\ 12_{30}$ $1236\ 13_{15}$ $1237\ 11_{15}$ $1237\ 13_{30}$ $1238\ 11_{30}$ $1238\ 12_{15}$ $1368\ 11_{15}$	tight, [102]
	....	....	$\mathbb{Z}_{15}$	....	....	

Table 2.16: 5-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Top. type	f-vector $(f_1, f_2, f_3, f_4, f_5)$	Automorphism group	Comb. type	List of orbits	Remarks
7	S^5	$(\underline{21},\underline{35},\underline{35},\underline{21},\underline{7})$	S_7	$^{5}7^{7}_{1}$	123456_{7}	$Bd\,\Delta_6$, regular
8	S^5	$(\underline{28},\underline{56},68,48,16)$	$[S_4^2]2$	$^{5}8^{47}_{1}$	123456_{16}	$BdC_6(8)$ $= Bd\,\Delta_3 * Bd\,\Delta_3$
9	S^5	$(\underline{36},81,108,81,27)$	$[S_3^3]S_3 = S_3 wr S_3$	$^{5}9^{31}_{1}$	123467_{27}	$BdTriC(1,2,5;8)$ $= 3*3*3$
		$(\underline{36},\underline{84},117,90,30)$	D_9	$^{5}9^{3}_{1}$	123456_{9} 123467_{18} 124578_{3}	$BdC_6(9)$
10	S^5	$(\underline{45},\underline{120},185,150,50)$	D_{10}	$^{5}10^{3}_{1}$	123456_{10} 123467_{20} 123478_{10} 124578_{10}	$BdC_6(10)$
11	S^5	$(\underline{55},154,242,198,66)$	D_{11}	$^{5}11^{2}_{2}$	123456_{11} 123467_{22} 123479_{11} 123678_{11} $12457\ 10_{11}$	$BdTriC(1,2,5;11)$
		$(\underline{55},\underline{165},275,231,77)$	$\mathbb{Z}_{11}$	$^{5}11^{1}_{1}$	123456_{11} 123467_{11} 123478_{11} $12348\ 10_{11}$ $12378\ 10_{11}$ $12379\ 10_{11}$ $12468\ 10_{11}$	
			D_{11}	$^{5}11^{2}_{1}$	123456_{11} 123467_{22} 123478_{22} 124578_{11} 124589_{11}	$BdC_6(11)$
12	S^5	$(60,160,240,192,64)$	$[2^6]S_6 = 2wrS_6$	$^{5}12^{293}_{1}$	$12468\ 10_{64}$	$BdC_6^{\Delta} = 4*4*4$, regular, nncs
		$(\underline{66},196,318,264,88)$	D_6	$^{5}12^{3}_{10}$	123456_{6} $12345\ 10_{12}$ $12346\ 11_{12}$ $1234\ 10\ 11_{12}$ $12356\ 10_{12}$ $1236\ 10\ 11_{12}$ $1245\ 10\ 11_{6}$ $12469\ 11_{6}$ $12569\ 10_{2}$ $14589\ 12_{2}$ $145\ 10\ 11\ 12_{6}$	
				$^{5}12^{3}_{11}$	123456_{6} $12345\ 11_{12}$ $12346\ 11_{12}$ $12356\ 10_{12}$ $1235\ 10\ 11_{12}$ $1236\ 10\ 11_{12}$ $1245\ 10\ 11_{6}$ $12469\ 11_{6}$ $12569\ 10_{2}$ $14589\ 12_{2}$ $145\ 10\ 11\ 12_{6}$	
			$t12n8(24) = S_4$	$^{5}12^{8}_{3}$	123468_{24} $12346\ 12_{24}$ $12348\ 12_{8}$ $12358\ 11_{24}$ $1236\ 10\ 11_{8}$	
		$(\underline{66},204,342,288,96)$	$\frac{1}{2}[3:2]4$	$^{5}12^{5}_{4}$	123457_{12} 123458_{12} 123468_{12} $12346\ 12_{12}$ $12347\ 12_{12}$ $12358\ 11_{12}$ 124578_{12} $1247\ 10\ 11_{12}$	
			$D_4 \times \mathbb{Z}_3$	$^{5}12^{14}_{1}$	123456_{24} 123467_{24} 123478_{24} $12348\ 11_{24}$	
			$t12n15(24)$	$^{5}12^{15}_{1}$	123456_{12} 123458_{24} $12346\ 11_{12}$ 123489_{12} $12356\ 12_{24}$ 124578_{12}	
			$S_4 \times S_3$	$^{5}12^{83}_{1}$	123456_{72} 123567_{24}	$BdTriC(1,2,5;12)$
		$(\underline{66},208,354,300,100)$	D_6	$^{5}12^{3}_{1}$	123456_{6} 123457_{6} 123468_{12} $12347\ 11_{12}$ $12348\ 12_{12}$ $1234\ 11\ 12_{6}$ $12356\ 10_{12}$ 123579_{12} $12368\ 12_{6}$ $12468\ 11_{6}$ $12469\ 11_{6}$ $12569\ 10_{2}$ $13579\ 11_{2}$	
				$^{5}12^{3}_{2}$	123456_{6} 123457_{6} 123468_{12} $12347\ 12_{12}$ $12348\ 11_{12}$ $1234\ 11\ 12_{6}$ 123579_{12} $12359\ 10_{12}$ $12368\ 12_{6}$ $12468\ 11_{6}$ $12469\ 11_{6}$ $12569\ 10_{2}$ $13579\ 11_{2}$	
				$^{5}12^{3}_{5}$	123456_{6} 123457_{6} $12346\ 10_{12}$ $12347\ 12_{12}$ $1234\ 10\ 12_{6}$ $12357\ 11_{12}$ $12369\ 10_{12}$ $12369\ 12_{6}$ $12378\ 11_{12}$ $12478\ 10_{6}$ $12569\ 10_{2}$ $12578\ 10_{6}$ $13579\ 11_{2}$	

Table 2.16: 5-dimensional combinatorial manifolds with vertex-transitive automorphism group.

				$^{5}12^{3}_{7}$	123456$_{6}$ 123458$_{12}$ 123468$_{12}$ 123568$_{12}$ 124589$_{12}$ 12459 10$_{12}$ 1245 10 11$_{6}$ 124689$_{12}$ 12469 11$_{6}$ 12569 10$_{2}$ 14589 12$_{2}$ 145 10 11 12$_{6}$	
				$^{5}12^{3}_{8}$	123456$_{6}$ 123459$_{12}$ 123469$_{12}$ 123569$_{12}$ 12459 11$_{12}$ 1245 10 11$_{6}$ 12469 11$_{6}$ 12569 10$_{2}$ 134589$_{12}$ 13458 10$_{12}$ 14589 12$_{2}$ 145 10 11 12$_{6}$	
				$^{5}12^{3}_{9}$	123456$_{6}$ 12345 10$_{12}$ 12346 10$_{12}$ 12356 10$_{12}$ 12459 11$_{12}$ 1245 10 11$_{6}$ 12469 11$_{6}$ 12569 10$_{2}$ 134589$_{12}$ 13458 10$_{12}$ 14589 12$_{2}$ 145 10 11 12$_{6}$	
				$^{5}12^{3}_{12}$	123456$_{6}$ 12345 11$_{12}$ 12346 11$_{12}$ 12356 11$_{12}$ 124589$_{12}$ 12459 10$_{12}$ 1245 10 11$_{6}$ 124689$_{12}$ 12469 11$_{6}$ 12569 10$_{2}$ 14589 12$_{2}$ 145 10 11 12$_{6}$	
		($\underline{66}$,216,378,324,108)	$\mathbb{Z}_{12}$	$^{5}12^{1}_{1}$	123456$_{12}$ 123467$_{12}$ 123478$_{12}$ 123489$_{12}$ 12349 11$_{12}$ 12389 11$_{12}$ 1238 10 11$_{12}$ 124689$_{12}$ 12469 11$_{12}$	
			D_{6}	$^{5}12^{3}_{3}$	123456$_{6}$ 123457$_{6}$ 123469$_{12}$ 12347 10$_{12}$ 12349 12$_{12}$ 1234 10 12$_{6}$ 12356 11$_{12}$ 123578$_{12}$ 12369 12$_{6}$ 12468 11$_{6}$ 12469 11$_{6}$ 12478 11$_{6}$ 13468 10$_{6}$	
				$^{5}12^{3}_{4}$	123456$_{6}$ 123457$_{6}$ 123469$_{12}$ 12347 12$_{12}$ 12349 10$_{12}$ 1234 10 12$_{6}$ 123578$_{12}$ 12358 11$_{12}$ 12369 12$_{6}$ 12468 11$_{6}$ 12469 11$_{6}$ 12478 11$_{6}$ 13468 10$_{6}$	
				$^{5}12^{3}_{6}$	123456$_{6}$ 123457$_{6}$ 12346 11$_{12}$ 12347 12$_{12}$ 1234 11 12$_{6}$ 12357 10$_{12}$ 12368 11$_{12}$ 12368 12$_{6}$ 12379 10$_{12}$ 12478 10$_{6}$ 12478 11$_{6}$ 12578 10$_{6}$ 13468 10$_{6}$	
			$\frac{1}{2}[3:2]4$	$^{5}12^{5}_{1}$	123456$_{12}$ 123457$_{12}$ 12346 11$_{12}$ 123478$_{12}$ 123489$_{12}$ 12349 12$_{12}$ 12356 11$_{12}$ 123578$_{12}$ 1239 10 12$_{12}$	
				$^{5}12^{5}_{2}$	123456$_{12}$ 123457$_{12}$ 12346 11$_{12}$ 12347 10$_{12}$ 12349 10$_{12}$ 12349 12$_{12}$ 12356 11$_{12}$ 123578$_{12}$ 1239 10 12$_{12}$	
				$^{5}12^{5}_{3}$	123457$_{12}$ 123458$_{12}$ 123468$_{12}$ 123469$_{12}$ 12347 12$_{12}$ 12349 12$_{12}$ 1235 10 11$_{12}$ 12369 12$_{12}$ 124578$_{12}$	
			$t12n8(24) = S_4$	$^{5}12^{8}_{1}$	123457$_{24}$ 12345 12$_{24}$ 123478$_{24}$ 12348 12$_{8}$ 123579$_{4}$ 123678$_{24}$	
				$^{5}12^{8}_{2}$	123458$_{24}$ 12345 12$_{24}$ 12348 12$_{8}$ 123578$_{24}$ 123579$_{4}$ 12369 10$_{24}$	
			$t12n13(24)$	$^{5}12^{13}_{1}$	123457$_{24}$ 123467$_{24}$ 123468$_{24}$ 123489$_{24}$ 12456 12$_{12}$	
				$^{5}12^{13}_{2}$	123457$_{24}$ 123478$_{24}$ 12348 11$_{24}$ 123578$_{24}$ 12456 12$_{12}$	
		($\underline{66}$,$\underline{220}$,390,336,112)	D_{12}	$^{5}12^{12}_{1}$	123456$_{12}$ 123467$_{24}$ 123478$_{24}$ 123489$_{12}$ 124578$_{12}$ 124589$_{24}$ 12569 10$_{4}$	$BdC_6(12)$

Table 2.16: 5-dimensional combinatorial manifolds with vertex-transitive automorphism group.

13	S^5	($\underline{78}$,247,416,351,117)	$\mathbb{Z}_{13}$	$^{5}13^{1}_{2}$	123456_{13} 123467_{13} 123478_{13} 123489_{13} $12349\,12_{13}$ $12389\,11_{13}$ $1239\,11\,12_{13}$ $12467\,12_{13}$ $12479\,12_{13}$	
			D_{13}	$^{5}13^{2}_{3}$	123456_{13} 123467_{26} 123479_{26} 123678_{13} 123789_{13} $12457\,12_{13}$ $12479\,12_{13}$	
		($\underline{78}$,260,455,390,130)	$\mathbb{Z}_{13}$	$^{5}13^{1}_{1}$	123456_{13} 123467_{13} 123478_{13} 123489_{13} $12349\,11_{13}$ $1234\,11\,12_{13}$ $12389\,11_{13}$ $1238\,10\,11_{13}$ $12457\,12_{13}$ $12479\,12_{13}$	
				$^{5}13^{1}_{5}$	123456_{13} 123467_{13} 123479_{13} $12349\,12_{13}$ 123678_{13} 123789_{13} $1238\,10\,11_{13}$ $1239\,11\,12_{13}$ $12467\,12_{13}$ $12479\,12_{13}$	
				$^{5}13^{1}_{7}$	123456_{13} 123467_{13} $12347\,12_{13}$ $12367\,11_{13}$ $1237\,11\,12_{13}$ $12459\,10_{13}$ $12459\,12_{13}$ $12467\,10_{13}$ $12479\,10_{13}$ $12479\,12_{13}$	
			D_{13}	$^{5}13^{2}_{2}$	123456_{13} 123467_{26} 123478_{26} $12348\,10_{13}$ 123789_{13} 124578_{13} $12458\,11_{13}$ $1248\,10\,11_{13}$	
			13:6	$^{5}13^{5}_{1}$	123456_{39} 123467_{78} $12347\,11_{13}$	
		($\underline{78}$,273,494,429,143)	$\mathbb{Z}_{13}$	$^{5}13^{1}_{3}$	123456_{13} 123467_{13} 123478_{13} $12348\,10_{13}$ $1234\,10\,12_{13}$ 123789_{13} $1239\,10\,12_{13}$ $1239\,11\,12_{13}$ $12468\,12_{13}$ $1248\,10\,12_{13}$ $13579\,11_{13}$	
				$^{5}13^{1}_{4}$	123456_{13} 123467_{13} 123478_{13} $12348\,11_{13}$ $1234\,11\,12_{13}$ $12378\,10_{13}$ $1238\,10\,11_{13}$ $12457\,12_{13}$ $12478\,10_{13}$ $12479\,10_{13}$ $12479\,12_{13}$	
				$^{5}13^{1}_{6}$	123456_{13} 123467_{13} $12347\,12_{13}$ 123678_{13} $12368\,11_{13}$ 123789_{13} $12379\,12_{13}$ $1238\,10\,11_{13}$ $1239\,11\,12_{13}$ $12467\,12_{13}$ $1257\,10\,12_{13}$	
			D_{13}	$^{5}13^{2}_{4}$	123456_{13} 123467_{26} $12347\,11_{13}$ $12367\,11_{26}$ $12457\,10_{26}$ $12459\,10_{13}$ $12479\,10_{13}$ $12479\,12_{13}$	
				$^{5}13^{2}_{5}$	123456_{13} 123469_{26} 123567_{13} 123578_{26} 123678_{13} 123689_{26} 124689_{13} $12479\,12_{13}$	
			13:3	$^{5}13^{3}_{1}$	123456_{39} 123467_{39} $12347\,11_{13}$ $12348\,11_{39}$ $1238\,11\,12_{13}$	
		($\underline{78}$,$\underline{286}$,533,468,156)	$\mathbb{Z}_{13}$	$^{5}13^{1}_{8}$	123456_{13} $12346\,12_{13}$ 123568_{13} $12358\,10_{13}$ $1235\,10\,11_{13}$ 123678_{13} 123679_{13} $12369\,11_{13}$ $1236\,11\,12_{13}$ 123789_{13} 124579_{13} $12469\,11_{13}$	
			D_{13}	$^{5}13^{2}_{1}$	123456_{13} 123467_{26} 123478_{26} 123489_{26} 124578_{13} 124589_{26} $12459\,10_{13}$ $12569\,10_{13}$	$BdC_6(13)$
				$^{5}13^{2}_{6}$	123456_{13} 123469_{26} 123567_{13} 123578_{26} 123678_{13} $12368\,11_{26}$ 124689_{13} $12478\,12_{26}$	
			13:6	$^{5}13^{5}_{2}$	123456_{39} $12346\,10_{78}$ $12347\,11_{13}$ 123569_{26}	
	X_{-1}	($\underline{78}$,$\underline{286}$,533,468,156)	13:3	$^{5}13^{3}_{2}$	123458_{39} 123459_{39} 123479_{39} 123569_{13} $12358\,10_{13}$ $12379\,12_{13}$	tight, [16], [102]
	$S^4 \underline{\times} S^1$	($\underline{78}$,195,260,195,65)	D_{13}	$^{5}13^{2}_{7}$	123457_{26} 123467_{26} 123567_{13}	min., tight, [88, M^5] [100, M^5_4], [102]

Table 2.16: 5-dimensional combinatorial manifolds with vertex-transitive automorphism group.

14	S^5	(84,266,448,378,126)	$L(2,7)\times\mathbb{Z}_2$	$^5 14^{19}_{1}$	123456_{84} $12346\,12_{42}$	
		(84,280,490,420,140)	$2[\frac{1}{2}]7:6$	$^5 14^{4}_{1}$	123456_{21} $12345\,14_{21}$ 123467_{42} $12347\,12_{7}$ $12356\,11_{42}$ $1236\,11\,14_{7}$	
			$7:6\times\mathbb{Z}_2$	$^5 14^{7}_{1}$	123456_{42} $12346\,12_{84}$ $12347\,12_{14}$	nncs
			$S_7\times\mathbb{Z}_2$	$^5 14^{49}_{1}$	123456_{140}	nncs
		($\underline{91}$,308,539,462,154)	D_{14}	$^5 14^{3}_{3}$	123456_{14} 123467_{28} 123479_{28} $12349\,10_{14}$ 123678_{14} 123789_{14} $12457\,13_{14}$ $12479\,10_{14}$ $1247\,10\,13_{14}$	
			$2[\frac{1}{2}]7:6$	$^5 14^{4}_{2}$	123456_{21} $12345\,14_{21}$ 123467_{42} $12347\,12_{7}$ $12356\,12_{42}$ $1235\,12\,14_{21}$	
				$^5 14^{4}_{3}$	123456_{21} $12345\,14_{21}$ $12346\,12_{42}$ $12347\,12_{7}$ $12356\,12_{42}$ $1235\,12\,14_{21}$	
				$^5 14^{4}_{4}$	$12346\,11_{42}$ $12346\,13_{21}$ $12348\,11_{21}$ $12369\,13_{42}$ $1236\,11\,14_{7}$ $12389\,14_{21}$	
		($\underline{91}$,322,581,504,168)	D_{14}	$^5 14^{3}_{5}$	123456_{14} 123467_{28} $12347\,12_{14}$ $12367\,11_{28}$ 124578_{14} $12458\,12_{14}$ $12478\,12_{28}$ $12569\,10_{14}$ $12569\,12_{14}$	
			$L(2,7):2$	$^5 14^{16}_{1}$	$12345\,12_{84}$ $12367\,10_{56}$ $12367\,14_{28}$	
		($\underline{91}$,336,623,546,182)	D_{14}	$^5 14^{3}_{6}$	123456_{14} 123468_{28} 123489_{28} $12349\,10_{14}$ 123567_{14} 123678_{14} 124679_{28} 124689_{14} $12479\,10_{14}$ $1247\,10\,13_{14}$	
				$^5 14^{3}_{7}$	123456_{14} 123469_{28} $12349\,10_{14}$ 123567_{14} 123578_{28} 123678_{14} 123689_{28} 124689_{14} $12479\,10_{14}$ $1247\,10\,13_{14}$	
		($\underline{91}$,350,665,588,196)	D_{14}	$^5 14^{3}_{2}$	123456_{14} 123467_{28} 123478_{28} $12348\,11_{14}$ $12378\,11_{28}$ 124578_{14} $12458\,12_{14}$ $1248\,11\,12_{28}$ $12569\,10_{14}$ $1256\,10\,11_{14}$	
				$^5 14^{3}_{4}$	123456_{14} 123467_{28} $12347\,11_{28}$ $12348\,11_{14}$ $12367\,10_{28}$ 124578_{14} $12458\,12_{14}$ $12478\,11_{28}$ $12569\,10_{14}$ $1256\,10\,11_{14}$	
		($\underline{91}$,$\underline{364}$,707,630,210)	D_{14}	$^5 14^{3}_{1}$	123456_{14} 123467_{28} 123478_{28} 123489_{28} 1234910_{14} 124578_{14} 124589_{28} $12459\,10_{28}$ $12569\,10_{14}$ $1256\,10\,11_{14}$	
			$2[\frac{1}{2}]7:6$	$^5 14^{4}_{5}$	123478_{42} $12347\,12_{7}$ 123489_{42} $12349\,10_{21}$ $12378\,10_{42}$ $12389\,10_{21}$ $123\,10\,11\,12_{21}$ $12489\,11_{14}$	
	$S^4\times S^1$	(84,210,280,210,70)	D_{14}	$^5 14^{3}_{8}$	123457_{28} 123467_{28} 123567_{14}	[100, $M^5_4(14)$]
		($\underline{91}$,252,371,294,98)	D_{14}	$^5 14^{3}_{10}$	123489_{28} $12348\,10_{28}$ 123789_{14} $12489\,11_{14}$ $1248\,10\,11_{14}$	
				$^5 14^{3}_{16}$	123789_{14} $12378\,10_{28}$ $12379\,10_{28}$ $12489\,11_{14}$ $1248\,10\,11_{14}$	

Table 2.16: 5-dimensional combinatorial manifolds with vertex-transitive automorphism group.

	$S^3 \times S^2$	(84,280,490,420,140)	D_{14}	$^{5}14^{3}_{9}$	123467_{28} $12346\,12_{28}$ 123567_{14} $12357\,11_{28}$ $12457\,13_{14}$ $1246\,10\,12_{28}$	
	$S^3 \times S^2$ or X_∞	($\underline{91}$,336,623,546,182)	D_{14}	$^{5}14^{3}_{14}$	123567_{14} $12356\,12_{28}$ 123579_{28} $12359\,11_{28}$ $1235\,11\,12_{28}$ $12459\,11_{14}$ 124689_{14} $12489\,11_{14}$ $1248\,10\,11_{14}$	
	$S^3 \times S^2$ or X_∞	($\underline{91}$,350,665,588,196)	D_{14}	$^{5}14^{3}_{13}$	123567_{14} 123569_{28} $12357\,13_{28}$ $12359\,11_{28}$ 123679_{28} 124589_{28} $12459\,11_{14}$ $12489\,11_{14}$ $1248\,10\,11_{14}$	
	$S^3 \times S^2$ or X_∞	($\underline{91}$,350,665,588,196)	D_{14}	$^{5}14^{3}_{15}$	123567_{14} $12356\,13_{28}$ 123579_{28} $12359\,11_{28}$ $1235\,11\,12_{28}$ 124589_{28} $12459\,11_{14}$ $12489\,11_{14}$ $1248\,10\,11_{14}$	
	$S^4 \underline{\times} S^1$	($\underline{91}$,252,371,294,98)	D_{14}	$^{5}14^{3}_{11}$	123489_{28} $12348\,11_{14}$ $12349\,10_{14}$ $12378\,10_{28}$ $12489\,11_{14}$	
				$^{5}14^{3}_{12}$	$12348\,10_{28}$ $12348\,11_{14}$ $12349\,10_{14}$ $12379\,10_{28}$ $12489\,11_{14}$	
	....	....	$\mathbb{Z}_{14}$, D_7	....	....	
15	S^5	(90,275,450,375,125)	$[D_5^3]S_3 = D_5 wr S_3$	$^{5}15^{60}_{1}$	123456_{125}	$5*5*5$
		($\underline{105}$,350,600,510,170)	$D_5 \times S_3$	$^{5}15^{7}_{3}$	123456_{30} $12346\,14_{30}$ $12356\,14_{60}$ 124689_{15} $12469\,14_{30}$ $1368\,11\,13_{5}$	
		($\underline{105}$,410,780,690,230)	$D_5 \times S_3$	$^{5}15^{7}_{1}$	123456_{30} 123467_{60} 123478_{60} $12348\,12_{15}$ $12378\,12_{30}$ 124578_{30} $1267\,11\,12_{5}$	
				$^{5}15^{7}_{2}$	123456_{30} 123467_{60} $12347\,12_{60}$ $12348\,12_{15}$ $12367\,11_{30}$ 124578_{30} $1267\,11\,12_{5}$	
	$S^4 \underline{\times} S^1$	($\underline{105}$,305,465,375,125)	$D_5 \times S_3$	$^{5}15^{7}_{4}$	123678_{30} $12367\,11_{30}$ $12368\,13_{30}$ $12378\,12_{30}$ $1368\,11\,13_{5}$	
	....	....	$\mathbb{Z}_{15}$, D_{15}, $D_5 \times \mathbb{Z}_3$ $\mathbb{Z}_5 \times S_3$	....	....	

Table 2.17: 6-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Top. type	f-vector $(f_1, f_2, f_3, f_4, f_5, f_6)$	Automorphism group	Comb. type	List of orbits	Remarks
8	S^6	$(\underline{28},\underline{56},\underline{70},\underline{56},\underline{28},\underline{8})$	S_8	${}^{6}8^{50}_{1}$	$1234567_{\,8}$	$Bd\,\Delta_7$, regular
10	S^6	$(\underline{45},\underline{120},205,222,140,40)$	$[2^5]D_5$	${}^{6}10^{23}_{1}$	$1234567_{\,40}$	
14	S^6	$(84,280,560,672,448,128)$	$[2^7]S_7$	${}^{6}14^{57}_{1}$	$1234567_{\,128}$	BdC_7^{Δ}, regular, nncs
		$(\underline{91},322,665,812,546,156)$	$2[\frac{1}{2}]7:6$	${}^{6}14^{4}_{1}$	$1234567_{\,42}$ $123457\,13_{\,42}$ $123467\,12_{\,14}$ $12356\,11\,12_{\,42}$ $12357\,11\,13_{\,14}$ $13579\,11\,13_{\,2}$	
		$(\underline{91},\underline{364},875,1190,840,240)$	$2[\frac{1}{2}]7:6$	${}^{6}14^{4}_{2}$	$1234568_{\,42}$ $123458\,14_{\,42}$ $12345\,11\,13_{\,42}$ $123567\,10_{\,42}$ $123567\,13_{\,42}$ $12356\,11\,14_{\,14}$ $12357\,11\,13_{\,14}$ $13579\,11\,13_{\,2}$	
	....	....	$\mathbb{Z}_{14}$, D_7	....	....	
15	$S^3 \times S^3$	$(\underline{105},435,1125,1605,1155,330)$	$D_5 \times S_3$	${}^{6}15^{7}_{2}$	$1234589_{\,60}$ $123458\,12_{\,30}$ $1234789_{\,60}$ $123479\,11_{\,60}$ $1235689_{\,60}$ $123568\,10_{\,60}$	
		$(\underline{105},450,1200,1740,1260,360)$	$D_5 \times S_3$	${}^{6}15^{7}_{1}$	$1234589_{\,60}$ $123458\,10_{\,60}$ $123459\,11_{\,60}$ $1234789_{\,60}$ $1235689_{\,60}$ $123568\,10_{\,60}$	
	....	....	$\mathbb{Z}_{15}$, D_{15}, $D_5 \times \mathbb{Z}_3$, $\mathbb{Z}_5 \times S_3$	....	....	

Table 2.18: 7-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Top. type	f-vector $(f_1, f_2, f_3, f_4, f_5, f_6, f_7)$	Automorphism group	Comb. type	List of orbits	Remarks
9	S^7	$(\underline{36},\underline{84},\underline{126},\underline{126},\underline{84},\underline{36},\underline{9})$	S_9	$^{7}9^{34}_{1}$	12345678$_{9}$	$Bd\,\Delta_8$, reg.
10	S^7	$(\underline{45},\underline{120},\underline{210},250,200,100,25)$	$[S_5^2]2$	$^{7}10^{43}_{1}$	12345678$_{25}$	$BdC_8(10)$ $= (Bd\,\Delta_4)^{*2}$
11	S^7	$(\underline{55},\underline{165},\underline{330},451,407,220,55)$	D_{11}	$^{7}11^{2}_{1}$	12345678$_{11}$ 12345689$_{22}$ 12346789$_{11}$ 1234679 10$_{11}$	$BdC_8(11)$
12	S^7	$(\underline{66},216,459,648,594,324,81)$	$[S_3^4]S_4 = S_3wrS_4$	$^{7}12^{289}_{1}$	12345678$_{81}$	$3*3*3*3$
		$(\underline{66},\underline{220},483,708,670,372,93)$	$t12n8(24) = S_4$	$^{7}12^{8}_{1}$	1234567 11$_{24}$ 1234567 12$_{24}$ 123457 11 12$_{24}$ 123459 11 12$_{12}$ 123467 10 12$_{6}$ 1246789 10$_{3}$	
		$(\underline{66},\underline{220},486,720,688,384,96)$	$[2^6]D_6 = 2wrD_6$	$^{7}12^{193}_{1}$	1234568 10$_{96}$	
		$(\underline{66},\underline{220},\underline{495},756,742,420,105)$	D_{12}	$^{7}12^{12}_{1}$	12345678$_{12}$ 12345689$_{24}$ 1234569 10$_{12}$ 12346789$_{12}$ 1234679 10$_{24}$ 123467 10 11$_{12}$ 1234789 10$_{6}$ 124578 10 11$_{3}$	$BdC_8(12)$
13	S^7	$(\underline{78},\underline{286},689,1092,1092,624,156)$	$\mathbb{Z}_{13}$	$^{7}13^{1}_{1}$	12345678$_{13}$ 12345689$_{13}$ 1234569 10$_{13}$ 123456 10 12$_{13}$ 123459 10 11$_{13}$ 12345 10 11 12$_{13}$ 1234678 12$_{13}$ 1234689 10$_{13}$ 123468 10 12$_{13}$ 1235679 10$_{13}$ 123567 10 11$_{13}$ 123579 10 11$_{13}$	
		$(\underline{78},\underline{286},702,1144,1170,676,169)$	D_{13}	$^{7}13^{2}_{2}$	12345678$_{13}$ 12345689$_{26}$ 1234569 10$_{26}$ 1234678 12$_{26}$ 1234689 10$_{26}$ 123468 10 12$_{13}$ 123567 10 11$_{13}$ 1235689 10$_{13}$ 123579 10 11$_{13}$	
				$^{7}13^{2}_{3}$	12345678$_{13}$ 12345689$_{26}$ 1234569 11$_{13}$ 1234589 11$_{26}$ 12346789$_{13}$ 1234679 12$_{26}$ 1234789 10$_{13}$ 1235689 11$_{26}$ 124578 10 11$_{13}$	
			$13:4$	$^{7}13^{4}_{1}$	12345678$_{26}$ 12345689$_{52}$ 1234569 10$_{52}$ 12346789$_{26}$ 1235689 10$_{13}$	
		$(\underline{78},\underline{286},\underline{715},1196,1248,728,182)$	D_{13}	$^{7}13^{2}_{1}$	12345678$_{13}$ 12345689$_{26}$ 1234569 10$_{26}$ 12346789$_{13}$ 1234679 10$_{26}$ 123467 10 11$_{26}$ 123467 11 12$_{13}$ 1234789 10$_{13}$ 123478 10 11$_{13}$ 124578 10 11$_{13}$	$BdC_8(13)$
			$13:4$	$^{7}13^{4}_{2}$	12345678$_{26}$ 12345689$_{52}$ 1234569 11$_{13}$ 1234589 10$_{52}$ 12346789$_{26}$ 1235689 10$_{13}$	
14	S^7	$(\underline{91},350,861,1372,1372,784,196)$	$[D_7^2]2 = D_7wr2$	$^{7}14^{20}_{1}$	12345678$_{49}$ 1234578 10$_{98}$ 1234789 10$_{49}$	$(BdC_4(7))^{*2}$
		$(\underline{91},\underline{364},973,1694,1806,1064,266)$	D_{14}	$^{7}14^{3}_{2}$	12345678$_{14}$ 12345689$_{28}$ 1234569 10$_{28}$ 123456 10 11$_{14}$ 12346789$_{14}$ 1234679 10$_{28}$ 123467 10 12$_{28}$ 123467 12 13$_{14}$ 12346 10 11 12$_{28}$ 123479 10 12$_{14}$ 123569 10 11$_{28}$ 123678 11 12$_{14}$ 124578 10 13$_{14}$	

Table 2.18: 7-dimensional combinatorial manifolds with vertex-transitive automorphism group.

				$^{7}14^{3}_{3}$	12345678_{14} 12345689_{28} $1234569\,10_{28}$ $123456\,10\,11_{14}$ $1234678\,13_{28}$ $1234689\,10_{28}$ $123468\,10\,11_{28}$ $123468\,11\,13_{14}$ $123489\,10\,11_{7}$ $123567\,10\,12_{14}$ $1235689\,10_{14}$ $12356\,10\,11\,12_{28}$ $123678\,11\,12_{14}$ $124689\,11\,13_{7}$	
				$^{7}14^{3}_{4}$	12345678_{14} 12345689_{28} $1234569\,10_{28}$ $123456\,10\,11_{14}$ $1234678\,13_{28}$ $1234689\,11_{28}$ $123468\,11\,13_{14}$ $123469\,10\,11_{28}$ $123489\,10\,11_{7}$ $123567\,10\,12_{14}$ $1235689\,10_{14}$ $12356\,10\,11\,12_{28}$ $123678\,11\,12_{14}$ $124689\,11\,13_{7}$	
				$^{7}14^{3}_{5}$	12345678_{14} 12345689_{28} $1234569\,12_{14}$ $1234589\,11_{28}$ 12346789_{14} $1234679\,13_{28}$ $123469\,10\,12_{28}$ $123469\,10\,13_{14}$ $1234789\,10_{14}$ $123489\,10\,11_{7}$ $1235689\,12_{28}$ $124578\,10\,12_{28}$ $124578\,10\,13_{14}$ $124589\,11\,12_{7}$	
		($\underline{91}$,$\underline{364}$,$\underline{1001}$,1806,1974,1176,294)	D_{14}	$^{7}14^{3}_{1}$	12345678_{14} 12345689_{28} $1234569\,10_{28}$ $123456\,10\,11_{14}$ 12346789_{14} $1234679\,10_{28}$ $123467\,10\,11_{28}$ $123467\,11\,12_{28}$ $123467\,12\,13_{14}$ $1234789\,10_{14}$ $123478\,10\,11_{28}$ $123478\,11\,12_{14}$ $123489\,10\,11_{7}$ $124578\,10\,11_{14}$ $124578\,11\,12_{14}$ $124589\,11\,12_{7}$	$BdC_{8}(14)$
	....	....	$\mathbb{Z}_{14}$, D_{7}	....	....	
15	S^{7}	($\underline{105}$,450,1260,2250,2430,1440,360)	$5:4\times\mathbb{Z}_{3}$	$^{7}15^{8}_{1}$	12345678_{60} 12345689_{60} $1234569\,10_{60}$ $123456\,10\,14_{60}$ $123459\,10\,12_{60}$ $1234678\,14_{60}$	
			$S_{5}\times S_{3}$	$^{7}15^{29}_{1}$	12345678_{360}	
		($\underline{105}$,450,1275,2310,2520,1500,375)	$5:4[\frac{1}{2}]S_{3}$	$^{7}15^{6}_{1}$	12345678_{60} 12345689_{60} $1234569\,12_{60}$ $123456\,12\,14_{60}$ $1234589\,11_{60}$ $123459\,11\,12_{60}$ $1234689\,12_{15}$	
				$^{7}15^{6}_{2}$	12345678_{60} $1234568\,10_{60}$ $123456\,10\,14_{60}$ 12345789_{60} $12345\,10\,12\,13_{60}$ $12345\,10\,13\,14_{60}$ $12348\,11\,12\,14_{15}$	
	....	....	$\mathbb{Z}_{15}$, D_{15}, $D_{5}\times\mathbb{Z}_{3}$ $\mathbb{Z}_{5}\times S_{3}$	....	....	

Table 2.19: 8-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Top. type	f-vector $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)$	Automorphism group	Comb. type	List of orbits	Remarks
10	S^8	$(\underline{45},\underline{120},\underline{210},\underline{252},\underline{210},\underline{120},\underline{45},\underline{10})$	S_{10}	${}^8 10^{45}_1$	123456789_{10}	$Bd\,\Delta_9$, regular
12	S^8	$(\underline{66},\underline{220},492,768,840,624,288,64)$	$[S_4^3]S_3 = S_4 wr S_3$	${}^8 12^{294}_1$	123456789_{64}	$(Bd\,\Delta_3)^{*3}$
14	S^8	$(\underline{91},\underline{364},987,1862,2408,2032,1008,224)$	$\mathbb{Z}_{14}$	${}^8 14^{1}_5$	123456789_{14} $12345679\ 11_{14}$ $1234567\ 11\ 13_{14}$ $12345689\ 10_{14}$ $1234569\ 10\ 11_{14}$ $123456\ 10\ 11\ 12_{14}$ $123456\ 11\ 12\ 13_{14}$ $1234579\ 11\ 13_{14}$ $12346789\ 11_{14}$ $1234678\ 11\ 13_{14}$ $1234689\ 10\ 11_{14}$ $123468\ 10\ 11\ 13_{14}$ $123468\ 10\ 12\ 13_{14}$ $1234789\ 11\ 13_{14}$ $1235679\ 10\ 11_{14}$ $123567\ 10\ 11\ 12_{14}$	
			D_7	${}^8 14^{2}_{14}$	123456789_{14} $12345679\ 11_{14}$ $1234567\ 11\ 13_{14}$ $12345689\ 10_{14}$ $1234569\ 10\ 11_{14}$ $12345789\ 14_{14}$ $1234579\ 11\ 13_{14}$ $1234589\ 10\ 14_{14}$ $12346789\ 11_{14}$ $1234678\ 11\ 13_{14}$ $1234689\ 10\ 11_{14}$ $123468\ 10\ 11\ 12_{14}$ $1235679\ 10\ 11_{14}$ $123567\ 10\ 11\ 12_{14}$ $123568\ 10\ 12\ 14_{14}$ $123578\ 10\ 12\ 14_{14}$	
			$[2^7]D_7 = 2wrD_7$	${}^8 14^{38}_1$	123456789_{224}	
		$(\underline{91},\underline{364},994,1904,2506,2144,1071,238)$	D_7	${}^8 14^{2}_2$	123456789_{14} $12345679\ 10_{14}$ $1234567\ 10\ 11_{14}$ $1234567\ 11\ 12_{14}$ $1234567\ 12\ 13_{14}$ $12345789\ 13_{14}$ $1234579\ 10\ 13_{14}$ $123457\ 10\ 11\ 13_{14}$ $123457\ 11\ 12\ 13_{14}$ $12346789\ 12_{14}$ $1234679\ 10\ 11_{14}$ $1234679\ 11\ 12_{14}$ $123469\ 10\ 11\ 12_{14}$ $1235679\ 10\ 11_{14}$ $1235689\ 11\ 14_{14}$ $123569\ 10\ 11\ 14_{14}$ $1235789\ 11\ 13_{14}$	
				${}^8 14^{2}_5$	123456789_{14} $12345679\ 10_{14}$ $1234567\ 10\ 11_{14}$ $1234567\ 11\ 13_{14}$ $12345689\ 10_{14}$ $12345789\ 13_{14}$ $1234579\ 10\ 13_{14}$ $123457\ 10\ 11\ 13_{14}$ $1234589\ 10\ 14_{14}$ $123459\ 10\ 11\ 13_{14}$ $12346789\ 12_{14}$ $1234678\ 12\ 13_{14}$ $1234679\ 10\ 12_{14}$ $123467\ 10\ 11\ 12_{14}$ $123568\ 10\ 11\ 14_{14}$ $1235789\ 11\ 13_{14}$ $123589\ 10\ 11\ 14_{14}$	
				${}^8 14^{2}_9$	123456789_{14} $12345679\ 10_{14}$ $1234567\ 10\ 13_{14}$ $12345689\ 11_{14}$ $1234569\ 10\ 12_{14}$ $12345789\ 11_{14}$ $1234578\ 11\ 13_{14}$ $1234579\ 10\ 11_{14}$ $123457\ 10\ 11\ 13_{14}$ $123459\ 10\ 11\ 12_{14}$ $12345\ 10\ 11\ 12\ 13_{14}$ $1234679\ 10\ 13_{14}$ $1234689\ 11\ 12_{14}$ $1235679\ 10\ 13_{14}$ $1235689\ 11\ 14_{14}$ $123569\ 10\ 12\ 13_{14}$ $1235789\ 11\ 13_{14}$	

Table 2.19: 8-dimensional combinatorial manifolds with vertex-transitive automorphism group.

				$^{8}14^{2}_{10}$	123456789$_{14}$ 12345679 10$_{14}$ 1234567 10 13$_{14}$ 12345689 12$_{14}$ 1234568 11 12$_{14}$ 12345789 10$_{14}$ 1234578 10 11$_{14}$ 1234578 11 13$_{14}$ 123457 10 11 13$_{14}$ 1234589 10 11$_{14}$ 1234589 11 12$_{14}$ 123459 10 11 13$_{14}$ 123459 11 12 13$_{14}$ 1235679 10 13$_{14}$ 1235689 11 12$_{14}$ 1235689 11 14$_{14}$ 1235789 11 13$_{14}$	
				$^{8}14^{2}_{11}$	123456789$_{14}$ 12345679 10$_{14}$ 1234567 10 13$_{14}$ 12345689 12$_{14}$ 1234568 11 12$_{14}$ 12345789 11$_{14}$ 1234578 11 13$_{14}$ 1234579 10 11$_{14}$ 123457 10 11 13$_{14}$ 1234589 11 12$_{14}$ 123459 10 11 12$_{14}$ 123459 10 12 13$_{14}$ 12345 10 11 12 13$_{14}$ 1235679 10 13$_{14}$ 1235689 11 12$_{14}$ 1235689 11 14$_{14}$ 1235789 11 13$_{14}$	
				$^{8}14^{2}_{12}$	123456789$_{14}$ 12345679 10$_{14}$ 1234567 10 13$_{14}$ 12345689 12$_{14}$ 1234568 11 12$_{14}$ 12345789 13$_{14}$ 1234579 10 13$_{14}$ 1234589 12 13$_{14}$ 123458 11 12 13$_{14}$ 12346789 12$_{14}$ 1234678 11 12$_{14}$ 1234679 10 12$_{14}$ 123467 10 11 12$_{14}$ 1235679 10 13$_{14}$ 1235679 11 12$_{14}$ 1235679 11 13$_{14}$ 1235789 12 13$_{14}$	
				$^{8}14^{2}_{15}$	123456789$_{14}$ 12345679 13$_{14}$ 12345689 10$_{14}$ 1234568 10 12$_{14}$ 1234569 10 12$_{14}$ 12345789 10$_{14}$ 1234578 10 12$_{14}$ 1234578 12 14$_{14}$ 1234579 10 12$_{14}$ 1234579 11 12$_{14}$ 1234579 11 13$_{14}$ 1234678 10 13$_{14}$ 1234679 10 13$_{14}$ 1235679 10 13$_{14}$ 123568 10 12 14$_{14}$ 123569 10 12 13$_{14}$ 123578 10 12 14$_{14}$	
				$^{8}14^{2}_{16}$	123456789$_{14}$ 12345679 13$_{14}$ 12345689 10$_{14}$ 1234568 10 12$_{14}$ 1234569 10 12$_{14}$ 12345789 10$_{14}$ 1234578 10 14$_{14}$ 1234579 10 12$_{14}$ 1234579 11 13$_{14}$ 1234579 11 14$_{14}$ 1234579 12 14$_{14}$ 1234678 10 13$_{14}$ 1234679 10 13$_{14}$ 1235679 10 13$_{14}$ 123568 10 12 13$_{14}$ 123569 10 12 13$_{14}$ 123579 10 12 13$_{14}$	
		(<u>91</u>,<u>364</u>,<u>1001</u>,1946,2604,2256,1134,252)	$\mathbb{Z}_{14}$	$^{8}14^{1}_{1}$	123456789$_{14}$ 12345679 10$_{14}$ 1234567 10 11$_{14}$ 1234567 11 12$_{14}$ 1234567 12 13$_{14}$ 12345789 10$_{14}$ 1234578 10 11$_{14}$ 1234578 11 12$_{14}$ 1234578 12 13$_{14}$ 1234589 10 11$_{14}$ 1234589 11 12$_{14}$ 1234589 12 13$_{14}$ 123459 10 11 13$_{14}$ 123459 11 12 13$_{14}$ 1235679 10 11$_{14}$ 1235679 11 13$_{14}$ 123569 10 11 13$_{14}$ 1235789 12 13$_{14}$	

Table 2.19: 8-dimensional combinatorial manifolds with vertex-transitive automorphism group.

				$^{8}14^{1}_{2}$	123456789$_{14}$ 12345679 10$_{14}$ 1234567 10 11$_{14}$ 1234567 11 13$_{14}$ 123456 10 11 12$_{14}$ 123456 11 12 13$_{14}$ 12345789 13$_{14}$ 1234579 10 11$_{14}$ 1234579 11 13$_{14}$ 12346789 11$_{14}$ 1234678 11 13$_{14}$ 1234678 12 13$_{14}$ 1234679 10 11$_{14}$ 1234689 10 11$_{14}$ 123468 10 11 13$_{14}$ 123468 10 12 13$_{14}$ 1234789 11 13$_{14}$ 123567 10 11 12$_{14}$	
				$^{8}14^{1}_{3}$	123456789$_{14}$ 12345679 10$_{14}$ 1234567 10 11$_{14}$ 1234567 11 13$_{14}$ 123456 10 11 12$_{14}$ 123456 11 12 13$_{14}$ 12345789 13$_{14}$ 1234579 10 11$_{14}$ 1234579 11 13$_{14}$ 12346789 13$_{14}$ 1234678 12 13$_{14}$ 1234679 10 11$_{14}$ 1234679 11 13$_{14}$ 1234689 10 11$_{14}$ 1234689 11 13$_{14}$ 123468 10 11 12$_{14}$ 123468 11 12 13$_{14}$ 123567 10 11 12$_{14}$	
				$^{8}14^{1}_{4}$	123456789$_{14}$ 12345679 10$_{14}$ 1234567 10 11$_{14}$ 1234567 11 13$_{14}$ 123456 10 11 13$_{14}$ 123456 10 12 13$_{14}$ 12345789 13$_{14}$ 1234579 10 11$_{14}$ 1234579 11 13$_{14}$ 123459 10 11 12$_{14}$ 12345 10 11 12 13$_{14}$ 1234678 10 11$_{14}$ 1234678 11 13$_{14}$ 1234678 12 13$_{14}$ 123468 10 11 13$_{14}$ 123468 10 12 13$_{14}$ 1234789 11 13$_{14}$ 123567 10 11 12$_{14}$	
				$^{8}14^{1}_{6}$	123456789$_{14}$ 12345679 11$_{14}$ 1234567 11 13$_{14}$ 12345689 10$_{14}$ 1234569 10 11$_{14}$ 123456 10 11 13$_{14}$ 123456 10 12 13$_{14}$ 1234579 11 13$_{14}$ 123459 10 11 12$_{14}$ 12345 10 11 12 13$_{14}$ 1234678 10 11$_{14}$ 1234678 11 13$_{14}$ 1234679 10 11$_{14}$ 123468 10 11 13$_{14}$ 123468 10 12 13$_{14}$ 1234789 11 13$_{14}$ 1235679 10 11$_{14}$ 123567 10 11 12$_{14}$	
			D_7	$^{8}14^{2}_{1}$	123456789$_{14}$ 12345679 10$_{14}$ 1234567 10 11$_{14}$ 1234567 11 12$_{14}$ 1234567 12 13$_{14}$ 12345789 10$_{14}$ 1234578 10 11$_{14}$ 1234578 11 12$_{14}$ 1234578 12 13$_{14}$ 1234589 10 11$_{14}$ 1234589 11 12$_{14}$ 1234589 12 13$_{14}$ 123459 10 11 13$_{14}$ 123459 11 12 13$_{14}$ 1235679 10 11$_{14}$ 1235679 11 13$_{14}$ 123569 10 11 13$_{14}$ 1235789 12 13$_{14}$	
				$^{8}14^{2}_{3}$	123456789$_{14}$ 12345679 10$_{14}$ 1234567 10 11$_{14}$ 1234567 11 13$_{14}$ 12345689 10$_{14}$ 12345789 13$_{14}$ 1234579 10 13$_{14}$ 123457 10 11 13$_{14}$ 1234589 10 11$_{14}$ 1234589 11 14$_{14}$ 123459 10 11 13$_{14}$ 12346789 12$_{14}$ 1234678 12 13$_{14}$ 1234679 10 12$_{14}$ 123467 10 11 12$_{14}$ 123568 10 11 14$_{14}$ 1235789 11 13$_{14}$ 1236789 12 13$_{14}$	

Table 2.19: 8-dimensional combinatorial manifolds with vertex-transitive automorphism group.

				$^{8}14^{2}_{4}$	123456789_{14} $12345679\,10_{14}$ $1234567\,10\,11_{14}$ $1234567\,11\,13_{14}$ $12345689\,10_{14}$ $12345789\,13_{14}$ $1234579\,10\,13_{14}$ $123457\,10\,11\,13_{14}$ $1234589\,10\,11_{14}$ $1234589\,11\,14_{14}$ $123459\,10\,11\,13_{14}$ $12346789\,13_{14}$ $1234679\,10\,12_{14}$ $1234679\,12\,13_{14}$ $123467\,10\,11\,13_{14}$ $123567\,10\,11\,14_{14}$ $1235789\,11\,13_{14}$ $123578\,10\,11\,14_{14}$	
				$^{8}14^{2}_{6}$	123456789_{14} $12345679\,10_{14}$ $1234567\,10\,13_{14}$ $12345689\,10_{14}$ $1234568\,10\,11_{14}$ $12345789\,13_{14}$ $1234579\,10\,13_{14}$ $1234589\,10\,11_{14}$ $1234589\,11\,13_{14}$ $123459\,10\,11\,13_{14}$ $12346789\,12_{14}$ $1234678\,12\,13_{14}$ $1234679\,10\,12_{14}$ $123467\,10\,11\,12_{14}$ $123467\,10\,11\,13_{14}$ $123567\,10\,11\,13_{14}$ $1235789\,11\,13_{14}$ $1236789\,12\,13_{14}$	
				$^{8}14^{2}_{7}$	123456789_{14} $12345679\,10_{14}$ $1234567\,10\,13_{14}$ $12345689\,10_{14}$ $1234568\,10\,11_{14}$ $12345789\,13_{14}$ $1234579\,10\,13_{14}$ $1234589\,10\,11_{14}$ $1234589\,11\,13_{14}$ $123459\,10\,11\,13_{14}$ $12346789\,13_{14}$ $1234679\,10\,12_{14}$ $1234679\,12\,13_{14}$ $123567\,10\,11\,13_{14}$ $123567\,10\,11\,14_{14}$ $123568\,10\,11\,14_{14}$ $1235789\,11\,13_{14}$ $123578\,10\,11\,14_{14}$	
				$^{8}14^{2}_{8}$	123456789_{14} $12345679\,10_{14}$ $1234567\,10\,13_{14}$ $12345689\,11_{14}$ $1234569\,10\,11_{14}$ $12345789\,11_{14}$ $1234578\,11\,13_{14}$ $1234579\,10\,11_{14}$ $123457\,10\,11\,13_{14}$ $12345\,10\,11\,12\,13_{14}$ $1234679\,10\,13_{14}$ $1234689\,11\,12_{14}$ $123469\,10\,11\,12_{14}$ $1234789\,11\,12_{14}$ $1235679\,10\,13_{14}$ $1235689\,11\,14_{14}$ $123569\,10\,12\,13_{14}$ $1235789\,11\,13_{14}$	
				$^{8}14^{2}_{13}$	123456789_{14} $12345679\,10_{14}$ $1234567\,10\,13_{14}$ $12345689\,13_{14}$ $1234568\,11\,12_{14}$ $12345789\,13_{14}$ $1234579\,10\,13_{14}$ $123458\,11\,12\,13_{14}$ $12346789\,12_{14}$ $1234678\,11\,12_{14}$ $1234679\,10\,12_{14}$ $123467\,10\,11\,12_{14}$ $123467\,10\,11\,13_{14}$ $1235679\,10\,13_{14}$ $1235679\,11\,12_{14}$ $1235679\,11\,13_{14}$ $1235689\,12\,13_{14}$ $1235789\,12\,13_{14}$	
15	$\sim\mathbb{HP}^{2}$	$(\underline{105},\underline{455},\underline{1365},\underline{3003},4515,4230,2205,490)$	A_5	$^{8}15^{5}_{1}$	$12345678\,12_{60}$ $12345678\,13_{60}$ $1234567\,12\,14_{60}$ $1234567\,13\,15_{15}$ $1234567\,14\,15_{15}$ $12345689\,12_{30}$ $12345689\,13_{30}$ $1234569\,13\,15_{60}$ $1234569\,14\,15_{60}$ $1234578\,10\,11_{20}$ $123459\,10\,13\,15_{10}$ $123459\,10\,14\,15_{30}$ $1234689\,10\,12_{30}$ $123479\,11\,14\,15_{10}$	minimal, tight, [43, M^{8}_{15}], [91], [102]
	....	....	$\mathbb{Z}_{15}$	....	....	

Table 2.20: 9-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Top. type	f-vector $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)$	Automorphism group	Comb. type	List of orbits	Remarks
11	S^9	(55,165,330,462,462,330,165,55,11)	S_{11}	${}^{9}11^{8}_{1}$	123456789 10 $_{11}$	$Bd\,\Delta_{10}$, reg.
12	S^9	(66,220,495,792,922,780,465,180,36)	$S_6 wr 2$	${}^{9}12^{299}_{1}$	123456789 10 $_{36}$	$BdC_{10}(12)$ $= (Bd\,\Delta_5)^{*2}$
13	S^9	(78,286,715,1287,1703,1638,1092,455,91)	D_{13}	${}^{9}13^{2}_{1}$	123456789 10 $_{13}$ 12345678 10 11 $_{26}$ 12345689 10 11 $_{26}$ 12345689 11 12 $_{13}$ 12346789 11 12 $_{13}$	$BdC_{10}(13)$
14	S^9	(91,364,1001,2002,2954,3136,2254,980,196)	D_{14}	${}^{9}14^{3}_{1}$	123456789 10 $_{14}$ 12345678 10 11 $_{28}$ 12345678 11 12 $_{14}$ 12345689 10 11 $_{28}$ 12345689 11 12 $_{28}$ 12345689 12 13 $_{14}$ 1234569 10 11 12 $_{14}$ 12346789 11 12 $_{28}$ 1234679 10 11 12 $_{14}$ 1234679 10 12 13 $_{14}$	$BdC_{10}(14)$
15	S^9	(105,450,1305,2673,3915,4050,2835,1215,243)	$S_3 wr S_5$	${}^{9}15^{93}_{1}$	123456789 10 $_{243}$	3^{*5}
		(105,455,1365,2985,4775,5400,4050,1800,360)	$\mathbb{Z}_5 \times S_3$	${}^{9}15^{4}_{1}$	123456789 10 $_{30}$ 12345678 10 11 $_{30}$ 12345678 11 12 $_{30}$ 12345678 12 14 $_{30}$ 1234567 11 12 13 $_{30}$ 1234567 12 13 14 $_{30}$ 12345689 10 14 $_{30}$ 1234568 10 11 12 $_{30}$ 1234568 10 12 14 $_{30}$ 12345789 10 11 $_{30}$ 1234578 12 13 14 $_{30}$ 123458 10 12 13 14 $_{30}$	
			$D_5 \times S_3$	${}^{9}15^{7}_{1}$	123456789 10 $_{30}$ 12345678 10 11 $_{60}$ 12345678 11 12 $_{60}$ 12345689 10 14 $_{60}$ 1234568 10 11 12 $_{60}$ 1234568 10 12 14 $_{30}$ 12345789 10 11 $_{30}$ 1234678 12 13 14 $_{30}$	
		(105,455,1365,3000,4850,5550,4200,1875,375)	$\mathbb{Z}_{15}$	${}^{9}15^{1}_{1}$	123456789 10 $_{15}$ 12345678 10 11 $_{15}$ 12345678 11 12 $_{15}$ 12345678 12 13 $_{15}$ 12345678 13 14 $_{15}$ 12345689 10 14 $_{15}$ 1234568 10 11 12 $_{15}$ 1234568 10 12 14 $_{15}$ 1234568 12 13 14 $_{15}$ 123456 10 11 12 14 $_{15}$ 12345789 10 11 $_{15}$ 12345789 11 12 $_{15}$ 12345789 12 13 $_{15}$ 1234579 11 12 13 $_{15}$ 1234589 10 11 12 $_{15}$ 1234589 10 12 13 $_{15}$ 1234589 10 13 14 $_{15}$ 123458 10 12 13 14 $_{15}$ 1234678 12 13 14 $_{15}$ 123489 10 12 13 14 $_{15}$ 1235679 10 11 13 $_{15}$ 1235679 10 13 14 $_{15}$ 1235679 11 12 13 $_{15}$ 123567 10 11 13 14 $_{15}$ 1235689 10 12 14 $_{15}$	
		(105,455,1365,3003,4865,5580,4230,1890,378)	D_{15}	${}^{9}15^{2}_{1}$	123456789 10 $_{15}$ 12345678 10 11 $_{30}$ 12345678 11 12 $_{30}$ 12345689 10 11 $_{30}$ 12345689 11 12 $_{30}$ 12345689 12 13 $_{30}$ 12345689 13 14 $_{15}$ 1234569 10 11 12 $_{30}$ 1234569 10 12 13 $_{15}$ 12346789 11 12 $_{30}$ 12346789 12 13 $_{15}$ 1234679 10 11 12 $_{15}$ 1234679 10 12 13 $_{30}$ 1234679 10 13 14 $_{30}$ 123467 10 11 12 13 $_{30}$ 124578 10 11 13 14 $_{3}$	$BdC_{10}(15)$

Table 2.21: 10-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Top. type	f-vector $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10})$	Automorphism group	Comb. type	List of orbits	Remarks
12	S^{10}	(66,220,495,792,924,792,495,220,66,12)	S_{12}	${}^{10}12^{301}_{1}$	123456789 10 11 $_{12}$	$Bd\,\Delta_{11}$, regular
14	S^{10}	(91,364,1001,2002,2996,3376,2814,1652,616,112)	$[2^7]D_7 = 2wrD_7$	${}^{10}14^{38}_{1}$	123456789 10 11 $_{56}$ 123456789 11 12 $_{56}$	

Table 2.22: 11-dimensional combinatorial manifolds with vertex-transitive automorphism group.

n	Top. type	f-vector $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11})$	Automorphism group	Comb. type	List of orbits	Remarks
13	S^{11}	(78,286,715,1287,1716,1716,1287,715,286,78,13)	S_{13}	${}^{11}13^{9}_{1}$	123456789 10 11 12 $_{13}$	$Bd\,\Delta_{12}$, reg.
14	S^{11}	(91,364,1001,2002,3003,3430,2989,1960,931,294,49)	$[S_7^2]2 = S_7wr2$	${}^{11}14^{61}_{1}$	123456789 10 11 12 $_{49}$	$BdC_{12}(14)$ $= (Bd\,\Delta_6)^{*2}$
15	S^{11}	(105,455,1365,3000,4975,6300,6075,4375,2250,750,125)	$[S_5^3]S_3 = S_5wrS_3$	${}^{11}15^{102}_{1}$	123456789 10 11 12 $_{125}$	$(Bd\,\Delta_4)^{*3}$
		(105,455,1365,3003,5000,6390,6255,4590,2403,810,135)	$[S_3^5]D_5 = S_3wrD_5$	${}^{11}15^{86}_{1}$	123456789 10 11 12 $_{135}$	
		(105,455,1365,3003,5005,6420,6330,4690,2478,840,140)	D_{15}	${}^{11}15^{2}_{1}$	123456789 10 11 12 $_{15}$ 123456789 10 12 13 $_{30}$ 12345678 10 11 12 13 $_{30}$ 12345678 10 11 13 14 $_{15}$ 12345689 10 11 12 13 $_{15}$ 12345689 10 11 13 14 $_{30}$ 12346789 11 12 13 14 $_{5}$	$BdC_{12}(15)$

2.6 List of Transitive Permutation Groups

The cyclic group actions, 10^1, 11^1, 12^1, 13^1, 14^1, and 15^1, and the dihedral group actions, 7^2, 8^6, 9^3, 10^3, 11^2, 12^{12}, 13^2, 14^3, and 15^2, have generators $a_n = (123\ldots n)$ and $b_n = (1\ 2\lfloor\frac{n}{2}\rfloor)(2\ 2\lfloor\frac{n}{2}\rfloor-1)\ldots(\lfloor\frac{n}{2}\rfloor\ \lfloor\frac{n}{2}\rfloor+1)$, with $\mathbb{Z}_n = \langle a_n\rangle$ and $D_n = \langle a_n, b_n\rangle$. The automorphism group of the boundary complex of a $(d+1)$-simplex trivially is the full symmetric group S_{d+2}. All other generators of the vertex-transitive automorphism groups that appeared in the previous sections can be found in Table 2.23.

Corollary 2.19 *The list of transitive permutation groups given in this section are precisely all permutation groups that occur as vertex-transitive automorphism groups of a combinatorial manifold on $n \leq 15$ vertices.*

Table 2.23: List of generators for the group actions.

Action	Group	Generators
6^{11}	$[2^3]S_3 = 2wrS_3$	$(135)(246)$, $(15)(24)$, (36)
6^{12}	A_5	(12346), $(14)(56)$
6^{13}	$[S_3^2]2 = S_3wr2$	$(14)(25)(36)$, (24), (246)
7^4	$7{:}6$	(1234567), (132645)
8^{15}	$t8n15(32)$	(12345678), $(15)(37)$, $(16)(25)(34)(78)$
8^{44}	$[2^4]S_4$	$(1238)(4567)$, $(18)(45)$, (48)
8^{47}	$[S_4^2]2$	(1238), $(15)(26)(37)(48)$, (23)
9^4	$S_3 \times \mathbb{Z}_3$	$(12)(45)(78)$, $(129)(345)(678)$, $(147)(258)(369)$
9^{13}	$\mathbb{Z}_3^2{:}\mathbb{Z}_6$	$(12)(35)(67)$, $(129)(345)(678)$, $(147)(258)(369)$, $(345)(687)$
9^{18}	$\mathbb{Z}_3^2{:}D_6$	$(12)(35)(67)$, $(12)(36)(48)(57)$, $(129)(345)(678)$, $(147)(258)(369)$, $(345)(687)$
9^{31}	$[S_3^3]S_3 = S_3wrS_3$	(12), (129), $(147)(258)(369)$, $(36)(47)(58)$
10^2	D_5	$(13579)(2468\mathit{10})$, $(14)(23)(5\mathit{10})(69)(78)$
10^4	$\frac{1}{2}[5{:}4]2$	$(1298)(3674)(5\mathit{10})$, $(13579)(2468\mathit{10})$
10^7	A_5	$(13579)(2468\mathit{10})$, $(19)(34)(5\mathit{10})(67)$
10^{21}	$[D_5^2]2$	$(16)(27)(38)(49)(5\mathit{10})$, $(2468\mathit{10})$, $(28)(46)$
10^{22}	$S_5 \times \mathbb{Z}_2$	$(13579)(2468\mathit{10})$, $(16)(27)(38)(49)(5\mathit{10})$, $(2\mathit{10})(57)$
10^{23}	$[2^5]D_5$	$(13579)(2468\mathit{10})$, $(19)(28)(37)(46)$, $(5\mathit{10})$
10^{39}	$[2^5]S_5$	$(13579)(2468\mathit{10})$, $(2\mathit{10})(57)$, $(5\mathit{10})$
10^{43}	$[S_5^2]2$	$(16)(27)(38)(49)(5\mathit{10})$, $(2468\mathit{10})$, $(2\mathit{10})$
12^2	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	$(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(17)(28)(39)(4\mathit{10})(5\mathit{11})(6\mathit{12})$, $(1\mathit{10})(25)(3\mathit{12})(47)(69)(8\mathit{11})$
12^3	D_6	$(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(1\mathit{11})(28)(39)(46)(57)(\mathit{10}\,\mathit{12})$, $(1\mathit{12})(23)(45)(67)(89)(\mathit{10}\,\mathit{11})$
12^4	A_4	$(195)(243)(687)(\mathit{10}\,\mathit{12}\,\mathit{11})$, $(1\mathit{11}6)(297)(3\mathit{10}5)(48\mathit{12})$
12^5	$\frac{1}{2}[3{:}2]4$	$(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(17)(28)(39)(4\mathit{10})(5\mathit{11})(6\mathit{12})$, $(1872)(369\mathit{12})(4\mathit{11}\,\mathit{10}5)$
12^6	$A_4(12) \times \mathbb{Z}_2$	$(17)(2\mathit{11})(3\mathit{12})(4\mathit{10})(58)(69)$, $(195)(243)(687)(\mathit{10}\,\mathit{12}\,\mathit{11})$, $(1\mathit{11}6)(297)(3\mathit{10}5)(48\mathit{12})$
12^7	$A_4(6) \times \mathbb{Z}_2$	$(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(1\mathit{12})(23)(45)(67)(89)(\mathit{10}\,\mathit{11})$, $(28)(39)(4\mathit{10})(5\mathit{11})$
12^8	$t12n8(24) = S_4$	$(12)(35)(46)(79)(8\mathit{10})$, $(136\mathit{12})(247\mathit{10})(58\mathit{11}9)$
12^9	$t12n9(24) = S_4$	$(12)(3\mathit{12})(4\mathit{11})(5\mathit{10})(69)(78)$, $(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(17)(39)(4\mathit{10})(6\mathit{12})$
12^{10}	$S_3 \times \mathbb{Z}_2^2$	$(15)(2\mathit{10})(48)(7\mathit{11})$, $(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(17)(28)(39)(4\mathit{10})(5\mathit{11})(6\mathit{12})$, $(1\mathit{10})(25)(3\mathit{12})(47)(69)(8\mathit{11})$
12^{11}	$S_3 \times \mathbb{Z}_4$	$(147\mathit{10})(258\mathit{11})(369\mathit{12})$, $(15)(2\mathit{10})(48)(7\mathit{11})$, $(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$
12^{13}	$t12n13(24)$	$(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(17)(28)(39)(4\mathit{10})(5\mathit{11})(6\mathit{12})$, $(1\mathit{10})(25)(3\mathit{12})(47)(69)(8\mathit{11})$, $(1\mathit{11})(2\mathit{10})(39)(48)(57)$
12^{14}	$D_4 \times \mathbb{Z}_3$	$(147\mathit{10})(258\mathit{11})(369\mathit{12})$, $(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(17)(39)(5\mathit{11})$
12^{15}	$t12n15(24)$	$(12)(3\mathit{12})(4\mathit{11})(5\mathit{10})(69)(78)$, $(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(17)(39)(5\mathit{11})$, $(28)(4\mathit{10})(6\mathit{12})$
12^{28}	$D_4 \times S_3$	$(147\mathit{10})(258\mathit{11})(369\mathit{12})$, $(15)(2\mathit{10})(48)(7\mathit{11})$, $(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(17)(39)(5\mathit{11})$
12^{54}	$t12n54(96)$	$(12)(3\mathit{12})(4\mathit{11})(5\mathit{10})(69)(78)$, $(159)(26\mathit{10})(37\mathit{11})(48\mathit{12})$, $(1\mathit{12})(23)$, $(23)(49)(58)(6\mathit{10})(7\mathit{11})$

Table 2.23: List of generators for the group actions.

12^{75}	$A_5 \times \mathbb{Z}_2$	(13579)(2468*12*), (1*11*)(28)(39)(*10 12*), (1*12*)(23)(45)(67)(89)(*10 11*)
12^{76}	$[2]A_5$	(12)(3*12*)(4*11*)(5*10*), (1*12*)(23)(45)(67)(89)(*10 11*), (2468*10*)(3579*11*), (4*10*)(5*11*)(68)(79)
12^{83}	$S_4 \times S_3$	(147*10*)(258*11*)(369*12*), (15)(2*10*)(48)(7*11*), (159)(26*10*)(37*11*)(48*12*), (1*10*)(25)(69)
12^{113}	$t12n113(192)$	(13)(2*12*)(4*10*)(5*11*)(68)(79), (13579*11*)(2468*10 12*), (1*12*)(23)(67)(89), (4*10*)(5*11*)(67)(89)
12^{124}	$[2]A_5:2$	(13*12*2)(4657)(8*11 9 10*), (1*12*)(23)(45)(67)(89)(*10 11*), (2468*10*)(3579*11*)
12^{125}	$[S_3^2]D_4 = D_6 wr 2$	(147*10*)(258*11*)(369*12*), (17)(39)(5*11*), (26*10*)(48*12*), (2*10*)(48)
12^{193}	$[2^6]D_6 = 2 wr D_6$	(13579*11*)(2468*10 12*), (1*11*)(28)(39)(46)(57)(*10 12*), (1*12*)
12^{289}	$[S_3^4]S_4 = S_3 wr S_4$	(147*10*)(258*11*)(369*12*), (1*10*)(25)(69), (48), (48*12*)
12^{293}	$[2^6]S_6 = 2 wr S_6$	(13)(2*12*), (13579*11*)(2468*10 12*), (1*12*)
12^{294}	$[S_4^3]S_3 = S_4 wr S_3$	(15)(2*10*)(48)(7*11*), (159)(26*10*)(37*11*)(48*12*), (369*12*), (69)
12^{299}	$[S_6^2]2 = S_6 wr 2$	(1*12*)(23)(45)(67)(89)(*10 11*), (2468*10 12*), (2*12*)
13^{3}	$13:3$	(123456789*10 11 12 13*), (139)(265)(4*12 10*)(78*11*)
13^{4}	$13:4$	(123456789*10 11 12 13*), (15*12*8)(2*10 11*3)(4796)
13^{5}	$13:6$	(123456789*10 11 12 13*), (143*12 9 10*)(286*11*57)
14^{2}	D_7	(13579*11 13*)(2468*10 12 14*), (1*12*)(2*11*)(3*10*)(49)(58)(67)(*13 14*)
14^{4}	$2[\frac{1}{2}]7:6$	(13579*11 13*)(2468*10 12 14*), (16)(25)(34)(7*14*)(8*13*)(9*12*)(*10 11*), (19*11*)(248)(3*13*5)(6*12 10*)
14^{5}	$7:3 \times \mathbb{Z}_2$	(13579*11 13*)(2468*10 12 14*), (18)(29)(3*10*)(4*11*)(5*12*)(6*13*)(7*14*), (19*11*)(248)(3*13*5)(6*12 10*)
14^{7}	$7:6 \times \mathbb{Z}_2$	(13579*11 13*)(2468*10 12 14*), (18)(29)(3*10*)(4*11*)(5*12*)(6*13*)(7*14*), (19*11*)(248)(3*13*5)(6*12 10*), (1*13*)(2*12*)(3*11*)(4*10*)(59)(68)
14^{16}	$L(2,7):2$	(18)(29)(3*10*)(4*11*)(5*12*)(6*13*)(7*14*), (19*11*)(248)(3*13*5)(6*12 10*), (1*13 11*9753)(2468*10 12 14*), (24)(5*13*)(6*12*)(9*11*)
14^{19}	$L(2,7) \times \mathbb{Z}_2$	(13579*11 13*)(2468*10 12 14*), (18)(29)(3*10*)(4*11*)(5*12*)(6*13*)(7*14*), (19*11*)(248)(3*13*5)(6*12 10*), (24)(5*13*)(6*12*)(9*11*)
14^{20}	$[D_7^2]2 = D_7 wr 2$	(18)(29)(3*10*)(4*11*)(5*12*)(6*13*)(7*14*), (2468*10 12 14*), (2*12*)(4*10*)(68)
14^{38}	$[2^7]D_7 = 2 wr D_7$	(13579*11 13*)(2468*10 12 14*), (1*13*)(2*12*)(3*11*)(4*10*)(59)(68), (7*14*)
14^{49}	$S_7 \times \mathbb{Z}_2$	(13579*11 13*)(2468*10 12 14*), (18)(29)(3*10*)(4*11*)(5*12*)(6*13*)(7*14*), (35)(*10 12*)
14^{57}	$[2^7]S_7$	(13579*11 13*)(2468*10 12 14*), (35)(*10 12*), (3*13*5)(6*12 10*), (7*14*)
14^{61}	$[S_7^2]2 = S_7 wr 2$	(18)(29)(3*10*)(4*11*)(5*12*)(6*13*)(7*14*), (2468*10 12 14*), (*10 12*)
15^{3}	$D_5 \times \mathbb{Z}_3$	(123456789*10 11 12 13 14 15*), (14)(28)(3*12*)(69)(7*13*)(*11 14*)
15^{4}	$\mathbb{Z}_5 \times S_3$	(123456789*10 11 12 13 14 15*), (1*11*)(27)(4*14*)(5*10*)(8*13*)
15^{5}	A_5	(14*10*)(258)(37*11*)(69*15*)(*12 14 13*), (19*10*3*14*)(2*15*7*12*6)(45*11 13*8)
15^{6}	$5:4[\frac{1}{2}]S_3$	(123456789*10 11 12 13 14 15*), (1248)(36*12*9)(5*10*)(7*14 13 11*), (14)(28)(3*12*)(69)(7*13*)(*11 14*)
15^{7}	$D_5 \times S_3$	(123456789*10 11 12 13 14 15*), (14)(28)(3*12*)(69)(7*13*)(*11 14*), (1*11*)(27)(4*14*)(5*10*)(8*13*)
15^{8}	$5:4 \times \mathbb{Z}_3$	(123456789*10 11 12 13 14 15*), (174*13*)(2*14*8*11*)(36*12*9)
15^{10}	S_5	(14)(26)(37)(5*15*)(89)(*12 13*), (14*10*)(258)(37*11*)(69*15*)(*12 14 13*), (19*10*3*14*)(2*15*7*12*6)(45*11 13*8)
15^{11}	$5:4 \times S_3$	(123456789*10 11 12 13 14 15*), (174*13*)(2*14*8*11*)(36*12*9), (1*11*)(27)(4*14*)(5*10*)(8*13*)
15^{15}	$[3]A_5 = GL(2,4)$	(12*15*)(456)(89*10*)(*12 13 14*), (14*10*)(258)(37*11*)(69*15*)(*12 14 13*), (19*10*3*14*)(2*15*7*12*6)(45*11 13*8)
15^{18}	$[5^2:2]S_3$	(14)(28)(3*12*)(69)(7*13*)(*11 14*), (16*11*)(27*12*)(38*13*)(49*14*)(5*10 15*), (1*11*)(27)(4*14*)(5*10*)(8*13*), (1*13 10*74)(258*11 14*)
15^{29}	$S_5 \times S_3$	(123456789*10 11 12 13 14 15*), (14)(69)(*11 14*), (1*11*)(27)(4*14*)(5*10*)(8*13*)
15^{60}	$[D_5^3]S_3 = D_5 wr S_3$	(16*11*)(27*12*)(38*13*)(49*14*)(5*10 15*), (1*11*)(27)(4*14*)(5*10*)(8*13*), (369*12 15*), (3*12*)(69)
15^{86}	$[S_3^5]D_5 = S_3 wr D_5$	(14)(28)(3*12*)(69)(7*13*)(*11 14*), (147*10 13*)(258*11 14*)(369*12 15*), (5*10*), (5*10 15*)
15^{93}	$[S_3^5]S_5 = S_3 wr S_5$	(14)(69)(*11 14*), (147*10 13*)(258*11 14*)(369*12 15*), (5*10*), (5*10 15*)
15^{102}	$[S_5^3]S_3 = S_5 wr S_3$	(16*11*)(27*12*)(38*13*)(49*14*)(5*10 15*), (1*11*)(27)(4*14*)(5*10*)(8*13*), (369*12 15*), (69)

Chapter 3

Combinatorial Pseudomanifolds with Transitive Automorphism Group on Few Vertices

A *combinatorial d-pseudomanifold* is a pure simplicial complex such that the link of every vertex is a (connected) combinatorial $(d-1)$-manifold.

The Euler characteristic of every odd-dimensional manifold is zero. Thus, if d is even, then Steps 1–6 of the enumeration algorithm from the previous chapter produce all candidates for combinatorial d-pseudomanifolds on n vertices that have a vertex-transitive automorphism group along with the candidates for vertex-transitive combinatorial d-manifolds on n vertices.

If d is odd, the link of a vertex has even dimension. With the described settings of the enumeration algorithm, the $(d-1)$-dimensional links of the candidates from Steps 1–6 are *Eulerian manifolds*, i.e., pure simplicial complexes that have the Euler characteristic of a $(d-1)$-dimensional sphere. In particular, a candidate could possibly be a non-PL manifold with a non-PL sphere as its vertex-link or a *homology manifold* with a homology sphere as the vertex-link. If we want to generate all vertex-transitive odd-dimensional d-pseudomanifolds, then we merely delete from the enumeration algorithm the requirement that the link of a vertex should have the Euler characteristic of a sphere.

In this chapter, we determine, up to combinatorial equivalence, all combinatorial pseudomanifolds with a vertex-transitive automorphism group on $n \leq 13$ vertices. With the exception of actions of groups of small order, $|G| = n$ for $n = 14$ and $|G| \leq 2n$ for $n = 15$ in dimensions $4 \leq d \leq 7$, we extend this result to $n \leq 15$ vertices. We also construct examples of 4-pseudomanifolds on $16 \leq n \leq 20$ vertices that have a large automorphism group.

On 13 vertices, the enumeration produced two 5-pseudomanifolds with a vertex-transitive automorphism group that have the combinatorial manifold $\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$, triangulated with 12 vertices, as their vertex-link. On 14 vertices there are at least two 5-pseudomanifolds that have the combinatorial manifold $S^2 \times S^2$ as vertex-link. Moreover, we present a vertex-minimal triangulation of $\mathbb{R}\mathbf{P}^4$ with 16 vertices, which was obtained by applying BISTELLAR flips to the vertex-link of the 5-dimensional combinatorial Kummer variety of KÜHNEL [89]. By the examples $\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$ and $\mathbb{R}\mathbf{P}^4$ we see that the vertex links of combinatorial pseudomanifolds can yield new examples of interesting combinatorial manifolds with few vertices.

3.1 Notations and Conventions

From now on, we use the term *combinatorial pseudomanifold* only for proper pseudomanifolds that are *not* manifolds.

We label, up to combinatorial equivalence, every vertex-transitive combinatorial pseudomanifold that we find with a unique symbol ${}^{d}_{p}n^{i}_{k}$ denoting the k-th example of a combinatorial pseudomanifold of dimension d with n vertices listed for the i-th transitive permutation group n^i of degree n.

3.2 Combinatorial Pseudomanifolds with Few Vertices

The link of a vertex-transitive combinatorial d-pseudomanifold with n vertices is a non-spherical $(d-1)$-dimensional manifold with at most $n-1$ vertices. In particular, there are no combinatorial 2-pseudomanifolds, and combinatorial pseudomanifolds with $n \leq 15$ vertices only exist in dimension $3 \leq d \leq 7$ by the BREHM-KÜHNEL bound (Theorem 2.1) for the numbers of vertices of combinatorial manifolds. The deduced lower bounds for the numbers of vertices of low-dimensional combinatorial pseudomanifolds are given in Table 3.1.

d	3	4	5	6	7	8
n	7	10	10	13	14	16

Table 3.1: Minimal number of vertices n of a combinatorial d-pseudomanifold.

Theorem 3.1 *There are precisely* 364 *combinatorial pseudomanifolds with* $n \leq 13$ *vertices that have a vertex-transitive automorphism group;* 362 *of these are* 3*-dimensional and two are* 5*-dimensional.*

3.2.1 3-Pseudomanifolds

We used the program MANIFOLD_VT, adapted to the search for pseudomanifolds, to enumerate all combinatorial 3-pseudomanifolds with $n \leq 15$ vertices.

Theorem 3.2 *There are exactly* 6935 *combinatorial* 3*-pseudomanifolds with* $n \leq 15$ *vertices that have a vertex-transitive automorphism group.*

The explicit numbers of combinatorial 3-pseudomanifolds with $n \leq 15$ vertices, in relation to the numbers of combinatorial 3-manifolds with $n \leq 15$ vertices, are given in Table 3.2.

n	5	6	7	8	9	10	11	12	13	14	15
# manifolds	1	1	1	2	2	10	6	39	15	55	34
# pseudomanifolds	–	–	0	3	2	6	10	209	132	1668	4905

Table 3.2: Manifolds and pseudomanifolds with vertex-transitive symmetry in dimension 3.

Table 3.3: 3-dimensional pseudomanifolds with vertex-transitive automorphism group.

n	Homology (H_0, H_1, H_2, H_3)	f-vector (f_1, f_2, f_3)	Or. of the link	Gen. of the link	Autom. group	Comb. type	List of orbits	Remarks
8	$(\mathbb{Z}, 0, \mathbb{Z}^8, \mathbb{Z})$	$(\underline{28},\underline{56},28)$	+	1	$PGL(2,7)$	${}^3_p8^{43}_1$	1235_{28}	design, [5, $\mathcal{T}$], [61, P_{28}], [91, p. 106]
	$(\mathbb{Z}, 0, \mathbb{Z}^3 \oplus \mathbb{Z}_2, 0)$	$(\underline{28},48,24)$	–	1	$[2^3]4$	${}^3_p8^{20}_1$	1234_8 1235_{16}	
	$(\mathbb{Z}, 0, \mathbb{Z}^3 \oplus \mathbb{Z}_2, 0)$	$(\underline{28},48,24)$	–	1	$\mathbb{Z}_2^3 : S_4$	${}^3_p8^{41}_1$	1245_{24}	[5, $\mathcal{P}$], [89]
9	$(\mathbb{Z}, 0, \mathbb{Z}^9, \mathbb{Z})$	$(\underline{36},72,36)$	+	1	$\mathbb{Z}_3^2 : 2D_4$	${}^3_p9^{19}_1$	1234_{36}	
	$(\mathbb{Z}, 0, \mathbb{Z}^8 \oplus \mathbb{Z}_2, 0)$	$(\underline{36},72,36)$	–	2	D_9	${}^3_p9^{3}_1$	1235_{18} 1248_9 1257_9	
10	$(\mathbb{Z}, 0, \mathbb{Z}^{10}, \mathbb{Z})$	$(\underline{45},90,45)$	+	1	D_{10}	${}^3_p10^{3}_1$	1234_{10} 1248_{20} 1256_{10} 1267_5	
	$(\mathbb{Z}, 0, \mathbb{Z}^{10}, \mathbb{Z})$	$(\underline{45},90,45)$	+	1	$\frac{1}{2}[5:4]2$	${}^3_p10^{4}_1$	1235_{20} 1247_{20} 1289_5	
	$(\mathbb{Z}, 0, \mathbb{Z}^9 \oplus \mathbb{Z}_2, 0)$	$(\underline{45},90,45)$	–	2	D_{10}	${}^3_p10^{3}_3$	1236_{20} 1245_{10} 1249_{10} 1267_5	
	$(\mathbb{Z}, 0, \mathbb{Z}^9 \oplus \mathbb{Z}_2, 0)$	$(\underline{45},90,45)$	–	2	$\frac{1}{2}[5:4]2$	${}^3_p10^{4}_2$	1245_5 1249_{20} 1268_{20}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{14} \oplus \mathbb{Z}_2, 0)$	$(\underline{45},100,50)$	–	3	D_5	${}^3_p10^{2}_1$	1234_5 1235_{10} 1247_{10} 1256_5 13510_{10} 1458_5 14510_5	
	$(\mathbb{Z}, 0, \mathbb{Z}^{19} \oplus \mathbb{Z}_2, 0)$	$(\underline{45},110,55)$	–	4	D_{10}	${}^3_p10^{3}_2$	1235_{20} 1249_{10} 1257_{20} 1267_5	
11	$(\mathbb{Z}, 0, \mathbb{Z}^{11}, \mathbb{Z})$	$(\underline{55},110,55)$	+	1	D_{11}	${}^3_p11^{2}_1$	1234_{11} 1245_{11} 1258_{22} 1267_{11}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{11}, \mathbb{Z})$	$(\underline{55},110,55)$	+	1	D_{11}	${}^3_p11^{2}_2$	1234_{11} 1246_{22} 1267_{11} 1369_{11}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{22}, \mathbb{Z})$	$(\underline{55},132,66)$	+	2	D_{11}	${}^3_p11^{2}_5$	1235_{22} 12410_{11} 1256_{11} 1268_{11} 1368_{11}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{10} \oplus \mathbb{Z}_2, 0)$	$(\underline{55},110,55)$	–	2	$\mathbb{Z}_{11}$	${}^3_p11^{1}_2$	1235_{11} 1236_{11} 1248_{11} 1268_{11} 1358_{11}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{10} \oplus \mathbb{Z}_2, 0)$	$(\underline{55},110,55)$	–	2	D_{11}	${}^3_p11^{2}_3$	1234_{11} 1246_{22} 1268_{11} 1368_{11}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{10} \oplus \mathbb{Z}_2, 0)$	$(\underline{55},110,55)$	–	2	$11:10$	${}^3_p11^{4}_1$	1245_{55}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{21} \oplus \mathbb{Z}_2, 0)$	$(\underline{55},132,66)$	–	4	$\mathbb{Z}_{11}$	${}^3_p11^{1}_3$	1235_{11} 1236_{11} 1249_{11} 1269_{11} 1359_{11} 1368_{11}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{21} \oplus \mathbb{Z}_2, 0)$	$(\underline{55},132,66)$	–	4	$\mathbb{Z}_{11}$	${}^3_p11^{1}_4$	1235_{11} 1238_{11} 1246_{11} 1256_{11} 1279_{11} 1368_{11}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{32} \oplus \mathbb{Z}_2, 0)$	$(\underline{55},154,77)$	–	6	$\mathbb{Z}_{11}$	${}^3_p11^{1}_1$	1234_{11} 1246_{11} 1258_{11} 12510_{11} 1267_{11} 1358_{11} 1368_{11}	
	$(\mathbb{Z}, 0, \mathbb{Z}^{32} \oplus \mathbb{Z}_2, 0)$	$(\underline{55},154,77)$	–	6	D_{11}	${}^3_p11^{2}_4$	1235_{22} 1247_{22} 1256_{11} 1268_{11} 1368_{11}	
13	$(\mathbb{Z}, 0, \mathbb{Z}^{78}, \mathbb{Z})$	$(\underline{78},\underline{286},143)$	+	6	$13:4$	${}^3_p13^{4}_1$	1234_{26} 12410_{52} 1258_{52} 1267_{13}	design, [50]
	$(\mathbb{Z}, 0, \mathbb{Z}^{78}, \mathbb{Z})$	$(\underline{78},\underline{286},143)$	+	6	$13:4$	${}^3_p13^{4}_6$	1235_{52} 12412_{13} 1259_{52} 1267_{13} 13710_{13}	design, [50]
	$(\mathbb{Z}, 0, \mathbb{Z}^{77} \oplus \mathbb{Z}_2, 0)$	$(\underline{78},\underline{286},143)$	–	12	$13:4$	${}^3_p13^{4}_3$	1234_{26} 12411_{52} 1259_{52} 1267_{13}	design, [50], [93, 10.5]
14	$(\mathbb{Z}, 0, \mathbb{Z}^{104} \oplus \mathbb{Z}_2, 0)$	$(\underline{91},\underline{364},182)$	–	15	D_7	${}^3_p14^{2}_{204}$	1234_7 1235_{14} 1249_{14} 1257_{14} 1269_{14} 12610_{14} 1358_{14} 1367_{14} 1379_{14} 13814_{14} 131012_7 14514_7 14710_7 14711_{14} 151014_7 16714_7	design

In Table 3.3, we list all combinatorial 3-pseudomanifolds with $n \leq 11$ vertices. For $12 \leq n \leq 15$ vertices we list only those pseudomanifolds that are designs. A *block design* $S_\lambda(t,k,n)$ is a collection of k-element subsets of an n-element set. These k-tuples (or *blocks*) need not to be distinct, but every t-tuple of elements of the n-element ground set has to be included in exactly λ blocks. For surveys on designs see [19] and [50]. Block designs that are, at the same time, (maximal faces of) combinatorial pseudomanifolds were studied in [93] and [101]. Every 3-neighborly 3-pseudomanifold with n vertices is automatically a design $S_2(3,4,n)$ without repeated blocks.

Corollary 3.3 *There are exactly five* 3*-neighborly* 3*-pseudomanifolds with* $n \leq 15$ *vertices that have a vertex-transitive automorphism group.*

REMARK: ALTSHULER [5] showed that every finite set $\mathcal{M}$ of triangulated surfaces is *pm-realizable*, i.e., for every $\mathcal{M}$ there exists a 3-pseudomanifold P such that the link of every vertex of P is a member of $\mathcal{M}$ and every surface of $\mathcal{M}$ is realized as the link of some vertex of P (see also [7]). Therefore, it comes as no surprise that we find a large number of combinatorially distinct 3-pseudomanifolds, even with few vertices and in addition with a vertex-transitive group action.

3.2.2 4-Pseudomanifolds

There is no vertex-transitive combinatorial 4-pseudomanifold with $n \leq 13$ vertices and also no such example with $14 \leq n \leq 15$ vertices and an automorphism group G of order $|G| > n$. This leaves three remaining actions, ${}^4 14\,^1$, ${}^4 14\,^2$, and ${}^4 15\,^1$, of the groups $\mathbb{Z}_{14}$, D_7, and $\mathbb{Z}_{15}$ of order $|G| = n$, respectively, that we couldn't process with MANIFOLD_VT.

On $16 \leq n \leq 20$ vertices, we found several combinatorial 4-pseudomanifolds that have a vertex-transitive automorphism group of large order. (We were not able to determine the full symmetry group of the example ${}^4_p 20\,^{[87]}_1$, but we believe that it is the group $20\,^{87}$.)

n	Homology (H_0,H_1,H_2,H_3,H_4)	f-vector (f_1,f_2,f_3,f_4)	Top. type of the link	Autom. group	Comb. type	List of orbits	Ref.
16	$(\mathbb{Z},0,\mathbb{Z}^6\oplus\mathbb{Z}_2^5,0,\mathbb{Z})$	$(\underline{120},400,480,192)$	$\mathbb{R}\mathbf{P}^3$	$t16n421(192)$	${}^4_p 16\,^{421}_1$	$129\;11\;13\;_{96}$ $1359\;14\;_{96}$	
	$(\mathbb{Z},0,\mathbb{Z}^6\oplus\mathbb{Z}_2^5,0,\mathbb{Z})$	$(\underline{120},400,480,192)$	$\mathbb{R}\mathbf{P}^3$	$t16n1033(576)$	${}^4_p 16\,^{1033}_1$	$1238\;12\;_{48}$ $127\;11\;16\;_{144}$	
	$(\mathbb{Z},0,\mathbb{Z}^6\oplus\mathbb{Z}_2^5,0,\mathbb{Z})$	$(\underline{120},400,480,192)$	$\mathbb{R}\mathbf{P}^3$	$t16n1328(1920)$	${}^4_p 16\,^{1328}_1$	$12358\;_{192}$	[89]
17	$(\mathbb{Z},0,\mathbb{Z}^{16}\oplus\mathbb{Z}_5,\mathbb{Z}_2,0)$	$(\underline{136},544,680,272)$	$S^2 \underline{\times} S^1$	$17\!:\!16$	${}^4_p 17\,^5_1$	$12489\;_{272}$	
19	$(\mathbb{Z},0,\mathbb{Z}^{18}\oplus\mathbb{Z}_{19},\mathbb{Z},\mathbb{Z})$	$(\underline{171},684,855,342)$	$S^2\times S^1$	$19\!:\!18$	${}^4_p 19\,^6_1$	$1245\;10\;_{342}$	
	$(\mathbb{Z},0,\mathbb{Z}^{38},\mathbb{Z}^2,\mathbb{Z})$	$(\underline{171},874,1140,456)$	$(S^2\times S^1)^{\#2}$	$19\!:\!6$	${}^4_p 19\,^4_1$	$1234\;11\;_{114}$ $1237\;10\;_{114}$ $1237\;11\;_{114}$ $1248\;11\;_{114}$	
20	$(\mathbb{Z},\mathbb{Z},\mathbb{Z}^{10},\mathbb{Z}^{10}\oplus\mathbb{Z}_2,0)$	$(180,520,600,240)$	$S^2\times S^1$	$[t20n87(320)]$	${}^4_p 20\,^{[87]}_1$	$1358\;14\;_{160}$ $135\;12\;14\;_{80}$	

Table 3.4: 4-dimensional pseudomanifolds with vertex-transitive automorphism group.

REMARK: We used the program BISTELLAR_EQUIVALENT to determine the topological types of the vertex-links of the 4-pseudomanifolds of Table 3.4.

3.2.3 5-Pseudomanifolds

Previously, only one combinatorial 5-pseudomanifold with a vertex-transitive automorphism group was known, namely the 5-dimensional combinatorial Kummer variety of Kühnel [89] with 31 vertices. We present four new examples with $n \leq 15$ vertices and examine their vertex-links. Table 3.5 gives an overview of the examples.

n	Homology $(H_0, H_1, H_2, H_3, H_4, H_5)$	f-vector $(f_1, f_2, f_3, f_4, f_5)$	Top. type of the link	Autom. group	Comb. type
13	$(\mathbb{Z}, 0, 0, \mathbb{Z}^{13}, 0, \mathbb{Z})$	$(\underline{78}, \underline{286}, 598, 546, 182)$	$\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$	$\mathbb{Z}_{13}$	${}^5_p13^1_1$
	$(\mathbb{Z}, 0, 0, \mathbb{Z}^{13}, 0, \mathbb{Z})$	$(\underline{78}, \underline{286}, 598, 546, 182)$	$\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$	D_{13}	${}^5_p13^2_1$
14	$(\mathbb{Z}, 0, 0, \mathbb{Z}^{14}, 0, \mathbb{Z})$	$(\underline{91}, \underline{364}, 777, 714, 238)$	$S^2 \times S^2$	D_{14}	${}^5_p14^3_1$
	$(\mathbb{Z}, 0, 0, \mathbb{Z}^{14}, 0, \mathbb{Z})$	$(\underline{91}, \underline{364}, 777, 714, 238)$	$S^2 \times S^2$	D_{14}	${}^5_p14^3_2$
	....	....	....	$\mathbb{Z}_{14}$, D_7	....
15	....	....	....	$\mathbb{Z}_{15}$, D_{15}, $D_5 \times \mathbb{Z}_3$ $\mathbb{Z}_5 \times S_3$	....

Table 3.5: 5-dimensional pseudomanifolds with vertex-transitive automorphism group.

Two Triangulations of $\mathbb{C}\mathbf{P}^2 \# -\mathbb{C}\mathbf{P}^2$ with 12 Vertices

With the exception of possible examples that might exist for the group actions ${}^514^1$, ${}^514^2$, ${}^515^1$, ${}^515^2$, ${}^515^3$, and ${}^515^4$, we found four 5-pseudomanifolds, ${}^5_p13^1_1$, ${}^5_p13^2_1$, ${}^5_p14^3_1$, and ${}^5_p14^3_2$, with a vertex-transitive automorphism group on $n \leq 15$ vertices. Both pseudomanifolds on 13 vertices are 3-neighborly and thus simply connected with f-vector $f = (\underline{13}, \underline{78}, \underline{286}, 598, 546, 182)$ and homology $H_* = (\mathbb{Z}, 0, 0, \mathbb{Z}^{13}, 0, \mathbb{Z})$. The example ${}^5_p13^2_1$ has dihedral automorphism group ${}^513^2$ and orbit representatives

123456_{13} 123469_{26} 123569_{26} 123589_{26} 124578_{13}
$12457\,12_{13}$ $12469\,11_{26}$ $12569\,11_{13}$ $12579\,11_{13}$ $13579\,11_{13}$.

We determine the homeomorphism type of the link L of vertex 13 (and thus, by transitivity, of any other vertex) of ${}^5_p13^2_1$. First of all, we compute the homology of L, $H_*(L) = (\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z})$. The f-vector of L is $f = (\underline{12}, \underline{66}, 184, 210, 84)$ and L is invariant under the involution

$$I = (1\,\mathit{12})(2\,\mathit{11})(3\,\mathit{10})(49)(58)(67),$$

which generates the 2-element stabilizer subgroup of the vertex 13. We split the 84 facets of L into two parts

L_1				L_2			
12345	1234 12	12358	1238 11	89 10 11 12	19 10 11 12	58 10 11 12	25 10 11 12
123 11 12	12458	12478	1247 12	12 10 11 12	589 11 12	569 11 12	169 11 12
1278 11	1345 10	13467	1346 11	256 11 12	389 10 12	679 10 12	279 10 12
1347 12	134 10 11	1358 10	1367 12	169 10 12	239 10 12	358 10 12	167 10 12
136 11 12	138 10 11	1458 10	14678	127 10 12	235 10 12	3589 12	5679 12
1468 10	1469 10	1469 11	149 10 11	3579 12	3479 12	2479 12	2349 12
1678 10	178 10 11	2345 10	2349 10	3567 12	2356 12	389 10 11	349 10 11
23568	2368 11	236 11 12	2458 10	578 10 11	257 10 11	127 10 11	3589 11
2468 10	2468 11	2469 10	2469 11	3579 11	2579 11	3479 11	2479 11
2478 11	2568 10	34678	3468 11	2569 11	3578 11	5679 10	2579 10
3478 11	35678			2569 10	5678 10		

where L_1 and L_2 are 4-manifolds with boundaries ∂L_1 and ∂L_2 respectively, and

$$L = L_1 \cup L_2.$$

We have that

$$L_1^I = L_2,$$

and

$$L_1 \cap L_2 = \partial L_1 = (\partial L_1)^I = \partial L_2$$

is a 3-sphere with facets

127 11	127 12	12 11 12	167 10	167 12	169 10	169 11	16 11 12
17 10 11	19 10 11	2349	234 12	2356	235 10	236 12	239 10
247 11	247 12	249 11	256 10	269 10	269 11	26 11 12	347 11
347 12	349 10	34 10 11	3567	3578	358 10	367 12	378 11
38 10 11	49 10 11	5678	568 10	678 10	78 10 11.		

If we add to L_1 (or to L_2) the cone over the boundary ∂L_1 with respect to a new vertex, we get a combinatorial manifold with 13 vertices, which is BISTELLAR_EQUIVALENT to the unique 9-vertex triangulation $\mathbb{C}\mathbf{P}^2_9$ of the complex projective plane. The involution I maps the simplex 12 11 12 of ∂L_1 onto itself, but interchanges the vertices pairwise by the permutation $(1\,\mathit{12})(2\,\mathit{11})$. As the number of transpositions is even, I preserves the orientation of ∂L_1. Recall that the *connected sum* $M\#N$ of two (compact) d-manifolds M and N is obtained by cutting out the interior of a d-ball in each of the manifolds and then pasting together the remainders via a homeomorphism of the boundary spheres of the balls (see [73, Ch. 3] and Chapter 2). If both manifolds are oriented, one requires that the homeomorphism reverses(!) the orientation of the boundary spheres, otherwise the connected sum is denoted by $M\#-N$. In our case, the link L therefore is a 12-vertex combinatorial triangulation of the manifold $\mathbb{C}\mathbf{P}^2\#-\mathbb{C}\mathbf{P}^2$.

Next, we examine the example ${}^5_p13^1_1$. It has cyclic automorphism group ${}^5 13^1$ and orbit representatives

123456_{13}	$12346\,12_{13}$	123568_{13}	$12358\,11_{13}$	123689_{13}
$12369\,12_{13}$	$12378\,11_{13}$	$12378\,12_{13}$	$1237\,11\,12_{13}$	$12389\,12_{13}$
124579_{13}	$12459\,10_{13}$	$1246\,10\,11_{13}$	$12479\,10_{13}$.	

The vertex-link of ${}^5_p13^1_1$ is BISTELLAR_EQUIVALENT to L, and thus gives also a 12-vertex triangulation of $\mathbb{C}\mathbf{P}^2\#-\mathbb{C}\mathbf{P}^2$.

Theorem 3.4 *There are precisely two 5-dimensional combinatorial pseudomanifolds, ${}^5_p13^1_1$ and ${}^5_p13^2_1$, on 13 vertices that have a vertex-transitive automorphism group. These two 5-pseudomanifolds with homology $(\mathbb{Z}, 0, 0, \mathbb{Z}^{13}, 0, \mathbb{Z})$ are 3-neighborly with f-vector $f = (\underline{13}, \underline{78}, \underline{286}, 598, 546, 182)$, and are thus simply connected.*

The vertex-links of both examples provide 12-vertex triangulations of the manifold $\mathbb{C}\mathbf{P}^2\#-\mathbb{C}\mathbf{P}^2$. Moreover, for the example ${}^5_p13^2_1$ there is an involution I that interchanges the two components of the connected sum decomposition of the link.

Two 5-Pseudomanifolds with 14 Vertices and Vertex-Link $S^2 \times S^2$

We found two 3-neighborly simply connected 5-pseudomanifolds on 14 vertices with dihedral automorphism group ${}^{5}14^{\,3}$. Both examples have homology $(\mathbb{Z}, 0, 0, \mathbb{Z}^{14}, 0, \mathbb{Z})$ and their vertex-links are BISTELLAR_EQUIVALENT to $S^2 \times S^2$. The orbit representatives of ${}^{5}_{p}14^{\,3}_{\,1}$ are

123456_{14}	123469_{28}	$12349\,10_{14}$	123569_{28}	123589_{28}	124578_{14}
$12457\,13_{14}$	$12469\,10_{28}$	$1246\,10\,12_{28}$	$12569\,10_{14}$	$1256\,10\,11_{14}$	$1257\,10\,11_{14}$

and those of ${}^{5}_{p}14^{\,3}_{\,2}$ are

123456_{14}	$12346\,10_{28}$	$12349\,10_{14}$	123569_{28}	$12369\,10_{28}$	124578_{14}
$12457\,13_{14}$	$1246\,10\,12_{28}$	$12478\,13_{28}$	$12569\,10_{14}$	$1256\,10\,11_{14}$	$1257\,10\,11_{14}$.

Vertex-Transitive 6- and 7-Pseudomanifolds with $n \leq 15$ Vertices?

Our enumeration produced no example of a vertex-transitive combinatorial 6-manifold with $n = 13$ vertices. Moreover, with the exception of combinatorial pseudomanifolds of dimension six or seven that might exist with a maximal symmetry group of type ${}^{6}14^{\,1}$, ${}^{6}14^{\,2}$, ${}^{6}15^{\,1}$, ${}^{6}15^{\,2}$, ${}^{6}15^{\,3}$, ${}^{6}15^{\,4}$, ${}^{7}14^{\,1}$, ${}^{7}14^{\,2}$, ${}^{7}15^{\,1}$, ${}^{7}15^{\,2}$, ${}^{7}15^{\,3}$, or ${}^{7}15^{\,4}$, there are otherwise no such examples with $n \leq 15$ vertices.

An S_6-Invariant Triangulation of $\mathbb{R}\mathbf{P}^4$ on 16 Vertices

In [89], Kühnel described an infinite series of triangulated Kummer varieties with the following properties. For every $d \geq 3$, there exists a 2-neighborly combinatorial pseudomanifold homeomorphic to the Kummer variety K^d with 2^d vertices, $d! \cdot 2^{d-1}$ facets, and a vertex-transitive automorphism group of order $(d+1)! \cdot 2^d$. The link of any vertex is a combinatorial $\mathbb{R}\mathbf{P}^{d-1}$ with $2^d - 1$ vertices and $\frac{1}{2}(d+1)!$ facets.

For $d = 3$, the construction yields ${}^{3}_{p}8^{\,41}_{\,1}$ with vertex-link $\mathbb{R}\mathbf{P}^2$. For $d = 4$ we get ${}^{4}_{p}16^{\,1328}_{\,1}$ with vertex-link $\mathbb{R}\mathbf{P}^3$. In dimension 5, the link provides us with a 31-vertex triangulation of $\mathbb{R}\mathbf{P}^4$ that has f-vector $f = (\underline{31}, 270, 780, 900, 360)$. We applied BISTELLAR flips to this object, and obtained a 16-vertex triangulation of $\mathbb{R}\mathbf{P}^4$ with f-vector $f = (\underline{16}, \underline{120}, 330, 375, 150)$.

Our combinatorial $\mathbb{R}\mathbf{P}^4_{16}$ has the symmetric group S_6 as its non-transitive automorphism group, which was established as follows. There are 10 vertices with Altshuler-Steinberg determinant 32301672103936 and 6 vertices with determinant 7666785058816. Thus, the automorphism group of $\mathbb{R}\mathbf{P}^4_{16}$ is a subgroup of $S_{10} \times S_6$. If we take the span (the induced subcomplex) in $\mathbb{R}\mathbf{P}^4_{16}$ of the 6 latter vertices, we get the complete graph K_6 with symmetry group S_6. The span of the first 10 vertices is a 3-dimensional simplicial complex. By a simple computer test, we determined the permutations in S_{10} that are symmetries of this complex. It turned out that the 3-dimensional complex has as well S_6 as its vertex-transitive automorphism group. Therefore, the automorphism group of $\mathbb{R}\mathbf{P}^4_{16}$ must be a subgroup of $S_6(10) \times S_6(6)$. Another computer test showed that it is in fact S_6.

With respect to a suitable labeling of the vertices, the action of S_6 on the $10 + 6$ vertices is generated by the two permutations $(12345\mathit{10})(689)(\mathit{11}\,\mathit{12}\,\mathit{13}\,\mathit{14}\,\mathit{15}\,\mathit{16})$ and $(27)(4\mathit{10})(56)(\mathit{11}\,\mathit{12})$. The 150 facets of $\mathbb{R}\mathbf{P}^4_{16}$ split into two orbits, $1245\,11_{30}$ and $124\,11\,13_{120}$, which have 30 respectively 120 facets.

Theorem 3.5 *There exists a 2-neighborly combinatorial triangulation* $\mathbb{RP}^4_{16}$ *of the 4-dimensional real projective space* $\mathbb{RP}^4$*, with f-vector* $f = (\underline{16}, \underline{120}, 330, 375, 150)$ *and* S_6 *as (non-transitive) automorphism group.*

ARNOUX and MARIN [13] proved lower bounds for the numbers of vertices of combinatorial triangulations of $\mathbb{RP}^d$ and $\mathbb{CP}^d$. These bounds are $n \geq ((d+1)(d+2))/2$ for $\mathbb{RP}^d$ and $n \geq (d+1)^2$ for $\mathbb{CP}^d$, where in both cases equality is possible only for $d = 2$.

Corollary 3.6 *The triangulation* $\mathbb{RP}^4_{16}$ *is vertex-minimal.*

Conjecture 3.7 *The triangulation* $\mathbb{RP}^4_{16}$ *is unique with* $f = (\underline{16}, \underline{120}, 330, 375, 150)$.

REMARK: Whenever we find a highly symmetric small triangulation of a manifold by *random* BISTELLAR flips, it seems to be very likely that this triangulation is minimal and that there are no other triangulations of this manifold with the same f-vector (see also Conjecture 2.10 and the remark on page 35).

3.3 List of Transitive Permutation Groups

We display the generators of the automorphism groups of all vertex-transitive combinatorial pseudomanifolds that appear in this chapter in Table 3.6. For further transitive group actions see the previous chapter.

Action	Group	Generators
8^{20}	$[2^3]4$	(1238)(4567), (26)(37)
8^{41}	$\mathbb{Z}_2^3{:}S_4$	(123)(465), (13)(28)(46)(57), (13)(4567), (15)(26)(37)(48), (18)(23)(45)(67)
8^{43}	$PGL(2,7)$	(1234568), (132645), (16)(23)(45)(78)
9^{3}	D_9	(123456789), (18)(27)(36)(45)
9^{19}	$\mathbb{Z}_3^2{:}2D_4$	(12)(35)(67), (129)(345)(678), (147)(258)(369), (16452387)
10^{2}	D_5	(13579)(2468*10*), (14)(23)(5*10*)(69)(78)
10^{3}	D_{10}	(123456789*10*), (1*10*)(29)(38)(47)(56)
10^{4}	$\frac{1}{2}[5{:}4]2$	(1298)(3674)(5*10*), (13579)(2468*10*)
11^{1}	$\mathbb{Z}_{11}$	(123456789*10 11*)
11^{2}	D_{11}	(123456789*10 11*), (1*10*)(29)(38)(47)(56)
11^{4}	11:10	(123456789*10 11*), (12485*10*9736)
13^{1}	$\mathbb{Z}_{13}$	(123456789*10 11 12 13*)
13^{2}	D_{13}	(123456789*10 11 12 13*), (1*12*)(2*11*)(3*10*)(49)(58)(67)
13^{4}	13:4	(123456789*10 11 12 13*), (15*12*8)(2*10 11*3)(4796)
14^{2}	D_7	(13579*11 13*)(2468*10 12 14*), (1*12*)(2*11*)(3*10*)(49)(58)(67)(*13 14*)
14^{3}	D_{14}	(123456789*10 11 12 13 14*), (1*14*)(2*13*)(3*12*)(4*11*)(5*10*)(69)(78)
16^{421}	$t16n421(192)$	(1*15*3*13*8*11*2*16*4*14*7*12*)(5*10*69), (1*10*3*16*)(294*15*)(5*11*7*13*)(6*12*8*14*), (1*15*4*14*8*11*)(2*16*3*13*7*12*)(5*10*)(69)
16^{1033}	$t16n1033(576)$	(1*16 10*7)(2*15*98)(3*13 12*6)(4*14 11*5), (1*10 16*39*14*4*11 13*2*12 15*)(5687)
16^{1328}	$t16n1328(1920)$	(16*11*5*13 15 14*94*10*2*16*)(37*12*8), (1457*15*)(2*14*3*12 11*)(6*13*9*10 16*)
17^{5}	17:16	(123456789*10 11 12 13 14 15 16 17*), (24*10 11 14*6*16 12 17 15*985*13*37)
19^{4}	19:6	(123456789*10 11 12 13 14 15 16 17 18 19*), (298*19 12 13*)(3*17 15 18*46)(5*14 10 16*7*11*)
19^{6}	19:18	(123456789*10 11 12 13 14 15 16 17 18 19*), (2359*17 14*8*15 10 19 18 16 12*47*13*6*11*)
20^{87}	$t20n87(320)$	(1*16*)(2*15*)(34)(5*11*)(6*12*)(7*19*)(8*20*)(9*17*)(*10 18*), (1*12*2*11*)(3*10 13 19*)(49*14 20*)(5867)(*15 18 16 17*), (1*12*)(2*11*)(3*13*)(4*14*)(78)(*17 18*)

Table 3.6: Generators for the group actions.

Chapter 4

Centrally Symmetric Spheres and ‘Products of Spheres’ with Few Vertices

Let M be a combinatorial d-manifold with n vertices. If M is invariant under an involution I of the vertex set, which fixes no face of M, then M is called *centrally symmetric*. Every centrally symmetric combinatorial d-manifold has an even number of vertices, $n = 2m$, and, without loss of generality, we assume that the involution is presented by the permutation $I = (1\ \ m+1)(2\ \ m+2)\cdots(m\ \ 2m)$. Crosspolytopes are centrally symmetric, and a subset $F \subseteq \{1, 2, \ldots, 2m\}$ is a face of the boundary complex BdC_m^Δ of the m-dimensional crosspolytope C_m^Δ if and only if it does not contain any ‘minimal non-face’ $\{i, i+m\}$. Hence, every centrally symmetric combinatorial manifold with $2m$ vertices is a subcomplex of the boundary complex of the m-dimensional crosspolytope.

4.1 Nearly Neighborly Centrally Symmetric Spheres

A centrally symmetric d-sphere S^d is *nearly neighborly* if every subset $F \subseteq \{1, 2, \ldots, 2m\}$ of cardinality $\lfloor \frac{d+1}{2} \rfloor$, which does not contain any minimal non-face, is indeed a face of S^d, that is, if S^d has the $\big(\lfloor \frac{d+1}{2} \rfloor - 1\big)$-skeleton of the crosspolytope C_m^Δ.

Stanley (cf. [78]) proved that the number of i-faces of a centrally symmetric d-sphere with $2m$ vertices is bounded above by the number of i-faces that nearly neighborly centrally symmetric d-spheres with $2m$ vertices have, *if* such spheres exist. The resulting upper bound on the number of i-faces is $2^{i+1}\binom{m}{i+1}$ for every $i \leq \lfloor \frac{(d-1)}{2} \rfloor$ and is determined by the Dehn-Sommerville equations for larger i’s.

This statement is analogous to the *upper bound theorem* for polytopal spheres by McMullen [120] and its generalization to simplicial spheres by Stanley [152]: The cyclic polytope $C_d(n)$ attains the maximal number of i-faces that any simplicial d-sphere with n vertices can have (see [164] for an exposition).

In the centrally symmetric case, the situation is more subtle. Grünbaum [66, p. 116] showed that there is no nearly neighborly centrally symmetric 4-polytope with 12 vertices, and due to Burton [45] there are no nearly neighborly centrally symmetric d-polytopes with many vertices. This does not rule out that there exist nearly neighborly centrally symmetric polytopal spheres for which the central symmetry cannot be realized or that there are non-polytopal nearly neighborly centrally symmetric combinatorial spheres.

Theorem 4.1 (JOCKUSCH [78]) *There is an infinite family J^3_{2m}, $m \geq 4$, of nearly neighborly centrally symmetric 3-spheres with $2m$ vertices.*

JOCKUSCH constructs this infinite class by induction. He starts with the boundary complex $J^3_8 = BdC^{\Delta}_4$ of the 4-dimensional crosspolytope, which has 8 vertices. For the induction step he first finds a 3-ball B_{2m} with image B^I_{2m} under the involution I such that the intersection $B_{2m} \cap B^I_{2m}$ does not contain any facet of S^3_{2m}. Then he cuts out the interiors of B_{2m} and B^I_{2m} and sews in two new balls $(m+1) * BdB_{2m}$ and $(2m+2) * BdB^I_{2m}$ to obtain J^3_{2m+2}. The way JOCKUSCH chooses the balls B_{2m}, he ensures that J^3_{2m+2} remains centrally symmetric and nearly neighborly in every step. By suspending each J^3_{2m}, JOCKUSCH also obtains an infinite class of nearly neighborly centrally symmetric 4-spheres.

We apply JOCKUSCH's idea to construct nearly neighborly centrally symmetric d-spheres S^d_{2d+4} with $2d+4$ vertices, d odd. For this, we start with the boundary complex BdC^{Δ}_{d+1} of the $(d+1)$-dimensional crosspolytope with $2d+2$ vertices. We compose the ball B_{2d+2} (and B^I_{2d+2}) as follows. Let the d-simplex $1\cdots(d+1)$ belong to B_{2d+2} and also all d-simplices $1\cdots k^I_1\cdots k^I_j\cdots(d+1)$ with $1 \leq k_1 < \ldots < k_j \leq d+1$ and $j = 1, \ldots, \frac{d-1}{2}$, with respect to the involution $I = (1\ \ d+2)(2\ \ d+3)\cdots(d+1\ \ 2d+2)$. This collection of simplices B_{2d+2} forms a ball, and, moreover, B_{2d+2} and B^I_{2d+2} have the desired properties. (The boundary of B_{2d+2} consists of all $(d-1)$-faces $1\cdots k^I_1\cdots\widehat{s}\cdots k^I_{(d-1)/2}\cdots(d+1)$ with vertex $s \in \{1, \ldots, d+1\}$, $s \neq k_i$, deleted.)

Lemma 4.2 *There is a family of nearly neighborly centrally symmetric d-spheres S^d_{2d+4} with $2d+4$ vertices, $d \geq 3$ odd.*

Proof: Let S^d_{2d+4} be the sphere that we obtain by removing the interiors of the balls B_{2d+2} and B^I_{2d+2} from BdC^{Δ}_{d+1} and then sewing in new balls $(2d+3) * BdB_{2d+2}$ and $(2d+4) * BdB^I_{2d+2}$. Clearly, the sphere S^d_{2d+4} is centrally symmetric with respect to the involution $I' = (1\ \ d+2)(2\ \ d+3)\cdots(d+1\ \ 2d+2)(2d+3\ \ 2d+4)$. Thus, we have to show that every subset $F \subseteq \{1, 2, \ldots, 2d+4\}$ of cardinality $\frac{d+1}{2}$ with $F \cap F^I = \emptyset$ is a face of S^d_{2d+4}. By construction, every F that does not contain the vertex $2d+3$ or the vertex $2d+4$ is a face of the boundary of the ball B_{2d+2} or of the boundary of the ball B^I_{2d+2} and hence is a face of S^d_{2d+4}. If F contains either of the vertices $v = 2d+3, 2d+4$, then $F \backslash v$ is a face of BdB_{2d+2} or of BdB^I_{2d+2}, and again F belongs to S^d_{2d+4}. □

REMARK: The spheres S^d_{2d+4} are polytopal and can alternatively be characterized as the boundary complexes of the $(d+1)$-polytopes $\text{conv}\{\pm e_i, \pm\mathbb{1}\}$, which are obtained from the crosspolytopes $\text{conv}\{\pm e_i\}$, $i = 1, \ldots, d+1$, by adding two extra vertices $\pm\mathbb{1}$, with $\mathbb{1}$ the sum of the unit vectors (G.M. Ziegler, personal communication). Furthermore, the examples S^3_{10} and S^5_{14} have vertex-transitive automorphism groups $S_5 \times \mathbb{Z}_2$ and $S_7 \times \mathbb{Z}_2$ respectively. We believe that, in general, S^d_{2d+4} has transitive symmetry group $S_{d+2} \times \mathbb{Z}_2$.

According to GRÜNBAUM [66, p. 116], there is a unique centrally symmetric 4-polytope with 10 vertices. GRÜNBAUM's example coincides with our example $S^3_{2\cdot 3+4}$, but is distinct from JOCKUSCH's example J^3_{10}. These two spheres are the only nearly neighborly centrally symmetric 3-spheres with 10 vertices, which are obtainable from the boundary complex of the 4-dimensional crosspolytope by the JOCKUSCH construction of cutting out and sewing in appropriate balls, as we checked by computer.

In Chapter 2, we found, besides BdC_4^Δ and the sphere $S^3_{2\cdot 3+4}$, two vertex-transitive nearly neighborly centrally symmetric 3-spheres with 12 vertices and one with 14 vertices. Apart from one example with 12-vertices, these spheres have a transitive cyclic automorphism group. It therefore seemed promising to search for nearly neighborly centrally symmetric spheres with a vertex-transitive cyclic or dihedral group action on more vertices and in higher dimensions d. Dihedral or cyclic group actions bring along a large number of small orbits of $(d+1)$-sets, but many of these orbits can be neglected if we are interested in centrally symmetric triangulations only. We deleted all orbits containing facets F, for which $F \cap F^I \neq \emptyset$, with respect to the involution $I = (12\cdots 2m)^m = (1\ m+1)\cdots(m\ 2m)$, in the preprocessing of Step 4 of the enumeration algorithm MANIFOLD_VT.

Every nearly neighborly centrally symmetric example that we found we labeled with a unique symbol ${}^{d}_{nn}n^{di/cy}_{k}$ denoting the k-th isomorphism type of a nearly neighborly centrally symmetric d-sphere listed for the dihedral/cyclic group action on $n = 2m$ vertices. The cyclic and dihedral group actions have generators $a_{2m} = (123\dots 2m)$ and $b_{2m} = (1\ 2m)(2\ 2m-1)\dots(m\ m+1)$, with $\mathbb{Z}_{2m} = \langle a_{2m}\rangle$ and $D_{2m} = \langle a_{2m}, b_{2m}\rangle$. For fixed d and $n = 2m$, we first processed the dihedral and then the cyclic action.

Theorem 4.3 *There exist nearly neighborly centrally symmetric d-spheres, d odd, with few vertices, $n = 2m$, that have a vertex-transitive cyclic group action. If d is even, then on few vertices the only nearly neighborly centrally symmetric d-sphere with a vertex-transitive cyclic group action is the boundary complex of the crosspolytope. The numbers of these spheres are*

$d\backslash n$	6	8	10	12	14	16	18
2	1	0	0	0	0	0	0
3	–	1	1	1	1	5	?
4	–	–	1	0	0	?	?
5	–	–	–	1	2	3	?
6	–	–	–	–	1	0	?
7	–	–	–	–	–	1	12

Question 4.4 *Are there nearly neighborly centrally symmetric d-spheres with a vertex-transitive cyclic group action for all even numbers of vertices $n \geq 2d+2$, $d \geq 3$ odd?*

In Table 4.1, we list all the spheres that we found (with the exception of ten cyclic 7-spheres with 18 vertices). From this list we see that nearly neighborly centrally symmetric spheres with a dihedral action do not exist for all parameters – for example not for $d = 3$, $n = 12$. If a sphere is *nearly k-neighborly*, that is, if it has the $(k-1)$-skeleton of the corresponding cross-polytope, then we note the k-entry of its f-vector in italic.

Table 4.1: Nearly neighborly centrally symmetric spheres with vertex-transitive cyclic (or dihedral) group action.

d	n	f-vector	Type	List of orbits	Remarks
2	6	(*12*,*8*)	${}^{2}_{nn}6^{di}_{1}$	123_{6} 135_{2}	BdC_3^{Δ}, [Ch. 2, ${}^{2}6^{11}_{1}$]
3	8	(*24*,*32*,*16*)	${}^{3}_{nn}8^{di}_{1}$	1234_{8} 1247_{8}	BdC_4^{Δ}, [Ch. 2, ${}^{3}8^{44}_{1}$]
	10	(*40*,60,30)	${}^{3}_{nn}10^{di}_{1}$	1234_{10} 1245_{10} 1258_{10}	[Ch. 2, ${}^{3}10^{22}_{1}$]
	12	(*60*,96,48)	${}^{3}_{nn}12^{cy}_{1}$	1234_{12} 1246_{12} $126\,11_{12}$ $135\,10_{12}$	[Ch. 2, ${}^{3}12^{1}_{1}$]
	14	(*84*,140,70)	${}^{3}_{nn}14^{cy}_{1}$	1234_{14} 1245_{14} $125\,10_{14}$ $126\,10_{14}$ $126\,12_{14}$	[Ch. 2, ${}^{3}14^{1}_{1}$]
	16	(*112*,192,96)	${}^{3}_{nn}16^{cy}_{1}$	1234_{16} 1246_{16} 1268_{16} $128\,15_{16}$ $135\,14_{16}$ $13\,10\,13_{16}$	
			${}^{3}_{nn}16^{cy}_{2}$	1234_{16} 1248_{16} 1268_{16} $126\,15_{16}$ 1357_{16} $138\,10_{16}$	
			${}^{3}_{nn}16^{cy}_{3}$	1234_{16} 1248_{16} $128\,15_{16}$ $135\,12_{16}$ $135\,14_{16}$ $137\,14_{16}$	
			${}^{3}_{nn}16^{cy}_{4}$	1234_{16} $124\,15_{16}$ 1357_{16} $136\,10_{16}$ $137\,14_{16}$ $13\,10\,13_{16}$	
			${}^{3}_{nn}16^{cy}_{5}$	1237_{16} 1238_{16} $126\,15_{16}$ $128\,15_{16}$ 1357_{16} $13\,10\,13_{16}$	
4	10	(*40*,*80*,*80*,*32*)	${}^{4}_{nn}10^{di}_{1}$	12345_{10} 12359_{10} 12458_{10} 13579_{2}	BdC_5^{Δ}, [Ch. 2, ${}^{4}10^{39}_{1}$]
5	12	(*60*,*160*,240,192,64)	${}^{5}_{nn}12^{di}_{1}$	123456_{12} $12346\,11_{12}$ $12356\,10_{24}$ $12469\,11_{12}$ $12569\,10_{4}$	BdC_6^{Δ}, [Ch. 2, ${}^{5}12^{293}_{1}$]
	14	(*84*,*280*,490,420,140)	${}^{5}_{nn}14^{di}_{1}$	123456_{14} 123467_{28} 1234712_{14} $12367\,12_{28}$ $12457\,10_{28}$ $1247\,10\,13_{14}$ $1256\,10\,11_{14}$	[Ch. 2, ${}^{5}14^{49}_{1}$]
			${}^{5}_{nn}14^{di}_{2}$	123456_{14} $12346\,12_{28}$ 1234712_{14} $12356\,11_{28}$ $12467\,10_{28}$ $1247\,10\,13_{14}$ $1256\,10\,11_{14}$	[Ch. 2, ${}^{5}14^{7}_{1}$]
	16	(*112*,*448*,864,768,256)	${}^{5}_{nn}16^{cy}_{1}$	123456_{16} 123467_{16} 123478_{16} $12348\,13_{16}$ $1234\,13\,15_{16}$ $12378\,12_{16}$ $1238\,12\,13_{16}$ $123\,12\,13\,15_{16}$ $123\,12\,14\,15_{16}$ $12468\,11_{16}$ $1246\,11\,15_{16}$ $1248\,11\,13_{16}$ $124\,11\,13\,15_{16}$ $1267\,11\,12_{16}$ $1268\,11\,13_{16}$ $1358\,10\,14_{16}$	
			${}^{5}_{nn}16^{cy}_{2}$	123456_{16} 123467_{16} 123478_{16} $12348\,13_{16}$ $1234\,13\,15_{16}$ $12378\,12_{16}$ $1238\,12\,13_{16}$ $123\,12\,13\,15_{16}$ $123\,12\,14\,15_{16}$ $12468\,15_{16}$ $1248\,13\,15_{16}$ $1267\,11\,12_{16}$ $1268\,11\,13_{16}$ $1268\,11\,15_{16}$ $128\,11\,13\,15_{16}$ $1358\,12\,14_{16}$	
			${}^{5}_{nn}16^{cy}_{3}$	123456_{16} 123467_{16} 123478_{16} $12348\,13_{16}$ $1234\,13\,15_{16}$ $12378\,12_{16}$ $1238\,12\,15_{16}$ $1238\,13\,15_{16}$ $123\,12\,14\,15_{16}$ $12467\,13_{16}$ $1246\,11\,13_{16}$ $1246\,11\,15_{16}$ $12478\,13_{16}$ $124\,11\,13\,15_{16}$ $1267\,11\,12_{16}$ $1357\,10\,12_{16}$	

Table 4.1: Nearly neighborly centrally symmetric spheres with vertex-transitive cyclic (or dihedral) group action.

6	14	(*84*,*280*,*560*,*672*,*448*,*128*)	${}^{6}_{nn}14^{di}_{1}$	1234567$_{14}$ 123457 13$_{14}$ 123467 12$_{28}$ 123567 11$_{14}$ 12357 11 13$_{14}$ 12367 11 12$_{14}$ 12457 10 13$_{14}$ 12467 10 12$_{14}$ 13579 11 13$_{2}$	BdC^{Δ}_{7}, [Ch. 2, ${}^{6}14^{57}_{1}$]
7	16	(*112*,*448*,*1120*,*1792*,*1792*,*1024*,*256*)	${}^{7}_{nn}16^{di}_{1}$	12345678$_{16}$ 1234568 15$_{16}$ 1234578 14$_{32}$ 1234678 13$_{32}$ 123468 13 15$_{16}$ 123478 13 14$_{16}$ 123568 12 15$_{32}$ 123578 12 14$_{32}$ 123678 12 13$_{16}$ 124578 11 14$_{16}$ 12468 11 13 15$_{16}$ 12478 11 13 14$_{16}$	BdC^{Δ}_{8}
	18	(*144*,*672*,*2016*,3780,4200,2520,630)	${}^{7}_{nn}18^{cy}_{1}$ $-{}^{7}_{nn}18^{cy}_{10}$		
			${}^{7}_{nn}18^{di}_{1}$	12345678$_{18}$ 12345689$_{36}$ 1234569 16$_{18}$ 1234589 16$_{36}$ 12346789$_{18}$ 1234679 14$_{36}$ 123467 14 17$_{36}$ 123469 14 17$_{18}$ 1234789 14$_{36}$ 123478 14 15$_{36}$ 123489 14 15$_{18}$ 1235689 13$_{36}$ 123569 13 16$_{36}$ 123589 13 16$_{36}$ 123678 13 14$_{18}$ 123689 13 14$_{36}$ 124578 12 15$_{18}$ 124579 12 15$_{36}$ 124589 15 16$_{18}$ 124679 12 17$_{36}$ 12479 12 14 15$_{18}$ 12479 12 14 17$_{18}$ 12569 12 13 16$_{18}$	
			${}^{7}_{nn}18^{di}_{2}$	12345678$_{18}$ 12345689$_{36}$ 1234569 16$_{18}$ 1234589 16$_{36}$ 12346789$_{18}$ 1234679 14$_{36}$ 123467 14 17$_{36}$ 123469 14 17$_{18}$ 1234789 14$_{36}$ 123478 14 15$_{36}$ 123489 14 15$_{18}$ 1235689 16$_{36}$ 123568 13 16$_{36}$ 123678 13 14$_{18}$ 123689 13 14$_{36}$ 123689 13 16$_{36}$ 124578 12 15$_{18}$ 124589 12 15$_{36}$ 124589 15 16$_{18}$ 124679 12 17$_{36}$ 12479 12 14 15$_{18}$ 12479 12 14 17$_{18}$ 12569 12 13 16$_{18}$	

4.2 'Products of Spheres'

Besides nearly neighborly centrally symmetric spheres, sphere products form a second interesting class of centrally symmetric combinatorial manifolds.

In this section, we present a list of such manifolds with few vertices. For some new examples that we found with the homology of products of spheres, it has not yet been possible to show that they are indeed homeomorphic to sphere products, although we believe that this is the case. Examples, for which the determination of their topological type is still open, we denote as *'products of spheres'*.

For general centrally symmetric combinatorial 2- and 4-manifolds the following lower bounds on the numbers of vertices hold.

Theorem 4.5 (Kühnel [92]) *Let M be a triangulated surface with $n = 2m$ vertices, which has a fixed point free involution. Then*

$$4^2 \binom{\frac{1}{2}(m-1)}{2} \geq -3\,(\chi(M) - 2), \tag{4.1}$$

with equality if and only if M contains the 1-skeleton of the crosspolytope BdC_m^Δ.

Theorem 4.6 (Sparla [150, 4.8], [151]) *Let M be a combinatorial 4-manifold with $n = 2m$ vertices, which has a fixed point free involution. Then*

$$4^3 \binom{\frac{1}{2}(m-1)}{3} \geq 10\,(\chi(M) - 2), \tag{4.2}$$

with equality if and only if M contains the 2-skeleton of the crosspolytope BdC_m^Δ.

Moreover, Sparla conjectured a generalization of these lower bounds to $2k$-dimensional centrally symmetric combinatorial manifolds.

Conjecture 4.7 (Sparla [150, 4.11], [151]) *Let M be a combinatorial $2k$-manifold with $n = 2m$ vertices, which has a fixed point free involution. Then*

$$4^{k+1} \binom{\frac{1}{2}(m-1)}{k+1} \geq (-1)^k \binom{2k+1}{k+1} (\chi(M) - 2), \tag{4.3}$$

with equality if and only if M contains the k-skeleton of the crosspolytope BdC_m^Δ.

In particular, the latter conjecture would imply that centrally symmetric combinatorial triangulations of sphere products $S^k \times S^k$ have at least $4k+4$ vertices, and any triangulation of $S^k \times S^k$ with precisely $4k+4$ vertices must contain the k-skeleton of BdC_m^Δ. The existence of such triangulations was conjectured by Sparla [150, 4.17].

In dimension two, a centrally symmetric triangulation of the 2-torus with 8 vertices is well known (cf. [Ch. 2, ${}^2 8^{15}_{1}$]). Centrally symmetric triangulations of the product $S^2 \times S^2$ were found by Sparla [150] and by Lassmann and Sparla [105]: There exist precisely three centrally symmetric triangulations of $S^2 \times S^2$ with 12 vertices that have a vertex-transitive cyclic group action.

Complement

by **Matthias Kreck**

Proposition: *Let M be a 1-connected smooth codimension 1 submanifold of S^{d+1} and $d > 4$. If M has the homology of $S^k \times S^{d-k}$ and $1 < k \leq d/2$, then M is diffeomorphic to $S^k \times S^{d-k}$. If $d = 4$, then M is homeomorphic to $S^2 \times S^2$.*

Proof: By the generalized Jordan separation theorem S^{d+1} decomposes as $X \cup Y$ with $\partial X = \partial Y = M$. By the van Kampen theorem X and Y are 1-connected. By the Mayer-Vietoris sequence (naming the two pieces appropriately) X has the homology of S^k and Y has the homology of S^{d-k}. Since $k \leq d/2$, one can, by the Whitney embedding theorem, represent the generator of $\pi_k(X) \cong H_k(X)$ (by the Hurewicz theorem) by an embedding of S^k in the interior of X. Denote the normal bundle of S^k in S^{d+1} by E and choose a tubular neighbourhood to identify the disk bundle of E with a neighbourhood of S^k in X. The next step in the proof is to check that the complement C of the interior of the disk bundle in X is an h-cobordism between the sphere bundle and $\partial X = M$. If $d > 4$ the h-cobordism theorem [2] implies that M is diffeomorphic to the sphere bundle of E. If $d = 4$ one uses instead Freedman's topological h-cobordism theorem [1] to conclude that M is homeomorphic to the sphere bundle of E.

To check that the complement is an h-cobordism one first uses the van Kampen theorem to show that C is 1-connected. Then one uses the Mayer-Vietoris sequence to show that the inclusions from the sphere bundle of E and from ∂X to C induce an isomorphism in homology up to dimension $n/2$. Then by Leftschetz duality the inclusion induces an isomorphism in all homology groups. By the Whitehead theorem both inclusions are homotopy equivalences.

To finish the proof we note that the bundle E has to be the trivial bundle. The reason is that E is the stable normal bundle of S^k in S^{d+1} and thus is stably trivial. Since $k \leq d/2$, the dimension of the vector bundle E is larger than k. If this is the case a stably trivial bundle is actually trivial. □

Remark: Since all the tools used in the proof above are available for PL and topological manifolds (using the fundamental work of Kirby and Siebenmann) the corresponding statement holds for PL respectively topological manifolds M, where one has to replace "diffeomorphism" in the statement by PL-isomorphism respectively homeomorphism.

Corollary: *Let M be a nearly 3-neighbourly centrally symmetric combinatorial d-manifold with $n = 2d + 4$ vertices and $d > 4$. If M has the homology of $S^k \times S^{d-k}$ and $1 < k \leq d/2$, then M is PL homeomorphic to $S^k \times S^{d-k}$. If $d = 4$, then M is homeomorphic to $S^2 \times S^2$.*

References

[1] M.H. Freedman. The topology of four-dimensional manifolds. *J. Differ. Geom.* **17**, 357–453 (1982).

[2] J.W. Milnor. *Lectures on the h-cobordism theorem.* Princeton, N.J.: Princeton University Press, 1965.

Our search with the program MANIFOLD_VT for nearly neighborly centrally symmetric spheres also produced new triangulations of 'products of spheres' with $n = 2d+4$ vertices, denoted by the symbols ${}^{d}_{\times}n^{di/cy}_{k}$. In fact, we completely enumerated all such manifolds with a vertex-transitive cyclic or dihedral group action for the parameters listed in Table 4.2. For 8-manifolds with 20 vertices, an enumeration was only possible for the dihedral group action.

Theorem 4.8 *There exist centrally symmetric (combinatorial) triangulations of the 'products of spheres'*

$$\begin{array}{ccccccc} S^1\times S^1, & S^2\times S^1, & S^3\times S^1, & S^4\times S^1, & S^5\times S^1, & S^6\times S^1, & S^7\times S^1, \\ & & S^2\times S^2, & S^3\times S^2, & & \sim S^5\times S^2, & \\ & & & & S^3\times S^3, & \sim S^4\times S^3, & \sim S^5\times S^3, \\ & & & & & & \sim S^4\times S^4 \end{array}$$

with a vertex-transitive dihedral group action on $n = 2d+4$ vertices. There are no 'products of spheres' $\sim S^4\times S^2$ and $\sim S^6\times S^2$ with a vertex-transitive cyclic group action on 16 respectively 20 vertices.

Proof: The manifolds of Theorem 4.8 are listed in Table 4.2. They are combinatorial manifolds, which we checked with the program BISTELLAR. Their homology was computed with the program HOMOLOGY by Heckenbach [71]. The topological types of the products $S^{d-1}\times S^1$ were determined in [100], and Sparla [150] showed that the examples ${}^{4}_{\times}12^{cy}_{1}$, ${}^{4}_{\times}12^{cy}_{2}$, and ${}^{4}_{\times}12^{di}_{1}$ are triangulations of $S^2\times S^2$. This leaves one 5-dimensional example, ${}^{5}_{\times}14^{di}_{2}$, with homology $H_* = (\mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}, 0, \mathbb{Z})$ and two 6-dimensional manifolds, ${}^{6}_{\times}16^{cy}_{1}$ and ${}^{6}_{\times}16^{cy}_{1}$, that have homology $H_* = (\mathbb{Z}, 0, 0, \mathbb{Z}^2, 0, 0, \mathbb{Z})$.

The example ${}^{5}_{\times}14^{di}_{2}$ is simply connected, since it has the 2-skeleton of the simply connected boundary sphere BdC^{Δ}_{7} of the crosspolytope C^{Δ}_{7}. According to the classification of all simply connected 5-manifolds by Barden [16], there are precisely two simply connected 5-manifolds with the homology of $S^3\times S^2$, namely X_∞ and $M_\infty = S^3\times S^2$. The manifold X_∞ has non-vanishing second Stiefel-Whitney class, but for our example the second Stiefel-Whitney class vanishes, which is verified as follows. The centrally symmetric combinatorial manifold ${}^{5}_{\times}14^{di}_{2}$ is embedded in the 6-dimensional boundary complex of the 7-dimensional crosspolytope C^{Δ}_{7} with 14 vertices and divides BdC^{Δ}_{7} into two connected components (Alexander duality), both having ${}^{5}_{\times}14^{di}_{2}$ as their common boundary. By a theorem of Pontrjagin, the Stiefel-Whitney numbers of a d-manifold that is the boundary of a smooth compact $(d+1)$-manifold are all zero (cf. [124, 4.9]). Hence, ${}^{5}_{\times}14^{di}_{2}$ is a triangulation of $S^3\times S^2$.

The manifolds ${}^{6}_{\times}16^{cy}_{1}$ and ${}^{6}_{\times}16^{cy}_{1}$ have the same 3-skeleton as the crosspolytope C^{Δ}_{8} and thus are 2-connected. Based on earlier work by Wall [159] on the (partial) classification of differentiable 2-connected 6-manifolds, Žubr [165] classified all simply connected 6-dimensional topological manifolds. In particular, he showed that any 2-connected topological 6-manifold is completely determined by its Euler characteristic. (We are grateful to M. Kreck for pointing this out to us.) Therefore, the two combinatorial 6-manifolds ${}^{6}_{\times}16^{cy}_{1}$ and ${}^{6}_{\times}16^{cy}_{1}$ both triangulate $S^3\times S^3$. □

Conjecture 4.9 *There is a centrally symmetric combinatorial triangulation of every product of spheres $S^{\lceil \frac{d}{2}\rceil}\times S^{\lfloor \frac{d}{2}\rfloor}$ with a vertex-transitive dihedral group action on $n = 2d+4$ vertices.*

We applied BISTELLAR flips to the 5-dimensional combinatorial manifold ${}^{5}_{\times}14^{di}_{2}$ and obtained a 12-vertex triangulation of $S^3 \times S^2$ with $f = (\underline{12}, \underline{66}, \underline{220}, 390, 336, 112)$ and facets ($a = 10$, $b = 11$, $c = 12$):

12346*a*	12346*b*	123478	12347*b*	12348*a*	12357*b*	12357*c*	12359*b*
12359*c*	1236*ab*	12378*c*	1238*ac*	1239*ab*	1239*ac*	124678	12467*b*
124689	12469*a*	12489*a*	1257*bc*	1259*bc*	12678*c*	1267*bc*	12689*c*
1269*ab*	1269*bc*	1289*ac*	134678	13467*b*	13468*a*	13579*b*	13579*c*
13678*c*	1367*bc*	1368*ac*	136*abc*	1379*ab*	1379*ac*	137*abc*	145689
14568*a*	14569*a*	14589*c*	1458*ac*	1459*ac*	1489*ac*	15689*b*	1568*ab*
1569*ab*	1579*ab*	1579*ac*	157*abc*	1589*bc*	158*abc*	1689*bc*	168*abc*
23456*a*	23456*c*	23458*a*	23458*b*	2345*bc*	2346*bc*	23478*b*	235678
23567*c*	2356*a*0	23578*b*	2359*bc*	23678*c*	2368*ac*	236*abc*	239*abc*
24567*a*	24567*c*	24578*a*	24578*b*	2457*bc*	246789	24679*a*	2467*bc*
24789*a*	25678*a*	26789*a*	2689*ac*	269*abc*	345679	34567*c*	345689
34568*a*	34579*c*	34589*b*	3459*bc*	346789	3467*bc*	34789*b*	3479*bc*
356789	35789*b*	379*abc*	45679*a*	4578*ab*	4579*ac*	457*abc*	4589*bc*
458*abc*	4789*ab*	479*abc*	489*abc*	56789*a*	5689*ab*	5789*ab*	689*abc*.

This triangulation is 3-neighborly and thus provides us with another proof that the manifold ${}^{5}_{\times}14^{di}_{2}$ is simply connected.

Corollary 4.10 *The triangulation of $S^3 \times S^2$ with 12 vertices has the minimal number of vertices that a non-spherical combinatorial 5-manifold can have by the* BREHM-KÜHNEL *bound of Theorem 2.1. In particular, the* BREHM-KÜHNEL *lower bound is sharp in dimension 5.*

Besides this minimal example, we found a second triangulation of $S^3 \times S^2$ with the same f-vector $f = (\underline{12}, \underline{66}, \underline{220}, 390, 336, 112)$ by applying BISTELLAR flips to a product triangulation of $S^3 \times S^2$ with $5 \cdot 4 = 20$ vertices. Both triangulations are combinatorially distinct; the above triangulation has Altshuler-Steinberg determinant $det = 4471184572226676864$ and the one obtained from the product triangulation has determinant $det = 4508595451809050112$. For a list of facets of the second example see [102]. Both triangulations have no symmetries, since the Altshuler-Steinberg determinants of their 12 vertex links are pairwise distinct.

In Chapter 2, we found two combinatorial 15-vertex 6-manifolds, ${}^{6}15^{7}_{1}$ and ${}^{6}15^{7}_{2}$, with the homology of $S^3 \times S^3$, which are invariant under a vertex-transitive group action of $D_5 \times S_3$. An application of BISTELLAR flips to these manifolds yielded triangulations with f-vector $f = (\underline{13}, \underline{78}, \underline{286}, \underline{715}, 1014, 728, 208)$ and Altshuler-Steinberg determinant $det = 7457141548234446198538 24$ in both cases. The two achieved triangulations with 13 vertices are combinatorially non-equivalent and have no symmetries, since the Altshuler-Steinberg determinants of their vertex links are pairwise distinct. According to the classification of 2-connected topological 6-manifolds by ŽUBR [165] (see the proof of Theorem 4.8), the two examples are triangulations of $S^3 \times S^3$.

Theorem 4.11 *There are (at least) two combinatorially distinct 13-vertex triangulations of $S^3 \times S^3$. These combinatorial triangulations are vertex-minimal by the* BREHM-KÜHNEL *bound of Theorem 2.1.*

Corollary 4.12 *The* BREHM-KÜHNEL *lower bound is sharp in dimension 6.*

The facets of the 13-vertex triangulation of $S^3 \times S^3$ that we found for ${}^{6}15^{7}_{1}$ are given in [102]. Here, we list the facets of the 13-vertex triangulation corresponding to ${}^{6}15^{7}_{2}$:

123456c	123456d	12345ab	12345ac	12345bd	12346cd	123479a	123479d
12347ad	12348ab	12348ad	12348bd	12349ac	12349cd	12356bc	12356bd
1235abc	1236bcd	12379ad	12389ab	12389ad	12389bd	1239abc	1239bcd
1245678	124567b	1245689	124569a	12456ac	12456bd	1245789	124579a
12457ab	124678b	124689b	12469ac	12469bd	12469cd	124789d	12478ab
12478ad	12489bd	125678c	12567bc	125689a	12568ac	125789a	12578ac
1257abc	12678ab	12678ac	1267abc	12689ab	1269abc	1269bcd	12789ad
1345679	134567d	134569a	13456ac	134579a	13457ad	1345abd	134679c
13467cd	13469ac	13479cd	1348abd	1356789	135678c	13567bc	13567bd
135689a	13568ac	135789c	13579ad	13579bc	13579bd	13589ad	13589bc
13589bd	1358abc	1358abd	136789c	1367bcd	13689ac	1379bcd	1389abc
1456789	14567bd	1457abd	146789c	14678bd	14678cd	14689bc	1468bcd
1469bcd	14789cd	1478abd	1489bcd	15789ad	15789cd	1578acd	1579bcd
157abcd	1589bcd	158abcd	1678abd	1678acd	167abcd	1689abc	168abcd
23456cd	23458bc	23458bd	23458cd	2345abc	23478ab	23478ad	23478bc
23478cd	23479ac	23479cd	2347abc	235678b	235678c	23567bc	235689b
235689d	23568cd	23569bd	23578bc	23589bd	23678ab	23678ad	23678cd
23679ab	23679ad	23679bd	2367bcd	23689ab	23689ad	2379abc	2379bcd
245678b	245689b	24569ac	24569bd	24569cd	245789c	24578bc	24579ac
2457abc	24589bd	24589cd	24789cd	25689ad	2568acd	2569acd	25789ad
25789cd	2578acd	2579acd	2678acd	2679abd	267abcd	269abcd	279abcd
345679a	34567ad	3456acd	3458abc	3458abd	3458acd	346789b	346789c
34678ab	34678ad	34678cd	34679ab	34689ab	34689ac	3468acd	34789bc
3479abc	3489abc	356789b	35679ad	35679bd	35689ad	3568acd	35789bc
456789b	45679ab	4567abd	4569abd	4569acd	45789bc	4579abc	4589bcd
458abcd	459abcd	4678abd	4689abc	468abcd	469abcd	5679abd	579abcd.

Remark: Both 13-vertex triangulations of $S^3 \times S^3$ are 4-neighborly and thus are tight, since equality holds in Proposition 4.6 of [91]; see also [102].

We also applied BISTELLAR flips to the product triangulation of $S^2 \times S^2$ on $4 \cdot 4 = 16$ vertices and obtained a triangulation of $S^2 \times S^2$ with $f = (\underline{11}, \underline{55}, 150, 170, 68)$ and facets:

12346	12347	12369	12379	12458	12459	12468	12479
12568	12569	13467	13567	13569	1357a	1359b	135ab
1379a	139ab	1458a	1459b	145ab	1467b	1468a	146ab
1479b	15678	1578a	1678b	168ab	178ab	179ab	23468
23478	2357a	2357b	235ab	2368a	2369a	2378b	2379a
238ab	24589	24789	2568b	2569a	256ab	25789	2578b
2579a	268ab	3467b	3468a	3469a	3469b	3478b	3489a
3489b	3567b	3569b	389ab	4569a	4569b	456ab	4589a
4789b	5678b	5789a	789ab.				

As mentioned in Chapter 2 on page 38, a triangulation of $S^2 \times S^2$ with 10 vertices does not exist. Hence, we have

Theorem 4.13 *There is a vertex-minimal triangulation of $S^2 \times S^2$ with 11 vertices.*

Remark: We as well applied BISTELLAR flips to the Sparla-Lassmann triangulations of $S^2 \times S^2$ with 12 vertices and obtained further minimal triangulations with 11 vertices, which are combinatorially distinct from the above example. All the examples that we found with 11 vertices are not symmetric.

While searching for nearly neighborly centrally symmetric 3-spheres that have a vertex-transitive dihedral or cyclic group action, we found various centrally symmetric non-spherical 3-manifolds with $n = 12, 14, 16$ vertices. Among them are many examples on $n = 12, 14, 16$ vertices that have the homology of $S^2 \times S^1$ or the homology of the 3-dimensional Klein bottle.

On 16 vertices, there is a centrally symmetric triangulation of the 3-dimensional torus with $f = (\mathit{16}, \mathit{112}, 192, 96)$. It has dihedral action and orbits

$$1237_{\,32}\;\; 1267_{\,16}\;\; 137\,10_{\,32}\;\; 148\,11_{\,16}.$$

Also, there are four centrally symmetric triangulations of the Lens space $L(3,1)$ with cyclic group action on 16 vertices.

Finally, there are two centrally symmetric triangulations of non-orientable 3-manifolds with homology $H_* = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$ and $f = (\mathit{16}, \mathit{112}, 192, 96)$, which have a vertex-transitive cyclic group action. The corresponding orbit representatives are

$$1236_{\,16}\;\; 1237_{\,16}\;\; 1257_{\,16}\;\; 136\,10_{\,16}\;\; 137\,14_{\,16}\;\; 13\,10\,13_{\,16}$$

for the first example and

$$1236_{\,16}\;\; 1237_{\,16}\;\; 1257_{\,16}\;\; 136\,13_{\,16}\;\; 137\,10_{\,16}\;\; 13\,10\,14_{\,16}$$

for the second example. We tried to find triangulations with fewer vertices for these two 3-manifolds by using BISTELLAR flips, but did not achieve any improvement. It therefore seems likely that the two examples are not homeomorphic to the connected sum $(S^2 \underline{\times} S^1)\#(S^2 \underline{\times} S^1)$, which can be triangulated with 12 vertices (cf. p. 38).

Table 4.2: Centrally symmetric 'products of spheres' with $n = 2d+4$ vertices and cyclic group action.

d	n	Top. type	f-vector	Comb. type	List of orbits	Remarks
2	8	$S^1 \times S^1$	(*24*,16)	${}^{2}_{\times}8^{di}_{1}$	123_{8} 136_{8}	[100, $M_1^2(8)$], [Ch. 2, ${}^{2}8^{15}_{1}$]
3	10	$S^2 \times S^1$	(*40*,60,30)	${}^{3}_{\times}10^{di}_{1}$	1235_{20} 1245_{10}	[158], [100, $M_2^3(10)$], [Ch. 2, ${}^{3}10^{3}_{2}$]
4	12	$S^3 \times S^1$	(*60*,120,120,48)	${}^{4}_{\times}12^{di}_{2}$	12346_{24} 12356_{24}	[100, $M_3^4(12)$], [Ch. 2, ${}^{4}12^{12}_{1}$]
		$S^2 \times S^2$	(*60*,*160*,180,72)	${}^{4}_{\times}12^{cy}_{1}$	12345_{12} 12356_{12} $1236\,11_{12}$ 12569_{12} $1269\,11_{12}$ $1358\,10_{12}$	[150, M_1], [Ch. 2, ${}^{4}12^{11}_{1}$]
				${}^{4}_{\times}12^{cy}_{2}$	12345_{12} 12356_{12} $1236\,11_{12}$ $1256\,10_{12}$ $1269\,11_{12}$ $1358\,10_{12}$	[150, $M = M_2$], [151], [Ch. 2, ${}^{4}12^{124}_{1}$]
				${}^{4}_{\times}12^{di}_{1}$	12345_{12} $1235\,10_{24}$ $1236\,10_{12}$ 12459_{12} $1358\,10_{12}$	[150, M_3], [Ch. 2, ${}^{4}12^{28}_{1}$]
5	14	$S^4 \times S^1$	(*84*,210,280,210,70)	${}^{5}_{\times}14^{di}_{1}$	123457_{28} 123467_{28} 123567_{14}	[100, $M_4^5(14)$] [Ch. 2, ${}^{5}14^{3}_{8}$]
		$S^3 \times S^2$	(*84*,*280*,490,420,140)	${}^{5}_{\times}14^{di}_{2}$	123467_{28} $12346\,12_{28}$ 123567_{14} $12357\,11_{28}$ $12457\,13_{14}$ $1246\,10\,12_{28}$	[Ch. 2, ${}^{5}14^{3}_{9}$]
6	16	$S^5 \times S^1$	(*112*,336,560,560,336,96)	${}^{6}_{\times}16^{di}_{2}$	1234568_{32} 1234578_{32} 1234678_{32}	[100, $M_5^6(16)$]
		$S^3 \times S^3$	(*112*,*448*,*1120*,1568,1120,320)	${}^{6}_{\times}16^{cy}_{1}$	1234567_{16} 1234578_{16} $123458\,15_{16}$ $123478\,13_{16}$ $12347\,13\,14_{16}$ $12348\,13\,15_{16}$ $1234\,13\,14\,15_{16}$ $123568\,15_{16}$ $123678\,12_{16}$ $12368\,12\,13_{16}$ $12368\,13\,15_{16}$ $12378\,12\,13_{16}$ $124578\,11_{16}$ $12457\,11\,14_{16}$ $12458\,11\,14_{16}$ $12478\,11\,13_{16}$ $1247\,11\,13\,14_{16}$ $1248\,11\,13\,15_{16}$ $1268\,11\,13\,15_{16}$ $1357\,10\,12\,14_{16}$	
				${}^{6}_{\times}16^{di}_{1}$	1234567_{16} 1234578_{32} $123458\,14_{16}$ $123478\,13_{32}$ $123567\,12_{16}$ $12356\,12\,15_{32}$ $123578\,12_{32}$ $12358\,12\,15_{16}$ $12378\,12\,13_{16}$ $12458\,11\,14_{16}$ $12467\,11\,13_{16}$ $12468\,11\,13_{32}$ $1247\,11\,13\,14_{32}$ $1357\,10\,12\,14_{16}$	

Table 4.2: Centrally symmetric 'products of spheres' with $n=2d+4$ vertices and cyclic group action.

7	18	$S^6\times S^1$	(*144*,504,1008,1260,1008,504,126)	$^{7}_{\times}18^{di}_{1}$	12345679$_{36}$ 12345689$_{36}$ 12345789$_{36}$ 12346789$_{18}$	[100, $M^7_6(18)$]
		$\sim S^5\times S^2$	(*144*,*672*,1764,2772,2688,1512,378)	$^{7}_{\times}18^{di}_{2}$	12345689$_{36}$ 1234568 16$_{36}$ 12345789$_{36}$ 1234579 15$_{36}$ 12346789$_{18}$ 1234679 17$_{36}$ 123468 14 16$_{36}$ 1235679 17$_{18}$ 123579 13 15$_{36}$ 123579 13 17$_{36}$ 124579 15 17$_{18}$ 12468 12 14 16$_{36}$	
		$\sim S^4\times S^3$	(*144*,*672*,*2016*,3780,4200,2520,630)	$^{7}_{\times}18^{di}_{3}$	1234579 15$_{36}$ 1234579 17$_{36}$ 1234679 14$_{36}$ 1234679 17$_{36}$ 123467 14 17$_{36}$ 1234689 14$_{36}$ 1234689 16$_{36}$ 123479 14 15$_{36}$ 1235679 13$_{36}$ 1235679 17$_{18}$ 1235689 13$_{36}$ 1235689 16$_{36}$ 123569 16 17$_{36}$ 1235789 13$_{36}$ 123589 15 16$_{36}$ 123679 13 14$_{36}$ 123689 13 14$_{36}$ 123789 13 15$_{18}$ 124589 15 16$_{18}$	
8	20	$S^7\times S^1$	(*180*,720,1680,2520,2520,1680,720,160)	$^{8}_{\times}20^{di}_{2}$	12345678 10$_{40}$ 12345679 10$_{40}$ 12345689 10$_{40}$ 12345789 10$_{40}$	[100, $M^8_7(20)$]
		$\sim S^5\times S^3$	(*180*,*960*,*3360*,7560,10920,9840,5040,1120)	$^{8}_{\times}20^{di}_{3}$	12345689 10$_{40}$ 12345689 17$_{40}$ 1234568 10 19$_{40}$ 1234569 10 18$_{40}$ 12345789 10$_{40}$ 1234578 10 16$_{40}$ 1234579 10 18$_{40}$ 123457 10 16 18$_{40}$ 1234678 10 19$_{40}$ 1234679 10 15$_{40}$ 123467 10 15 19$_{40}$ 123469 10 15 18$_{40}$ 123469 15 17 18$_{40}$ 123479 10 15 18$_{40}$ 1235679 10 18$_{40}$ 1235689 14 17$_{40}$ 123568 14 17 19$_{40}$ 123569 14 17 18$_{40}$ 123578 10 16 19$_{40}$ 123679 10 15 18$_{40}$ 124579 10 16 18$_{40}$ 124579 13 16 18$_{40}$ 12458 10 13 16 17$_{40}$ 124679 13 15 18$_{40}$ 12469 10 13 15 18$_{40}$ 12478 10 13 15 16$_{40}$ 12478 10 13 15 19$_{40}$ 12478 10 15 16 19$_{40}$	
		$\sim S^4\times S^4$	(*180*,*960*,*3360*,*8064*,12600,12000,6300,1400)	$^{8}_{\times}20^{di}_{1}$	123456789$_{20}$ 12345679 10$_{40}$ 1234567 10 18$_{20}$ 1234569 10 17$_{40}$ 12345789 16$_{40}$ 1234578 16 19$_{40}$ 1234579 10 16$_{40}$ 123457 10 16 19$_{20}$ 123459 10 16 17$_{20}$ 12346789 15$_{20}$ 1234679 10 18$_{40}$ 1234679 15 18$_{40}$ 123469 10 17 18$_{40}$ 1234789 15 16$_{40}$ 123479 10 16 18$_{40}$ 1235679 10 14$_{40}$ 123567 10 14 18$_{20}$ 1235689 10 14$_{40}$ 123568 10 14 17$_{40}$ 12356 10 14 17 19$_{40}$ 12356 10 14 18 19$_{20}$ 1235789 14 16$_{20}$ 123578 14 16 19$_{40}$ 123579 10 14 16$_{40}$ 12357 10 14 16 19$_{40}$ 123589 10 14 16$_{40}$ 123589 10 16 17$_{40}$ 12358 10 14 16 19$_{20}$ 123679 10 14 18$_{40}$ 123679 14 15 18$_{40}$ 12368 10 14 15 17$_{40}$ 12369 10 14 15 18$_{20}$, 123789 14 15 16$_{20}$ 124579 10 13 16$_{40}$ 12457 10 13 16 19$_{20}$ 12459 10 13 16 17$_{20}$ 12467 10 13 15 18$_{20}$ 124689 13 15 17$_{20}$ 12468 10 13 15 17$_{40}$ 12469 10 15 17 18$_{40}$ 12469 13 15 17 18$_{40}$ 12479 10 13 15 18$_{40}$ 13579 12 14 16 18$_{20}$	
		....	....	$^{8}_{\times}20^{cy}_{\dots}$	....	

Chapter 5

Neighborly and Regular Maps with Few Vertices

We listed all triangulated surfaces that have a vertex-transitive automorphism group on $n \leq 15$ vertices in Chapter 2. For the particular classes of vertex-transitive neighborly and regular simplicial maps we now extend the enumeration to $n \leq 22$ vertices.

5.1 Neighborly Symmetric Maps

Neighborly maps have attracted attention in various ways. By Theorem 2.7, they provide examples of minimal triangulations of surfaces. At the same time, every neighborly map M with n vertices is an example of a minimal graph embedding of the complete graph $K_n \to M$.

Theorem 5.1 (RINGEL and YOUNGS [135]) *For every abstract surface M, different from the 2-sphere and with the exception of the Klein bottle, there exists an embedding $K_n \to M$ if and only if $n \leq \frac{1}{2}(7+\sqrt{49-24\chi(M)})$. If equality holds, then the embedding of the complete graph induces a triangulation of M. For the Klein bottle, there exist embeddings for $n \leq 6$.*

The existence of such embeddings establishes the *Map Color Theorem*, first formulated by HEAWOOD [70], i.e., it gives the answer to the question for the number of colors $\gamma(M)$, the *chromatic number* of M, that is needed to color any map on a (non-spherical) surface M. HEAWOOD proved that $\gamma(M) \leq \lfloor\frac{7+\sqrt{1+48g_+}}{2}\rfloor$ for all orientable surfaces with genus $g_+ \geq 1$. (In the case of orientable surfaces M, the genus g_+ is related to the Euler characteristic $\chi(M)$ by $g_+ = (2-\chi(M))/2$, for non-orientable surfaces we have $g_- = (2-\chi(M))$.) Theorem 5.1 implies equality $\gamma(M) = \lfloor\frac{1}{2}(7+\sqrt{49-24\chi(M)})\rfloor$ for all (non-spherical) orientable and non-orientable abstract surfaces, with the exception of the Klein bottle, where $\gamma(M) = 6$. For the 2-sphere, $\gamma(M) = 4$ is the essence of the famous *Four Color Theorem* of APPEL and HAKEN [12]; see also [138].

Another interest in neighborly maps stems from the realizability problem posed by GRÜNBAUM [66, Ch. 13.2]: *Can every triangulated orientable 2-manifold be embedded geometrically in $\mathbb{R}^3$, i.e., with flat triangles and without self intersections?*

By STEINITZ' theorem (cf. [164]), every combinatorial 2-sphere is realizable as the boundary complex of a convex 3-dimensional polytope. For the 2-torus with genus $g_+ = 1$

the realizability problem is open for general triangulations. A first geometric realization of the minimal 7-vertex triangulation of the 2-torus was found by Császár [55] (see Figure 5.1). It was also possible to construct geometric realizations of triangulated orientable 2-manifolds with genera $g_+ = 2, 3, 4$ that have the minimal number of vertices $n = 10, 10, 11$, respectively (cf. [25], [26], [37], [39]; see [27] for a complete list of polyhedral realizations of the 7-vertex torus).

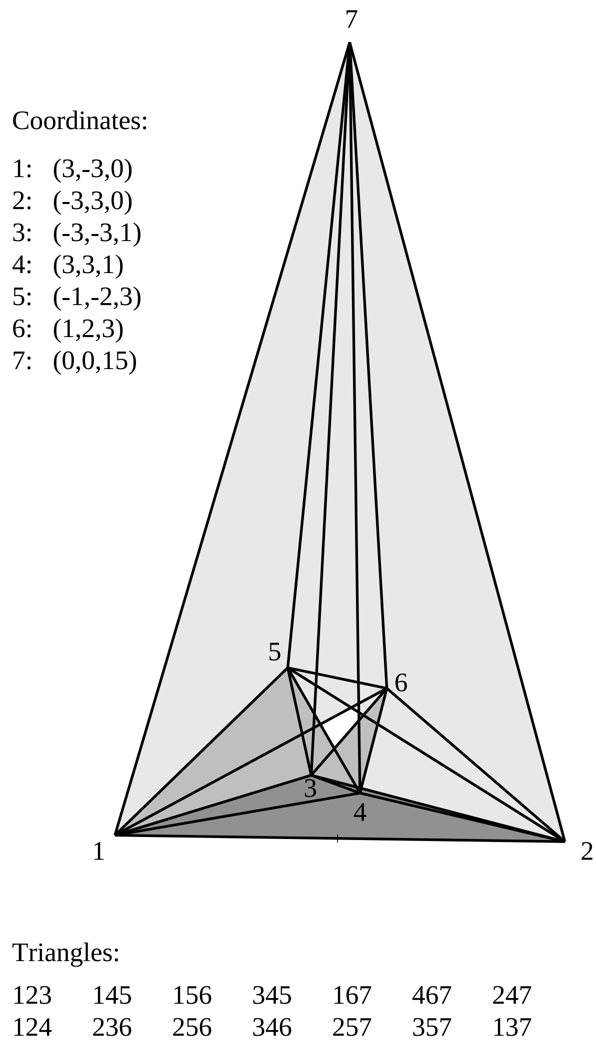


Figure 5.1: Császár's torus.

Neighborly maps of higher genus were considered as candidates for counter-examples to the Grünbaum realization problem for a while (cf. [31, p. 137]). Neighborly orientable maps have genus $g_+ = (n-3)(n-4)/12$ and therefore $n \equiv 0, 3, 4, 7 \bmod 12$ vertices, with $g_+ = 6$ and $n = 12$ as the first interesting case beyond the tetrahedron and the 7-vertex torus. Most recently, Bokowski and Guedes de Oliveira [29] succeeded to show that there indeed is a non-realizable neighborly triangulation of a map of genus 6 with 12 vertices, the map No. 54 in Altshuler's list [6].

We used the program MANIFOLD_VT to enumerate all neighborly maps with $n \leq 22$ vertices that have a vertex-transitive automorphism group. In Step 4 of the program we took only those sum vectors as result vectors that have entries 2 only. (If there were an entry 0, then the resulting map would not be neighborly.) This restriction made it possible to extend the enumeration beyond $n = 15$ vertices.

By Theorem 5.1, $n = \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})$ holds for every neighborly map, which is equivalent to $\chi(M) = n(7-n)/6$. Table 5.1 lists the numbers of vertex-transitive neighborly maps, orientable and non-orientable, which we found with $n \leq 22$ vertices. The maps themselves are listed in Table 5.2 for $n \leq 15$ and for $16 \leq n \leq 22$ if they are orientable.

n	$\chi(M)$	g_+	#	g_-	#
4	2	0	1	0	0
6	1	–	–	1	1
7	0	1	1	2	0
9	−3	–	–	5	1
10	−5	–	–	7	1
12	−10	6	1	12	0
13	−13	–	–	15	2
15	−20	11	0	22	0
16	−24	13	46	26	36
18	−33	–	–	35	0
19	−38	20	32	40	46
21	−49	–	–	51	913
22	−55	–	–	57	0

Table 5.1: Numbers of orientable and non-orientable neighborly maps with vertex-transitive action.

Every vertex-transitive neighborly map that we found is labeled with a unique symbol ${}^{2}_{nb}n^{i}_{k}$, which denotes the k-th isomorphism type of a neighborly map with n vertices listed for the i-th transitive permutation group n^i of degree n.

Heffter [72] and Ringel [134] showed that, apart from the tetrahedron, neighborly orientable maps with a transitive cyclic group action of order n exist for $n \equiv 7 \bmod 12$ only (cf. [93, p. 296] and [135, Thm. 2.8]). For the construction of triangulations with respect to other groups see [135].

Table 5.2: Neighborly 2-manifolds with vertex-transitive automorphism group.

n	Or.	Gen. g	Automorphism group	Type	List of orbits	Remarks
4	+	0	S_4	${}^{2}_{nb}4^{5}_{1}$	123_{4}	tetrahedron
6	–	1	A_5	${}^{2}_{nb}6^{12}_{1}$	123_{10}	$\mathbb{R}\mathbf{P}^{2}_{6}$
7	+	1	$7:6$	${}^{2}_{nb}7^{4}_{1}$	124_{14}	Möbius' torus, [55], [125]
9	–	5	$S_3 \times \mathbb{Z}_3$	${}^{2}_{nb}9^{4}_{1}$	124_{18} 138_{6}	[8, $N(9,2)$], [53], [Ch. 2, ${}^{2}9^{4}_{1}$]
10	–	7	A_5	${}^{2}_{nb}10^{7}_{1}$	123_{30}	[8, $N(10,13)$], [34, 380, (i)], [53], [Ch. 2, ${}^{2}10^{7}_{1}$]
12	+	6	A_4	${}^{2}_{nb}12^{4}_{1}$	124_{12} 127_{12} 138_{4} 159_{4} $15\,11_{12}$	[6, N^{12}_{58}], [Ch. 2, ${}^{2}12^{4}_{5}$]

Table 5.2: Neighborly 2-manifolds with vertex-transitive automorphism group.

13	−	15	$\mathbb{Z}_{13}$	${}^{2}_{nb}13^{1}_{1}$	123_{13} 137_{13} 148_{13} 149_{13}	[Ch. 2, ${}^{2}13^{1}_{1}$]
			$13\!:\!3$	${}^{2}_{nb}13^{3}_{1}$	123_{39} 138_{13}	[Ch. 2, ${}^{2}13^{3}_{1}$]
16	+	13	$\mathbb{Z}_4\times\mathbb{Z}_4$	${}^{2}_{nb}16^{4}_{1}$	123_{16} 137_{16} 149_{16} 159_{16} 167_{16}	
			$\mathbb{Z}_2\times\mathbb{Z}_8$	${}^{2}_{nb}16^{5}_{1}$	123_{16} 137_{16} 149_{16} $15\,10_{16}$ $16\,11_{16}$	
				${}^{2}_{nb}16^{5}_{2}$	123_{16} 137_{16} 149_{16} $15\,14_{16}$ $16\,11_{16}$	
				${}^{2}_{nb}16^{5}_{3}$	123_{16} 137_{16} $14\,11_{16}$ $15\,10_{16}$ $16\,11_{16}$	
				${}^{2}_{nb}16^{5}_{4}$	123_{16} 137_{16} $14\,11_{16}$ $15\,14_{16}$ $16\,11_{16}$	
				${}^{2}_{nb}16^{5}_{5}$	123_{16} 138_{16} 149_{16} $15\,10_{16}$ $15\,12_{16}$	
				${}^{2}_{nb}16^{5}_{6}$	123_{16} 138_{16} 149_{16} $15\,12_{16}$ $15\,14_{16}$	
				${}^{2}_{nb}16^{5}_{7}$	123_{16} 138_{16} $14\,11_{16}$ $15\,10_{16}$ $15\,12_{16}$	
				${}^{2}_{nb}16^{5}_{8}$	123_{16} 138_{16} $14\,11_{16}$ $15\,12_{16}$ $15\,14_{16}$	
				${}^{2}_{nb}16^{5}_{9}$	123_{16} 139_{16} $14\,13_{16}$ $15\,10_{16}$ $16\,11_{16}$	
				${}^{2}_{nb}16^{5}_{10}$	123_{16} 139_{16} $14\,13_{16}$ $15\,14_{16}$ $16\,11_{16}$	
				${}^{2}_{nb}16^{5}_{11}$	123_{16} 139_{16} $14\,14_{16}$ $15\,10_{16}$ $15\,11_{16}$	
				${}^{2}_{nb}16^{5}_{12}$	123_{16} 139_{16} $14\,14_{16}$ $15\,11_{16}$ $15\,14_{16}$	
				${}^{2}_{nb}16^{5}_{13}$	123_{16} $13\,12_{16}$ $14\,13_{16}$ $15\,10_{16}$ $16\,11_{16}$	
				${}^{2}_{nb}16^{5}_{14}$	123_{16} $13\,12_{16}$ $14\,13_{16}$ $15\,14_{16}$ $16\,11_{16}$	[135, p. 90]
				${}^{2}_{nb}16^{5}_{15}$	123_{16} $13\,12_{16}$ $14\,14_{16}$ $15\,10_{16}$ $15\,11_{16}$	
				${}^{2}_{nb}16^{5}_{16}$	123_{16} $13\,12_{16}$ $14\,14_{16}$ $15\,11_{16}$ $15\,14_{16}$	
			$t16n6(16)$	${}^{2}_{nb}16^{6}_{1}$	123_{16} 136_{16} $14\,10_{16}$ 159_{16} $15\,12_{16}$	
				${}^{2}_{nb}16^{6}_{2}$	123_{16} 136_{16} $14\,10_{16}$ $15\,12_{16}$ $15\,13_{16}$	
				${}^{2}_{nb}16^{6}_{3}$	123_{16} 136_{16} $14\,11_{16}$ 159_{16} $15\,12_{16}$	
				${}^{2}_{nb}16^{6}_{4}$	123_{16} 136_{16} $14\,11_{16}$ $15\,12_{16}$ $15\,13_{16}$	
				${}^{2}_{nb}16^{6}_{5}$	123_{16} 137_{16} $14\,10_{16}$ 159_{16} $15\,12_{16}$	
				${}^{2}_{nb}16^{6}_{6}$	123_{16} 137_{16} $14\,10_{16}$ $15\,12_{16}$ $15\,13_{16}$	
				${}^{2}_{nb}16^{6}_{7}$	123_{16} 137_{16} $14\,11_{16}$ 159_{16} $15\,12_{16}$	
				${}^{2}_{nb}16^{6}_{8}$	123_{16} 137_{16} $14\,11_{16}$ $15\,12_{16}$ $15\,13_{16}$	
				${}^{2}_{nb}16^{6}_{9}$	123_{16} 138_{16} 146_{16} $14\,10_{16}$ 159_{16}	
				${}^{2}_{nb}16^{6}_{10}$	123_{16} 138_{16} 146_{16} $14\,10_{16}$ $15\,13_{16}$	
				${}^{2}_{nb}16^{6}_{11}$	123_{16} 138_{16} 146_{16} $14\,11_{16}$ 159_{16}	
				${}^{2}_{nb}16^{6}_{12}$	123_{16} 138_{16} 146_{16} $14\,11_{16}$ $15\,13_{16}$	
				${}^{2}_{nb}16^{6}_{13}$	123_{16} $13\,10_{16}$ 146_{16} $14\,16_{16}$ 159_{16}	
				${}^{2}_{nb}16^{6}_{14}$	123_{16} $13\,10_{16}$ 146_{16} $14\,16_{16}$ $15\,13_{16}$	
				${}^{2}_{nb}16^{6}_{15}$	123_{16} $13\,12_{16}$ 146_{16} $14\,16_{16}$ 159_{16}	
				${}^{2}_{nb}16^{6}_{16}$	123_{16} $13\,12_{16}$ 146_{16} $14\,16_{16}$ $15\,13_{16}$	
			$t16n8(16)$	${}^{2}_{nb}16^{8}_{1}$	123_{16} 137_{16} 149_{16} 157_{16} 169_{16}	
				${}^{2}_{nb}16^{8}_{2}$	123_{16} 137_{16} 149_{16} 157_{16} $16\,10_{16}$	
				${}^{2}_{nb}16^{8}_{3}$	123_{16} 137_{16} 149_{16} 158_{16} 169_{16}	
				${}^{2}_{nb}16^{8}_{4}$	123_{16} 137_{16} 149_{16} 158_{16} $16\,10_{16}$	
			$t16n12(16)$	${}^{2}_{nb}16^{12}_{1}$	123_{16} 139_{16} 145_{16} 146_{16} $15\,13_{16}$	
				${}^{2}_{nb}16^{12}_{2}$	123_{16} 139_{16} 146_{16} $14\,16_{16}$ $15\,13_{16}$	
				${}^{2}_{nb}16^{12}_{3}$	123_{16} $13\,12_{16}$ 145_{16} 146_{16} $15\,13_{16}$	
				${}^{2}_{nb}16^{12}_{4}$	123_{16} $13\,12_{16}$ 146_{16} $14\,16_{16}$ $15\,13_{16}$	
			$t16n14(16)$	${}^{2}_{nb}16^{14}_{3}$	123_{16} 137_{16} $14\,11_{16}$ 157_{16} $15\,13_{16}$	
				${}^{2}_{nb}16^{14}_{6}$	123_{16} 137_{16} $14\,11_{16}$ 158_{16} $15\,13_{16}$	
				${}^{2}_{nb}16^{14}_{7}$	123_{16} 137_{16} $14\,11_{16}$ 159_{16} $15\,11_{16}$	
				${}^{2}_{nb}16^{14}_{8}$	123_{16} 137_{16} $14\,11_{16}$ $15\,10_{16}$ $15\,11_{16}$	
			$t16n63(48)$	${}^{2}_{nb}16^{63}_{1}$	125_{48} $15\,10_{16}$ $16\,16_{16}$	
19	+	20	$\mathbb{Z}_{19}$	${}^{2}_{nb}19^{1}_{41}$	124_{19} 126_{19} $13\,10_{19}$ $14\,10_{19}$ $15\,12_{19}$ $16\,14_{19}$	
				${}^{2}_{nb}19^{1}_{42}$	124_{19} 126_{19} $13\,10_{19}$ $14\,10_{19}$ $15\,13_{19}$ $16\,12_{19}$	
				${}^{2}_{nb}19^{1}_{43}$	124_{19} 126_{19} $13\,10_{19}$ $14\,14_{19}$ $15\,12_{19}$ $16\,12_{19}$	[135, p. 25]
				${}^{2}_{nb}19^{1}_{44}$	124_{19} 126_{19} $13\,10_{19}$ $14\,14_{19}$ $15\,13_{19}$ $16\,12_{19}$	

Table 5.2: Neighborly 2-manifolds with vertex-transitive automorphism group.

				$^{2}_{nb}19^{1}_{45}$	124_{19} 126_{19} $13\,13_{19}$ $14\,10_{19}$ $15\,12_{19}$ $16\,12_{19}$	
				$^{2}_{nb}19^{1}_{46}$	124_{19} 126_{19} $13\,13_{19}$ $14\,10_{19}$ $15\,12_{19}$ $16\,14_{19}$	
				$^{2}_{nb}19^{1}_{47}$	124_{19} 127_{19} 139_{19} $14\,11_{19}$ $15\,10_{19}$ $15\,12_{19}$	
				$^{2}_{nb}19^{1}_{48}$	124_{19} 127_{19} 139_{19} $14\,13_{19}$ $15\,10_{19}$ $15\,12_{19}$	
				$^{2}_{nb}19^{1}_{49}$	124_{19} 127_{19} $13\,10_{19}$ 149_{19} $15\,11_{19}$ $15\,12_{19}$	
				$^{2}_{nb}19^{1}_{50}$	124_{19} 127_{19} $13\,10_{19}$ 149_{19} $15\,13_{19}$ $15\,14_{19}$	
				$^{2}_{nb}19^{1}_{51}$	124_{19} 127_{19} $13\,10_{19}$ $14\,15_{19}$ $15\,11_{19}$ $15\,12_{19}$	
				$^{2}_{nb}19^{1}_{52}$	124_{19} 127_{19} $13\,10_{19}$ $14\,15_{19}$ $15\,12_{19}$ $15\,14_{19}$	
				$^{2}_{nb}19^{1}_{53}$	124_{19} 127_{19} $13\,13_{19}$ 149_{19} $15\,11_{19}$ $15\,12_{19}$	
				$^{2}_{nb}19^{1}_{54}$	124_{19} 127_{19} $13\,13_{19}$ 149_{19} $15\,13_{19}$ $15\,14_{19}$	
				$^{2}_{nb}19^{1}_{55}$	124_{19} 127_{19} $13\,13_{19}$ $14\,15_{19}$ $15\,11_{19}$ $15\,13_{19}$	
				$^{2}_{nb}19^{1}_{56}$	124_{19} 127_{19} $13\,13_{19}$ $14\,15_{19}$ $15\,13_{19}$ $15\,14_{19}$	
				$^{2}_{nb}19^{1}_{57}$	124_{19} 127_{19} $13\,14_{19}$ $14\,11_{19}$ $15\,10_{19}$ $15\,12_{19}$	
				$^{2}_{nb}19^{1}_{58}$	124_{19} 127_{19} $13\,14_{19}$ $14\,13_{19}$ $15\,10_{19}$ $15\,13_{19}$	[134]
				$^{2}_{nb}19^{1}_{59}$	124_{19} 128_{19} $13\,12_{19}$ $14\,16_{19}$ $15\,15_{19}$ $16\,12_{19}$	
				$^{2}_{nb}19^{1}_{60}$	124_{19} 129_{19} 137_{19} $14\,13_{19}$ $15\,15_{19}$ $16\,12_{19}$	
				$^{2}_{nb}19^{1}_{61}$	124_{19} 129_{19} $13\,16_{19}$ $14\,13_{19}$ $15\,10_{19}$ $16\,14_{19}$	
				$^{2}_{nb}19^{1}_{62}$	124_{19} $12\,10_{19}$ 138_{19} 148_{19} $15\,14_{19}$ $16\,12_{19}$	
				$^{2}_{nb}19^{1}_{63}$	124_{19} $12\,12_{19}$ 138_{19} 148_{19} $15\,11_{19}$ $16\,14_{19}$	
				$^{2}_{nb}19^{1}_{64}$	124_{19} $12\,14_{19}$ $13\,11_{19}$ 148_{19} $15\,10_{19}$ $16\,14_{19}$	
				$^{2}_{nb}19^{1}_{70}$	125_{19} 129_{19} $13\,10_{19}$ $13\,17_{19}$ $15\,11_{19}$ $16\,12_{19}$	
				$^{2}_{nb}19^{1}_{71}$	125_{19} 129_{19} $13\,10_{19}$ $13\,17_{19}$ $15\,11_{19}$ $16\,14_{19}$	
			19 : 3	$^{2}_{nb}19^{3}_{2}$	125_{57} 129_{19} 136_{19} $15\,11_{19}$	
				$^{2}_{nb}19^{3}_{3}$	125_{57} 129_{19} 136_{19} $15\,14_{19}$	
				$^{2}_{nb}19^{3}_{4}$	125_{57} 129_{19} $13\,17_{19}$ $15\,14_{19}$	
				$^{2}_{nb}19^{3}_{5}$	125_{57} $12\,13_{19}$ 136_{19} $15\,11_{19}$	
				$^{2}_{nb}19^{3}_{6}$	125_{57} $12\,13_{19}$ $13\,17_{19}$ $15\,11_{19}$	
				$^{2}_{nb}19^{3}_{7}$	125_{57} $12\,13_{19}$ $13\,17_{19}$ $15\,14_{19}$	

Table 5.3 lists the generators of all the group actions that appear in Table 5.2.

Table 5.3: List of generators for the group actions.

Action	Group	Generators
4^{5}	S_4	(14), (24), (34)
6^{12}	A_5	(12346), (14)(56)
7^{4}	7 : 6	(1234567), (132645)
9^{4}	$S_3 \times \mathbb{Z}_3$	(12)(45)(78), (129)(345)(678), (147)(258)(369)
10^{7}	A_5	(13579)(2468 *10*), (19)(34)(5 *10*)(67)
12^{4}	A_4	(195)(243)(687)(*10 12 11*), (1 *11* 6)(297)(3 *10* 5)(48 *12*)
13^{1}	$\mathbb{Z}_{13}$	(123456789 *10 11 12 13*)
13^{3}	13 : 3	(123456789 *10 11 12 13*), (139)(265)(4 *12 10*)(78 *11*)
16^{4}	$\mathbb{Z}_4 \times \mathbb{Z}_4$	(175 *11*)(286 *12*)(39 *15 13*)(4 *10 16 14*), (1 *13* 2 *14*)(3847)(596 *10*)(*11 15 12 16*)
16^{5}	$\mathbb{Z}_2 \times \mathbb{Z}_8$	(14589 *11 13 16*)(2367 *10 12 14 15*), (1 *15 13 12* 9753)(2 *16 14 11 10* 864)
16^{6}	$t16n6(16)$	(1358 *10 12 14 15*)(24679 *11 13 16*), (19)(2 *10*)(34)(5 *13*)(6 *14*)(78)(*11 12*)(*15 16*)
16^{8}	$t16n8(16)$	(186 *12*)(275 *11*)(39 *15 14*)(4 *10 16 13*), (1 *15* 63)(2 *16* 54)(7 *14 11* 9)(8 *13 12 10*)
16^{12}	$t16n12(16)$	(13579 *12 14 15*)(2468 *10 11 13 16*), (16)(25)(3 *11*)(4 *12*)(78)(9 *13*)(*10 14*)(*15 16*)
16^{14}	$t16n14(16)$	(1 *12* 2 *11*)(394 *10*)(5867)(*13 15 14 16*), (1 *13* 2 *14*)(3 *12* 4 *11*)(596 *10*)(7 *15* 8 *16*)
16^{63}	$t16n63(48)$	(124)(5 *16 12*)(6 *14* 9)(7 *13 11*)(8 *15 10*), (192 *10*)(3 *11* 4 *12*)(5 *14* 6 *13*)(7 *16* 8 *15*)
19^{1}	$\mathbb{Z}_{19}$	(123456789 *10 11 12 13 14 15 16 17 18 19*)
19^{3}	19 : 3	(123456789 *10 11 12 13 14 15 16 17 18 19*), (28 *12*)(3 *15* 4)(5 *10* 7)(6 *17 18*)(9 *19 13*)(*11 14 16*)

5.2 Regular Simplicial Maps

We mentioned regular maps briefly in Chapter 2. In fact, there are different concepts for *regularity* of maps. One way of defining *regular* is that the polygonal regions of a map should be all of the same type, say, they are all m-gons for a fixed m, and, in addition, all vertices are required to have equal degree. With respect to this weak definition, e.g. any neighborly simplicial map is regular.

For our part, we call a map *regular* (or *flag-transitive*) if and only if it has a flag-transitive automorphism group. The book by Coxeter and Moser [54, Ch. 8] gives an extensive treatment of the theory of flag-transitive maps. A list of all regular maps with up to 100 edges (!) can be found in the dissertation of Wilson [161].

Any flag-transitive action of a group on a triangulated surface is transitive on the vertices and the triangles (and the edges) of the surface. In particular, when we classified all vertex-transitive simplicial maps with up to $n \leq 15$ vertices in Chapter 2, the resulting list contains all regular simplicial maps on these numbers of vertices. Since regular simplicial maps have only one orbit of triangles, we can easily extend their enumeration to more than 15 vertices. For this, we first generate all orbits of triangles for transitive group actions on $n \leq 22$ vertices. Then we check whether the triangles of any single orbit form a closed surface, and, if so, test if the group action is transitive on the flags of that surface.

Proposition 5.2 *There are exactly* 14 *regular simplicial maps on* $n \leq 22$ *vertices;* 10 *of them are orientable and* 4 *are non-orientable.*

Table 5.4 gives a list of these maps, and the corresponding permutation groups can be found in Table 5.5.

n	Or.	Gen. g	Automorphism Group	Type	Orbit	Remarks
4	+	0	S_4	${}^{2}_{reg}4^{5}_{1}$	123_{4}	tetrahedron
6	+	0	$[2^3]S_3 = 2wrS_3$	${}^{2}_{reg}6^{11}_{1}$	123_{8}	octahedron
	−	1	A_5	${}^{2}_{reg}6^{12}_{1}$	123_{10}	$\mathbb{RP}^2_6$
9	+	1	$\mathbb{Z}_3^2 : D_6$	${}^{2}_{reg}9^{18}_{1}$	136_{18}	[3], [35], [54, Ch. 8], [161], [Ch. 2, ${}^{2}9^{18}_{1}$]
12	+	0	$[2]A_5$	${}^{2}_{reg}12^{76}_{1}$	126_{20}	icosahedron
		1	$S_4 \times S_3$	${}^{2}_{reg}12^{83}_{1}$	123_{24}	[3], [35], [54, Ch. 8], [161], [Ch. 2, ${}^{2}12^{83}_{1}$]
		3	$t12n113(192)$	${}^{2}_{reg}12^{113}_{1}$	124_{32}	Dyck's regular map [58], [59]; [24], [32], [38], [146], [161], [Ch. 2, ${}^{2}12^{113}_{1}$]
15	+	6	$[5^2 : 2]S_3$	${}^{2}_{reg}15^{18}_{1}$	123_{50}	[161], [Ch. 2, ${}^{2}15^{18}_{1}$]
16	+	1	$t16n431(192)$	${}^{2}_{reg}16^{431}_{1}$	159_{32}	[3], [35], [54, Ch. 8], [161]
18	+	10	$t18n153(432)$	${}^{2}_{reg}18^{153}_{1}$	$1\,7\,13_{72}$	
	−	14	$t18n145(360)$	${}^{2}_{reg}18^{145}_{1}$	159_{60}	[161]
21	+	15	$t21n23(588)$	${}^{2}_{reg}21^{23}_{1}$	$1\,8\,16_{98}$	
	−	9	$t21n20(336)$	${}^{2}_{reg}21^{20}_{1}$	124_{56}	[161]
				${}^{2}_{reg}21^{20}_{2}$	$1\,12\,19_{56}$	[161]

Table 5.4: Regular simplicial maps with $n \leq 22$ vertices.

Action	Group	Generators
4^{5}	S_4	(14), (24), (34)
6^{11}	$[2^3]S_3 = 2wrS_3$	(135)(246), (15)(24), (36)
6^{12}	A_5	(12346), (14)(56)
9^{18}	$\mathbb{Z}_3^2{:}D_6$	(12)(35)(67), (12)(36)(48)(57), (129)(345)(678), (147)(258)(369), (345)(687)
12^{76}	$[2]A_5$	(12)(3*12*)(4*11*)(5*10*), (1*12*)(23)(45)(67)(89)(*10 11*), (2468*10*)(3579*11*), (4*10*)(5*11*)(68)(79)
12^{83}	$S_4 \times S_3$	(147*10*)(258*11*)(369*12*), (15)(2*10*)(48)(7*11*), (159)(26*10*)(37*11*)(48*12*), (1*10*)(25)(69)
12^{113}	$t12n113(192)$	(13)(2*12*)(4*10*)(5*11*)(68)(79), (13579*11*)(2468*10 12*), (1*12*)(23)(67)(89), (4*10*)(5*11*)(67)(89)
15^{18}	$[5^2{:}2]S_3$	(14)(28)(3*12*)(69)(7*13*)(*11 14*), (16*11*)(27*12*)(38*13*)(49*14*)(5*10 15*), (1*11*)(27)(4*14*)(5*10*)(8*13*), (1*13 10*74)(258*11 14*)
16^{431}	$t16n431(192)$	(142)(5*14 12*)(6*15 10*)(7*13*9)(8*16 11*), (15)(26)(37)(48)(9*14*)(*10 13*)(*11 16*)(*12 15*), (23)(5*11*)(69)(7*12*)(8*10*)(*13 16*)
18^{145}	$t18n145(360)$	(1*13 16 10*4)(2*15 17 12*53*14 18 11*6)(89), (1*18 10*3*16 12*)(2*17 11*)(49*13*67*15*)(58*14*)
18^{153}	$t18n153(432)$	(1*14*4*15*5*18*2*13*3*16*6*17*)(7*12*98*11 10*), (1*14 11*2*13 12*)(3*18*76*15 10*)(4*17*85*16*9)
21^{20}	$t21n20(336)$	(1*16*593*10*4*17*)(2*19*68)(7*13 21 11 12 14 20 18*), (1*17*2*18 13 11 19*)(3*12 15 20*9*10*5)(48*14*6*16*7*21*)
21^{23}	$t21n23(588)$	(14)(23)(57)(8*20 11 15 14 17 10 19 13 21*9*16 12 18*), (1*10*69482*14*7*13*5*12*3*11*)(*15 19 16 20 17 21 18*)

Table 5.5: List of generators for the group actions.

Chapter 6

Some Results Related to the Evasiveness Conjecture

In this and the following chapter, we discuss particular vertex-homogeneous (i.e., vertex-transitive) simplicial complexes and set systems, which are closely related to complexity problems of graph properties, and which are, at the same time, of interest in the context of topological fixed point theorems.

6.1 Topological Aspects of Graph Properties

Let $\mathcal{P}$ be any graph property on a fixed set of nodes V of size $n := |V|$, and let E denote the set of all edges on V, with $m := |E| = \binom{n}{2}$. We identify $\mathcal{P}$ with the set system

$$\mathcal{F}_{\mathcal{P}} := \{A \subseteq E : \text{Graph } (V, A) \text{ has property } \mathcal{P}\} \subseteq 2^{E},$$

and for an unknown graph $\mathcal{G} = (V, A)$ on V we consider the *decision problem* whether $\mathcal{G}$ has the property $\mathcal{P}$ or not. In order to find out if the edge set A of $\mathcal{G}$ belongs to $\mathcal{F}_{\mathcal{P}}$, we ask questions of the type "Is $e \in A$?", and an oracle answers (correctly) YES or NO.

The number of elements of E that we will have to test in the worst case, if we proceed according to some optimal strategy, is called the *argument complexity* $c(\mathcal{F}_{\mathcal{P}})$ of $\mathcal{P}$. It is $0 \le c(\mathcal{F}_{\mathcal{P}}) \le m$, and $\mathcal{P}$ is *trivial* if $c(\mathcal{F}_{\mathcal{P}}) = 0$ and *non-trivial* if $c(\mathcal{F}_{\mathcal{P}}) > 0$. If $c(\mathcal{F}_{\mathcal{P}}) = m$, then $\mathcal{P}$ is called *evasive*, and *non-evasive* if $c(\mathcal{F}_{\mathcal{P}}) < m$. For general set systems $\mathcal{F} \subseteq 2^{E}$, these terms are defined analogously.

In the early seventies Richard Karp proposed the following remarkable conjecture.

Evasiveness Conjecture for Graph Properties: *Every non-trivial monotone graph property $\mathcal{P}$ is evasive.*

Extensive work has been done on determining the argument complexity of particular graph properties (see e.g. [1], [14], [33], [162], and references contained therein).

The first successful approach to Karp's Conjecture was carried through by Kahn, Saks, and Sturtevant [80] in 1984. Using methods from algebraic topology, in particular, a fixed point theorem by Oliver [129], they were able to settle the case when n is a prime power (and the case $n = 6$). For this, they restated Karp's Conjecture in the language of simplicial complexes: If $\mathcal{P}$ is a monotone graph property, then the corresponding set system $\mathcal{F}_{\mathcal{P}}$ is a (finite abstract) simplicial complex with the vertex set E. We call $\mathcal{F}_{\mathcal{P}}$ the *graph complex* associated with $\mathcal{P}$. Invariance under permutation of the nodes of V (what one naturally requires for $\mathcal{P}$ to be a graph property) gives rise to an induced action of the symmetric group S_n on the edge set E, and thus on the simplicial complex $\mathcal{F}_{\mathcal{P}}$. Clearly, the action of S_n is transitive on E.

By allowing the symmetry group to be any finite group G, one obtains the following more general situation.

Evasiveness Conjecture for Simplicial Complexes [80]: *If $\mathcal{F}$ is a non-evasive vertex-homogeneous simplicial complex* (VHSC) *on the vertex set $E = \{1, \ldots, m\}$ with vertex-transitive action by some group G, then it is the standard $(m-1)$-simplex Δ_{m-1}.*

To be "non-evasive" is in fact a rather strong *topological* requirement. The following sequence of implications holds for finite simplicial complexes (cf. [80]; for an exposition of topological methods in combinatorics see [22]):

$$\text{non-evasive} \Rightarrow \text{collapsible} \Rightarrow \text{contractible} \Rightarrow \mathbb{Z}\text{-acyclic} \Rightarrow \mathbb{Q}\text{-acyclic} \Rightarrow \tilde{\chi} = 0$$

and leads to further generalizations of the above conjectures (cf. Figure 6.1). ($\tilde{\chi}$ denotes the reduced Euler characteristic of a simplicial complex.)

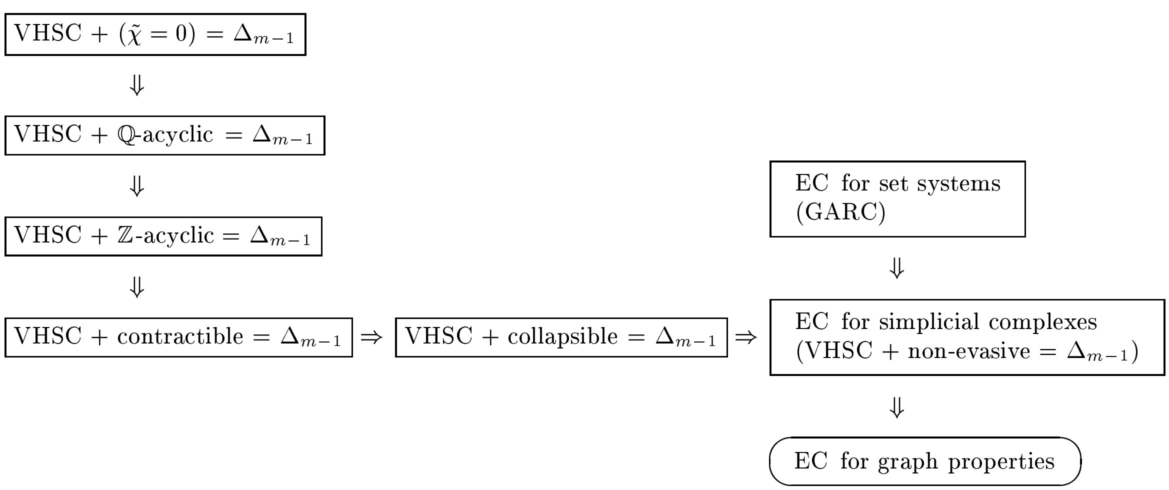


Figure 6.1: Generalizations of the Evasiveness Conjecture (EC) for graph properties.

Instead of relaxing the condition "non-evasive" one can alternatively remove the monotonicity. The resulting Evasiveness Conjecture for set systems is known as the

Generalized Aanderaa-Rosenberg Conjecture (GARC) [137] *Let $\mathcal{F} \subseteq 2^E$ be a set system with induced transitive symmetry group $G \subseteq S_E$. If $\emptyset \in \mathcal{F}$, but $E \notin \mathcal{F}$, then $\mathcal{F}$ is evasive.*

Albeit the latter conjecture as well as the Conjecture $[\text{VHSC} + (\tilde{\chi}=0) = \Delta_{m-1}]$ were proved by RIVEST and VUILLEMIN [137] for sets E of prime power cardinality, $m = q^s$, ILLIES [77] provided a counterexample to GARC for $m = 12$, and there is an abundance of counterexamples to the Conjecture $[\text{VHSC} + (\tilde{\chi}=0) = \Delta_{m-1}]$ for $m \neq q^s$.

In the next section we will review some fixed point theorems and their applications. In Section 6.3 we show that there is, apart from the simplex, no $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex with $m = 6$, 10, or 12 vertices. This fact implies the non-existence of 2- and 3-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes, different from a simplex.

Furthermore, we construct in Section 6.4 an *infinite class* of counterexamples to the Generalized Aanderaa-Rosenberg Conjecture for $m = u(u+1)$, $u \geq 3$ odd, with the Illies example as the smallest member of the class.

6.2 Fixed Point Theorems and Group Actions

Recall that if the vertex-transitive action of a (finite) group G on a (finite) simplicial complex K (with m vertices) has a fixed point, then K is a simplex. This can be seen geometrically by regarding K as a subcomplex of the $(m-1)$-dimensional simplex Δ_{m-1} with vertices $e_1, \ldots, e_m$. Any point x of K has a unique representation $x = \sum_{i=1}^{m} \lambda_i e_i$, with $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$. The group G then acts by permuting the coordinates, $gx = \sum_{i=1}^{m} \lambda_i e_{g(i)}$, $g \in G$. If G is transitive, then for every i, j there is some $g \in G$ such that $e_j = e_{g(i)}$. If, in addition, the action of G has a fixed point y, then $gy = y$ for every group element g, and therefore $\lambda_1 = \ldots = \lambda_m = \frac{1}{m}$. But $y = \frac{1}{m} \sum_{i=1}^{m} e_i$ is a point of K if and only if K is a simplex.

This simple fact, in combination with fixed point theorems from algebraic topology, provides an important tool for the study of group actions on simplicial complexes. It was shown by Smith [149] that if a p-group P, i.e., a group with prime power order $|P| = p^t$, acts on a $\mathbb{Z}_p$-acyclic complex, then the fixed point set for this action is $\mathbb{Z}_p$-acyclic as well. In particular, the fixed point set is not empty – hence, there are no vertex-transitive group actions of a p-group on a $\mathbb{Z}_p$-acyclic simplicial complex (that is not a simplex).

The theorem by Smith has been generalized by Oliver.

Theorem 6.1 (Oliver [129]) *Let G be a finite group with subsequent normal subgroups $P \lhd Q \lhd G$ such that*

(i) *P is a p-group,*

(ii) *G/Q is a q-group, and*

(iii) *Q/P is cyclic.*

If G acts on a $\mathbb{Z}_p$-acyclic complex K, then the Euler characteristic of the fixed point set K^G is $\chi(K^G) \equiv 1 \bmod q$.

We say that a group G is of *Oliver (p,q)-type* if it has the properties of Theorem 6.1. If a group G of Oliver (p,q)-type acts vertex-transitively on a $\mathbb{Z}_p$-acyclic simplicial complex K, then K is a simplex. In fact, if H is an Oliver (p,q)-type vertex-transitive subgroup of some group G, which acts on a $\mathbb{Z}_p$-acyclic complex K, then already K is a simplex.

Theorem 6.2 (Kahn, Saks, and Sturtevant [80]) *Let $\mathcal{F}_{\mathcal{P}_n}$ be the graph complex associated with some (non-trivial) graph property $\mathcal{P}_n$ on $n = p^t$ nodes, with p prime. Then $\mathcal{F}_{\mathcal{P}_n}$ is not $\mathbb{Z}_p$-acyclic.*

Proof: Let $G = \mathit{Aff}(GF(p^t)) < S_n$ be the group of affine transformations of $GF(p^t)$. The group G is 2-transitive on $\{1 \ldots n\}$ and therefore transitive on the edge set E. Furthermore, G is of Oliver $(p,1)$-type (choose $Q := G$ and $P := \{x \mapsto x+b : b \in GF(p^t)\}$). Hence, G is a vertex-transitive Oliver $(p,1)$-type subgroup of the symmetric group S_n with induced action on all graph complexes $\mathcal{F}_{\mathcal{P}_n}$. But then either $\mathcal{F}_{\mathcal{P}_n}$ is a simplex, and thus $\mathcal{P}_n$ is trivial, or $\mathcal{F}_{\mathcal{P}_n}$ is not $\mathbb{Z}_p$-acyclic by Theorem 6.1. □

If a graph complex is not $\mathbb{Z}_p$-acyclic, then it cannot be non-evasive.

Corollary 6.3 (KAHN, SAKS, and STURTEVANT [80]) *The Evasiveness Conjecture for graph properties holds for every prime power number of nodes.*

6.3 The Evasiveness Conjecture in Dimension 2 and 3

We will show in the following that (non-trivial) non-evasive vertex-homogeneous simplicial complexes do not exist in dimension 2 and 3.

Proposition 6.4 (BJÖRNER) *Let E be a finite set of cardinality $m = |E| = q_1^{\alpha_1} \cdots q_r^{\alpha_r}$ (primepower-decomposition) and $M = \max\{q_1^{\alpha_1}, \ldots, q_r^{\alpha_r}\}$. If $K \subseteq 2^E$ is a vertex-homogeneous simplicial complex on the vertex set E with reduced Euler characteristic $\tilde{\chi}(K) = 0$, then $\dim(K) \geq M - 1$.*

Proof: By the transitivity of the group action, every element of E is contained the same number of times, s, in the k-sets of every orbit $\mathcal{O}$ of $(k-1)$-dimensional faces, i.e., $k \cdot |\mathcal{O}| = s \cdot |E|$. For $M = q_i^{\alpha_i}$ this implies that $q_i \mid |\mathcal{O}|$ if $k < M$. Now, if $\dim(K) < M-1$, then, with the exception of the orbit of the empty set, which has size 1, the size of every orbit of $(k-1)$-faces of K is divisible by q_i. Hence, $\tilde{\chi}(K) \equiv -1 \bmod q_i \neq 0$. □

Corollary 6.5 (RIVEST and VUILLEMIN [137]) *Conjecture* [VHSC + $(\tilde{\chi} = 0) = \Delta_{m-1}$] *holds if $m = q^s$ is a prime power.*

dim	# vertices
1	-
2	6
3	6, 12
4	10, 12, 15, 20, 30, 60
5	10, 12, 15, 20, 30, 60
6	10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 60, 70, 84, 105, 140, 210, 420
7	10, 12, 14, 15, 20, 21, 24, 28, 30, 35, 40, 42, 56, 60, 70, 84, 105, 120, 140, 168, 210, 280, 420, 840
8	12, 14, 15, 18, 20, 21, 24, 28, 30, 35, 36, 40, 42, 45, 56, 60, 63, 70, 72, 84, 90, 105, 120, 126, 140, 168, 180, 210, 252, 280, 315, 360, 420, 504, 630, 840, 1260, 2520

Table 6.1: Numbers of vertices for low-dimensional vertex-homogeneous simplicial complexes with $\tilde{\chi} = 0$.

It follows from Proposition 6.4 that for every d-dimensional vertex-homogeneous simplicial complex K with reduced Euler characteristic $\tilde{\chi} = 0$ one has $M \leq d + 1$. In particular, the cardinality m of the vertex set E of K can only attain finitely many different values. Furthermore, $m > d + 2$. This follows from the fact that for $m = d + 2$ there is, by transitivity, only one orbit of d-faces. But the boundary complex of a simplex is a sphere with $\tilde{\chi} \neq 0$. Table 6.1 displays the vertex-numbers that are possible for $d \leq 8$.

As a result of the above, we get a lower bound for the dimension of graph complexes $\mathcal{F}_{\mathcal{P}_n}$ with reduced Euler characteristic $\tilde{\chi} = 0$ for graph properties $\mathcal{P}_n$ on n nodes. See Table 6.2 for small $n \neq p^t$.

# nodes n	# vertices $m = \binom{n}{2}$	dim $\geq$	# nodes n	# vertices $m = \binom{n}{2}$	dim $\geq$
6	15	4	26	325	24
10	45	8	28	378	26
12	66	10	30	435	28
14	91	12	33	528	15
15	105	6	34	561	16
18	153	16	35	595	16
20	190	18	36	630	8
21	210	6	38	703	36
22	231	10	39	741	18
24	276	22	40	780	12

Table 6.2: Lower bounds for the dimension of graph complexes ($\neq$ simplex) with $\tilde{\chi} = 0$.

The 'smallest example' of a graph property $\mathcal{P}$ with $\tilde{\chi}(\mathcal{F}_{\mathcal{P}}) = 0$ can be found on 6 nodes. It is the property $\mathcal{P}_{\mathrm{A}}$ of being a subgraph of any of the first four graphs of Figure 6.2. The next listed three respective two graphs describe examples of higher-dimensional graph complexes on 6 nodes with $\tilde{\chi}(\mathcal{F}_{\mathcal{P}}) = 0$. All three examples have nontrivial

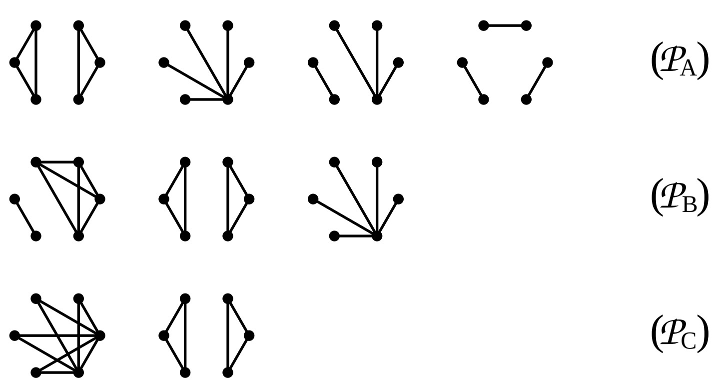


Figure 6.2: Three examples of graph properties with $\tilde{\chi}(\mathcal{F}_{\mathcal{P}}) = 0$.

reduced homology, $\tilde{H}_*(\mathcal{F}_{\mathcal{P}_{\mathrm{A}}}) = (0, 0, \mathbb{Z}^{15}, \mathbb{Z}^{15}, 0, 0)$, $\tilde{H}_*(\mathcal{F}_{\mathcal{P}_{\mathrm{B}}}) = (0, 0, \mathbb{Z}^{15}, \mathbb{Z}^{15}, 0, 0, 0)$, and $\tilde{H}_*(\mathcal{F}_{\mathcal{P}_{\mathrm{C}}}) = (0, 0, 0, \mathbb{Z}^{20} \oplus \mathbb{Z}_3, \mathbb{Z}^{20}, 0, 0, 0, 0)$.

Remark: Although there are many examples of graph complexes with $\tilde{\chi}(\mathcal{F}_{\mathcal{P}}) = 0$, there seems to be no example known of a nontrivial $\mathbb{Q}$-acyclic graph complex.

We turn back to general vertex-homogeneous simplicial complexes. As we mentioned in Chapter 2, there exists a classification of all transitive permutation groups of degree $m \leq 15$. We determined for all transitive permutation groups G of degree $m = 6, 10, 12, 14, 15$ whether they are of Oliver (p,q)-type (for some p and some q) and, in addition, for the groups that are *not* of Oliver (p,q)-type if they have a transitive subgroup $H < G$ of Oliver (p,q)-type. Table 6.3 gives a statistics.

# Vertices m	# Transitive group actions	# Groups G not of Oliver (p,q)-type	# Groups without trans. $H < G$ of Oliver (p,q)-type
6	16	4	0
10	45	21	3
12	301	107	1
14	63	34	2
15	104	64	5

Table 6.3: Transitive permutation groups without subgroups of Oliver (p,q)-type.

One example of a permutation group of degree 6 is the group action of the alternating group A_5. Although A_5 is not of Oliver (p,q)-type as it is simple, on 6 vertices A_5 has A_4 as a vertex-transitive subgroup of Oliver $(2,1)$-type.

Lemma 6.6 *There is, apart from the 2-simplex, no 2-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex.*

Proof: If there were a non-trivial 2-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex K, then it would have 6 vertices by Proposition 6.4. But every transitive permutation group of degree 6 is of Oliver (p,q)-type or has a vertex-transitive subgroup of Oliver (p,q)-type. Thus K cannot be $\mathbb{Z}$-acyclic by Theorem 6.1. $\square$

There exists a transitive permutation group, $[2^4]\,3^2{:}4$, on 12 vertices, which is not of Oliver (p,q)-type and that has no vertex-transitive subgroup of Oliver (p,q)-type. Thus, we cannot use the above argument in the case of 3-dimensional complexes. In this case, we proceed as follows. We will first determine all simplicial complexes on 12 vertices with a vertex-transitive action of the group $[2^4]\,3^2{:}4$ for which $\tilde{\chi} = 0$ and then compute their homology. It will turn out that the homology is non-trivial for each of the complexes.

Definition 6.7 *Let G be a transitive permutation group of the set $E = \{1, 2, \dots, m\}$. Then G acts on the system of sets 2^E, and we denote the set of orbits for this action by $\mathrm{Orb}_G(2^E)$. We define a partial order "$<$" on $\mathrm{Orb}_G(2^E)$: For $\mathcal{O}_1, \mathcal{O}_2 \in \mathrm{Orb}_G(2^E)$, $\mathcal{O}_2 < \mathcal{O}_1$ if and only if*

(i) *$\mathcal{O}_2$ and $\mathcal{O}_1$ are orbits of $(k-1)$-sets and k-sets respectively, and*

(ii) *for any $B \in \mathcal{O}_2$ and $g \in G$, $gB \subseteq A$ for some $A \in \mathcal{O}_1$.*

The partially ordered set $(\mathrm{Orb}_G(2^E), <)$ is the ***orbit poset*** *corresponding to the action of G on 2^E.*

It follows that simplicial complexes $K \subseteq 2^E$, which are invariant under the induced action of a transitive permutation group G on E, are in one-to-one correspondence with the ideals of the orbit poset $\mathrm{Orb}_G(2^E)$.

Since our aim is to determine vertex-homogeneous simplicial complexes with reduced Euler characteristic $\tilde{\chi} = 0$, it is natural to consider *weighted orbit posets*, where we label each orbit of k-sets by its size times the sign $(-1)^{k+1}$. Complexes with reduced Euler characteristic $\tilde{\chi} = 0$ then correspond to ideals of weighted orbit posets with $\sum_{\mathcal{O} \in I} w_{\mathcal{O}} = 0$ for the weights $w_{\mathcal{O}}$ of the orbits $\mathcal{O}$ in the ideal I.

The weighted orbit posets for the A_5-action and its transitive A_4-subaction on 6 vertices are depicted in Figure 6.3. The respective ideals I with $\sum_{\mathcal{O} \in I} w_{\mathcal{O}} = 0$ of these two orbit posets all correspond to the 6-vertex triangulation of the real projective plane. For the case of the A_4-action we shaded the orbit with 4 triangles in Figure 6.4 in grey, the other orbit of 6 triangles is in white.

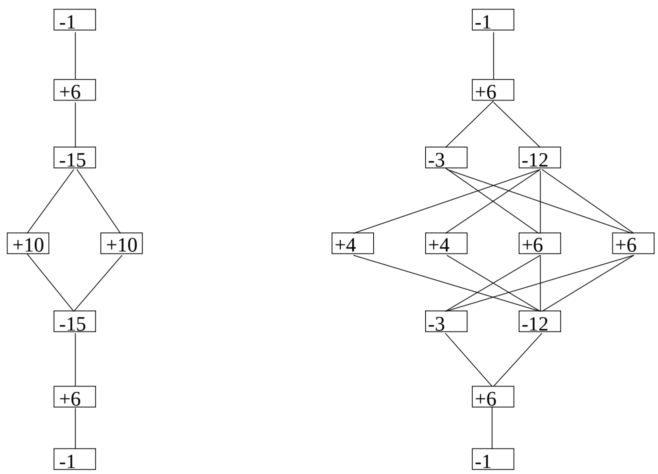


Figure 6.3: Weighted orbit posets of the transitive A_5- and A_4-actions on 6 vertices.

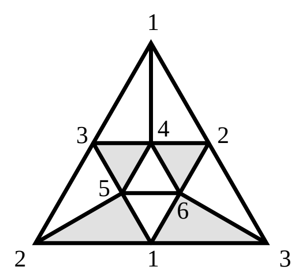


Figure 6.4: 6-vertex triangulation of the real projective plane.

As listed in Table 6.3, there are three transitive permutation groups on 10 vertices that are not of Oliver type and contain no transitive Oliver type subgroup. These groups are $A_5 < A_6 < M_{10}$ (with inclusions as transitive permutation groups). On 12 vertices there is only one such group, $[2^4]\,3^2\!:\!4$. There are two groups on 14 vertices, $PSL_2(7)$ and $PSL_2(13)$, which are not included in each other; and on 15 vertices we have five groups, with the inclusions $A_5 < S_5, A_6$; $S_5 < S_6, A_7$; and $A_6 < S_6, A_7$. If we want to generate all vertex-homogeneous simplicial complexes with $\tilde{\chi} = 0$ for these actions, it would be sufficient to do this for the actions $A_5(10)$, $[2^4]\,3^2 : 4$, $PSL_2\,(7)$, $PSL_2\,(13)$, and $A_5(15)$, since these are transitive subgroups of the other groups. The weighted orbit poset for $A_5(10)$ is displayed in Figure 6.5. Nevertheless, we only succeeded with the

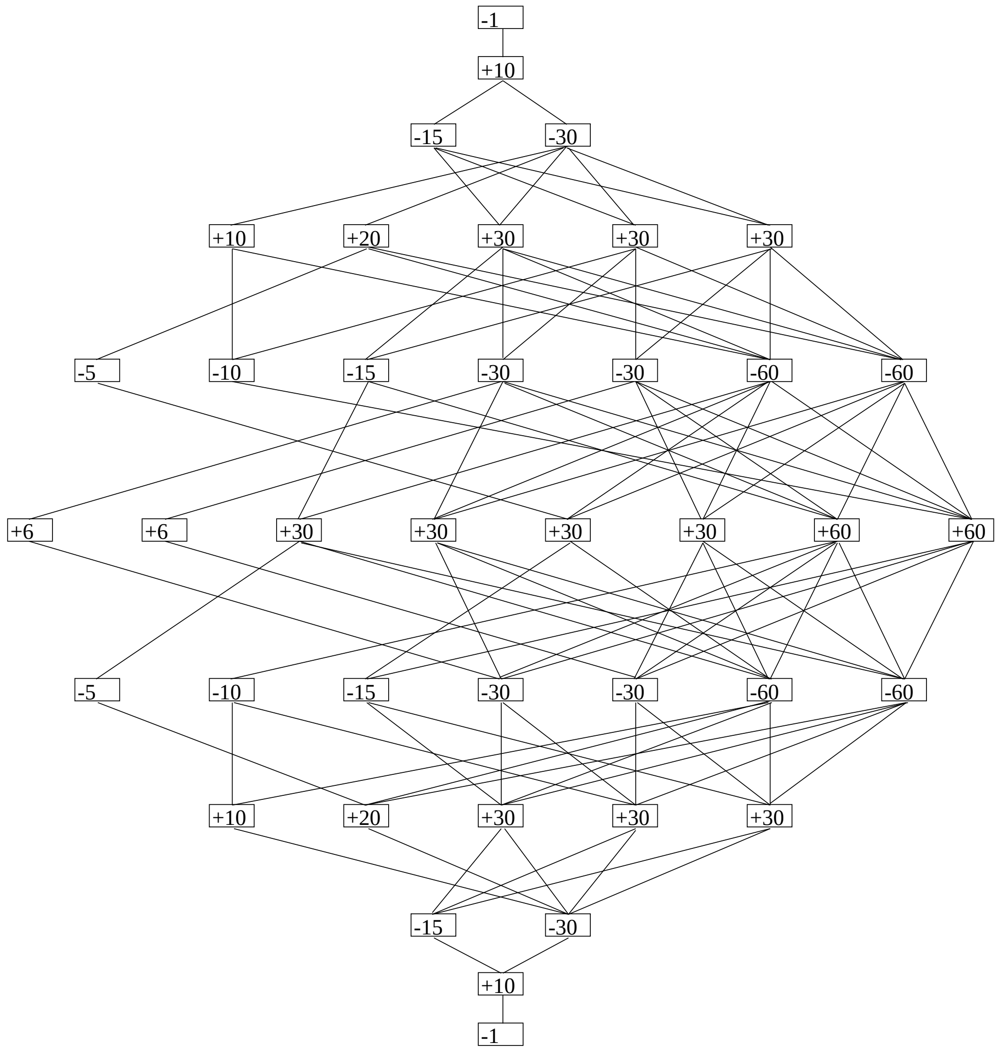


Figure 6.5: Weighted orbit poset of the transitive A_5-action on 10 vertices.

generation of all ideals (simplicial complexes) with $\sum = 0$ ($\tilde{\chi} = 0$) for the three actions on 10 vertices, for the action of $[2^4]\,3^2 : 4$ on 12 vertices, of $PSL_2\,(13)$ on 14 vertices, and of A_7 on 15 vertices. Table 6.4 lists the number of complexes that we found with a GAP-program.

Several of the complexes are combinatorially isomorphic, but none of the complexes with 10 or 12 vertices is $\mathbb{Z}$-acyclic, which we checked with the C-program HOMOLOGY by HECKENBACH [71]. (For most of the complexes with 14 and 15 vertices, it was not possible to determine their homology with the program.)

Theorem 6.8 *There is, other than a simplex, no $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex on $m = 6,\ 10,\ 12$ vertices. In particular, there is no non-trivial 2- and 3-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex.*

# Vertices m	Weighted orbit poset	# Complexes with $\tilde{\chi} = 0$	# $\mathbb{Z}$-acyclic complexes
10	A_5	112	0
	A_6	8	0
	M_{10}	0	0
12	$[2^4]\,3^2\!:\!4$	336	0
14	$PSL_2\,(7)$	?	?
	$PSL_2\,(13)$	140	?
15	A_5	?	?
	S_5	?	?
	A_6	?	?
	S_6	?	?
	A_7	42	?

Table 6.4: Transitive permutation groups that do not have any subgroup of Oliver (p,q)-type.

Corollary 6.9 *The Evasiveness Conjecture holds for* 2- *and* 3-*dimensional simplicial complexes.*

Remark: In the next Chapter, we construct a 5-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex on 30 vertices, with f-vector $f = (1, 30, 195, 340, 255, 96, 15)$. Further examples of higher dimension exist on 30 and 60 vertices. The first such example of dimension 11 on 60 vertices was found by Oliver [80]. We believe that there is, apart from the 4-simplex, no $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex of dimension 4.

6.4 An Infinite Class of Counterexamples to the Generalized Aanderaa-Rosenberg Conjecture

We construct an infinite class of counterexamples to the Generalized Aanderaa-Rosenberg Conjecture (see p. 100) in three steps.

Let $m = u(u+1)$, for $u \geq 3$ odd. As group of symmetries we consider $G = \mathbb{Z}_m$, with action on the ground set $E = \{1, 2, \ldots, m\}$ by translation (mod m).

1. Let $\mathcal{A}_m$ be the set system of all subsets of the sets of the $\mathbb{Z}_m$-orbit

$$\begin{array}{ll}
\{\,1,\, u{+}1,\, 2u{+}1,\, \ldots\,,\, (u{-}1)u{+}1,\, uu{+}1\,\}, & \{\,1,4,7,10\,\},\\
\{\,2,\, u{+}2,\, 2u{+}2,\, \ldots\,,\, (u{-}1)u{+}2,\, uu{+}2\,\}, & \{\,2,5,8,11\,\},\\
\{\,3,\, u{+}3,\, 2u{+}3,\, \ldots\,,\, (u{-}1)u{+}3,\, uu{+}3\,\}, & \\
\qquad\ldots & \\
\{\,u,\, u{+}u,\, 2u{+}u,\, \ldots\,,\, (u{-}1)u{+}u,\, uu{+}u\,\}, & \{\,3,6,9,12\,\}
\end{array}$$

with u sets of $u{+}1$ elements each. (On the right hand side, we note Illies' example [77] for $u = 3$.)

2. Let $\mathcal{B}_m$ be the set system of all subsets of the sets of the $\mathbb{Z}_m$-orbit

$$\begin{array}{ll}
\{\,1,\, 2u{+}1,\, 4u{+}1,\, \ldots\,,\, (u{-}1)u{+}1\,\}, & \{\,1,7\,\},\\
\{\,2,\, 2u{+}2,\, 4u{+}2,\, \ldots\,,\, (u{-}1)u{+}2\,\}, & \{\,2,8\,\},\\
\{\,3,\, 2u{+}3,\, 4u{+}3,\, \ldots\,,\, (u{-}1)u{+}3\,\}, & \{\,3,9\,\},\\
\qquad\ldots & \ \ldots\\
\{\,2u,\, 2u{+}2u,\, 4u{+}2u,\, \ldots\,,\, (u{-}1)u{+}2u\,\}, & \{\,6,12\,\}
\end{array}$$

with $2u$ sets of $(u{+}1)/2$ elements each.

3. Let $\mathcal{C}_m$ be the set system of all subsets of the sets of the $\mathbb{Z}_m$-orbit

$$\begin{array}{ll}
\{\,1,\, (u{+}1){+}1,\, 2(u{+}1){+}1,\, \ldots\,,\, (u{-}1)(u{+}1){+}1\,\}, & \{\,1,5,9\,\},\\
\{\,2,\, (u{+}1){+}2,\, 2(u{+}1){+}2,\, \ldots\,,\, (u{-}1)(u{+}1){+}2\,\}, & \{\,2,6,10\,\},\\
\{\,3,\, (u{+}1){+}3,\, 2(u{+}1){+}3,\, \ldots\,,\, (u{-}1)(u{+}1){+}3\,\}, & \{\,3,7,11\,\},\\
\qquad\ldots & \\
\{\,(u{+}1),\, (u{+}1){+}(u{+}1),\, 2(u{+}1){+}(u{+}1),\, \ldots\,,\, (u{-}1)(u{+}1){+}(u{+}1)\,\}, & \{\,4,8,12\,\}
\end{array}$$

with $(u{+}1)$ sets of u elements each.

Proposition 6.10 *The set system $\mathcal{F}_m = (\mathcal{A}_m \backslash \mathcal{B}_m) \cup \mathcal{C}_m$ is non-evasive and thus provides an infinite class of counterexamples to the Generalized Aanderaa-Rosenberg Conjecture.*

Proof: Let $A \in 2^E$. We want to determine whether A is in $\mathcal{F}_m$ or not by asking questions "Is $e \in A$?". An oracle answers YES or NO. In order to show that $\mathcal{F}_m$ is non-evasive, we give a decision tree of depth $m-1$.

Case I ($A \cap \{1, 2, \ldots, m-2u\} \neq \emptyset$)

We start with elements in $\{1, 2, \ldots, m-2u\}$ and ask successively "Is $1 \in A$?", "Is $2 \in A$?", "Is $3 \in A$?", ... If none of these elements is in A, we have checked that $\{1, 2, \ldots, m-2u\} \cap A = \emptyset$, and this case will be discussed later. Otherwise, the first time we get YES as an answer, say, for "Is $r \in A$?" with $r \leq m-2u$, we next test the elements in $\{r+u, r+3u, \ldots\} \subseteq \{1, 2, \ldots, m\}$, consecutively. As soon as we are successful, we see by the construction of the set system $\mathcal{F}_m$ that, since $r \leq m-2u$, there exists at least one element, namely, $r+2u$, such that either both the sets A and $A \cup \{ r+2u \}$ lie in $\mathcal{F}_m$, or both do not. Hence, we do not have to ask for $r+2u$ in order to check whether A is in $\mathcal{F}_m$ or not. This gives a leaf of the decision tree, which we depict in Figure 6.6 by an oval, containing all the elements we do not have to ask for. Such a leaf has a depth of altogether at most $m-1$. Finally, if we have tested all the elements $\{ r+u, r+3u, \dots, r+(2k+1)u \}$, with k the greatest integer such that $r+(2k+1)u \leq m$, and none of them is contained in A, then for A to lie in $\mathcal{F}_m$ it has to be in $\mathcal{C}_m$. But then there is again at least one element that we do not have to ask for, namely $r+(u+1)$. Thus, this leaf has also at most depth $m-1$.

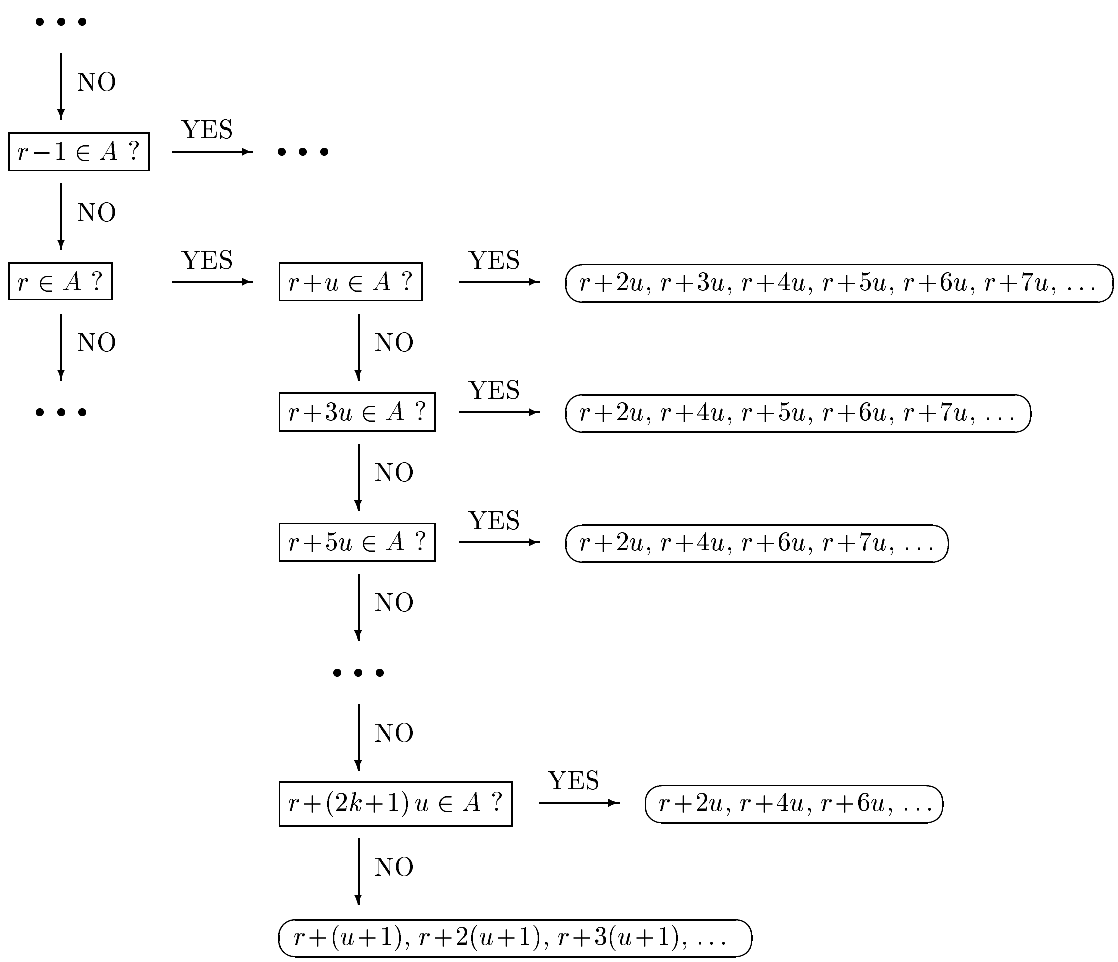


Figure 6.6: Decision tree for $r \leq m-2u$.

Case II $(A \cap \{ 1, 2, \dots, m-2u \} = \emptyset)$

In this part, we further analyze A in the case where $A \cap \{ 1, 2, \dots, m-2u \} = \emptyset$. For this, we can restrict $\mathcal{F}_m$ to sets that do not contain the elements $1, 2, \dots, m-2u$. The corresponding sets in $\mathcal{A}_m \setminus \mathcal{B}_m$ are

$$\begin{array}{ll}
\{\,(m-2u)+1,(m-u)+1\,\}, & \{\,7,10\,\},\\
\{\,(m-2u)+2,(m-u)+2\,\}, & \{\,8,11\,\},\\
\cdots & \\
\{\,(m-2u)+u,(m-u)+u\,\}, & \{\,9,12\,\},
\end{array}$$

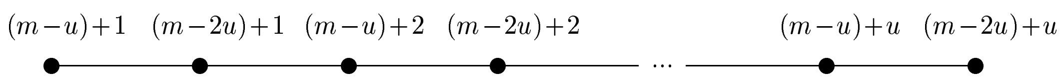
and the remaining sets of $\mathcal{C}_m$ are

$$\begin{array}{ll}
\{\,(m-2u)+1,(m-u)+1+1\,\}, & \{\,7,11\,\},\\
\{\,(m-2u)+2,(m-u)+1+2\,\}, & \\
\cdots & \\
\{\,(m-2u)+(u-1),(m-u)+1+(u-1)\,\} & \{\,8,12\,\}
\end{array}$$

$$\begin{array}{ll}
\{\,(m-2u)+1\,\}, & \{\,7\,\},\\
\{\,(m-2u)+2\,\}, & \{\,8\,\},\\
\cdots & \cdots\\
\{\,(m-2u)+2u\,\}, & \{\,12\,\},
\end{array}$$

$$\begin{array}{ll}
\{\ \}, & \{\ \}.
\end{array}$$

If we denote by $\overline{\mathcal{A}_m}$, $\overline{\mathcal{B}_m}$, and $\overline{\mathcal{C}_m}$ the restrictions of $\mathcal{A}_m$, $\mathcal{B}_m$, and $\mathcal{C}_m$ to the set of remaining elements $\{\,m-2u+1,\ldots,m\,\}$ respectively, then the restriction $\overline{\mathcal{F}_m} = (\overline{\mathcal{A}_m} \setminus \overline{\mathcal{B}_m}) \cup \overline{\mathcal{C}_m}$ of $\mathcal{F}_m$ is a path



Since paths are non-evasive and can be tested in # vertices – 1 steps, the depth of the leaf corresponding to the above path is again $m-1$. Altogether, it follows that $\mathcal{F}_m$ is non-evasive. □

Chapter 7

Examples of $\mathbb{Z}$-Acyclic and Contractible Vertex-Homogeneous Simplicial Complexes

In the preceding chapter, we saw that there are no (non-trivial) 2- and 3-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes. We will now construct a 5-dimensional example, and further examples in dimension 11, one of which is Oliver's example, the only previously known non-contractible $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex. We also present methods to obtain infinite series of contractible vertex-homogeneous simplicial complexes by starting with one of the $\mathbb{Z}$-acyclic examples.

7.1 Topological Tools

Before we begin with the construction of $\mathbb{Z}$-acyclic and contractible vertex-homogeneous simplicial complexes, we briefly discuss some topological tools that will be needed.

7.1.1 The Nerve Operation

Let K be a (finite abstract) simplicial complex with the collection of facets $\mathcal{F} = (F_j)_{j\in J}$ and corresponding index set J. We call the covering of K by its maximal faces $\mathcal{F}$ the *standard covering* of K.

Theorem 7.1 (Nerve Theorem, Borsuk, cf. [22]) *Let $\mathcal{N}(K)$ be the* ***nerve complex*** *of K (with respect to the standard covering $\mathcal{F} = (F_j)_{j\in J}$), that is, $\mathcal{N}(K)$ is the simplicial complex on the vertex set J such that $\Delta \subseteq J$ is a simplex of $\mathcal{N}(K)$ if and only if $\bigcap_{j\in\Delta} F_j \neq \emptyset$. Then K and $N := \mathcal{N}(K)$ are homotopy equivalent.*

The nerve complex of a simplex of any dimension is a point, and Grünbaum describes in [67] the class of all simplicial complexes having the same nerve. Moreover, by a theorem of Mani (cf. [67]), there is for every simplicial complex N some simplicial complex K such that $N = \mathcal{N}(K)$.

Definition 7.2 *Let E be the vertex set of K, and for every vertex e of K let $e^1, \dots, e^n$ be n distinct copies. The* ***n-th multiple nK*** *of K is the simplicial complex on the vertex set $nE = \bigcup_{r=1}^{n} E^r$, where E^r denotes the r-th copy of E, that has as its facets the sets $nF = \{e_1^1, e_1^2, \dots, e_1^n, \dots, e_k^1, e_k^2, \dots, e_k^n\}$ for the facets $F = \{e_1, \dots, e_k\}$ of K.*

By construction, $\bigcap_{j\in\Delta} F_j \neq \emptyset$ if and only if $\bigcap_{j\in\Delta} nF_j \neq \emptyset$. Hence, $\mathcal{N}(nK) = \mathcal{N}(K)$, and nK is homotopy equivalent to K.

Although the above examples demonstrate that there are always non-isomorphic simplicial complexes, which have the same nerve complex, the nerve operation is injective on a large class of simplicial complexes. Grünbaum [67] calls a simplicial complex *taut* if every vertex is the intersection of the facets containing it.

Lemma 7.3 (Duality, [67]) *If K is taut, then $\mathcal{N}(K)$ is taut and $K = \mathcal{N}(\mathcal{N}(K))$.*

Proof: Let K be taut with standard covering $\mathcal{F} = (F_j)_{j\in J}$. The nerve $N = \mathcal{N}(K)$ has one vertex j for every facet F_j of K, $j \in J$. If $\Delta_e \subseteq J$ is the collection of all j's so that the corresponding facets F_j contain the vertex $e \in E$, then, by the tautness of K, $\bigcap_{j\in\Delta_e} F_j = \{\, e \,\}$ and Δ_e is a facet of N. Hence, to any vertex e of K there uniquely corresponds a facet $\Delta_e = \{\, j : e \in F_j \,\}$ of N.

On the other hand, let $j \in J$. Suppose there exists some $j' \in J$, $j' \neq j$, such that $j' \in \bigcap_{e\in E, j\in\Delta_e} \Delta_e$. But then $F_j \subset F_{j'}$, which is a contradiction to the maximality of F_j. Thus, the nerve N is taut, and $\mathcal{N}(\mathcal{N}(K)) = K$. □

The nerve complex $\mathcal{N}(K)$ of a simplicial complex K need not to be taut. For example, the nerve complex of a path of length m is a path of length $(m-1)$. A point is taut, and m-gons C_m are taut for $m \geq 3$.

7.1.2 The Join and the Dual Join Product

Recall that the join $K * K'$ of two simplicial complexes K and K' (with disjoint vertex sets) is defined as $K * K' := \{\Delta \cup \Delta' \,:\, \Delta \in K, \Delta' \in K'\}$.

Lemma 7.4 *If K and K' are taut simplicial complexes, different from a point, then their join product $K * K'$ is taut as well.*

Proof: The vertex set of $K * K'$ is $E \cup E'$. Let e be a vertex of $K * K'$ with $e \in E$. If $\mathcal{V} = (F_j \cup F'_{j'})_{(j\in J, j'\in J')}$ denotes the collection of maximal faces of $K * K'$ for the facets $\mathcal{F} = (F_j)_{j\in J}$ of K and the facets $\mathcal{F}' = (F'_{j'})_{j'\in J'}$ of K', then

$$\bigcap_{\substack{j\,\in\, J,\, j'\,\in\, J',\\ e\,\in\, F_j\,\cup\, F'_{j'}}} F_j \cup F'_{j'} = \bigcap_{\substack{j\,\in\, J,\\ e\,\in\, F_j}} F_j = \{e\},$$

and the same is true for $e \in E'$. Therefore, $K * K'$ is taut. □

Definition 7.5 *Let K and K' be (finite abstract) simplicial complexes. Then the* ***dual join product*** *of K and K' is the product*

$$K \bowtie K' := \mathcal{N}(\,\mathcal{N}(K) * \mathcal{N}(K')\,). \tag{7.1}$$

The dual join product of two (non-trivial) taut complexes K and K' is taut by Lemma 7.3 and Lemma 7.4. In particular, the following equality holds for (non-trivial) taut complexes,

$$\mathcal{N}(K \bowtie K') = \mathcal{N}(K) * \mathcal{N}(K'). \tag{7.2}$$

7.2 Vertex-Homogeneous Simplicial Complexes

7.2.1 The Nerve of a Vertex-Homogeneous Simplicial Complex

Let G be a permutation group of the vertex set E, and let $K \subseteq 2^E$ be a simplicial complex which is invariant under the given G-action.

Lemma 7.6 *The action of G on K induces an action of G on the nerve $\mathcal{N}(K)$ of K.*

Proof: Let $\mathcal{F} = (F_j)_{j \in J}$ be the standard covering of K. The action of G on the set J of facets of K gives rise to an action of G on the nerve $\mathcal{N}(K)$, since $\bigcap_{j \in \Delta} gF_j \neq \emptyset$ if and only if $\bigcap_{j \in \Delta} F_j \neq \emptyset$ for any $\Delta \in \mathcal{N}(K)$. □

Let, from now on, K be a vertex-homogeneous simplicial complex. Then the induced action of G on the nerve N of K is, in general, not vertex-homogeneous anymore. More precisely, the action of G on N is transitive on the set of vertices of N if and only if K has exactly one orbit of maximal faces.

Lemma 7.7 (Characterization of vertex-homogeneous simplicial complexes)

(i) *If K is vertex-homogeneous, then its nerve $N = \mathcal{N}(K)$ is facet-homogeneous.*

(ii) *If N is facet-homogeneous, then its nerve $K = \mathcal{N}(N)$ is vertex-homogeneous.*

Proof: (i) Let K be a vertex-homogeneous simplicial complex on m vertices.

We first show that $N = \mathcal{N}(K)$ is pure. Let $\operatorname{Max}(K)$ denote the collection of orbits of maximal faces of K. By transitivity, every vertex $e \in E$ is contained the same number of times, $r_{\mathcal{O}}$, in the k-element sets of any particular orbit $\mathcal{O}$ of facets of K, i.e., $k \cdot |\mathcal{O}| = r_{\mathcal{O}} \cdot m$. Altogether, every vertex e is contained in precisely $r = \sum_{\mathcal{O} \in \operatorname{Max}(K)} r_{\mathcal{O}}$ distinct facets of K, i.e., $\dim(\Delta_e) = r - 1$ for every e. In particular, $\dim(\mathcal{N}(K)) = r - 1$.

Let Δ and Δ' be two different facets of N. Then Δ and Δ' correspond to two distinct sets of r maximal faces of K respectively. Let e be some element in the intersection of the r maximal faces of K corresponding to Δ (there can be more than one such element!), and let e' be an element of E representing Δ'. Since the action of G is transitive on E, there exists a group element $g \in G$ such that $e' = g * e$. But then g maps the facets corresponding to e to the facets corresponding to e', and hence the action of G on N is transitive on the facets of N.

(ii) Trivial, since any facet-homogeneous simplicial complex has only one orbit of maximal faces. □

Example 1: The n-gon C_n, $n \geq 3$, with rotations by elements of $\mathbb{Z}_n$ is a taut vertex-homogeneous and facet-homogeneous simplicial complex with $C_n = \mathcal{N}(C_n)$.

If we replace every edge of the circle C_n by an m-simplex, $m \geq 2$, then the resulting (n, m)-necklace C_n^m is a facet-homogeneous pure simplicial complex with $C_n = \mathcal{N}(C_n^m)$. Thus, (n, m)-necklaces form an infinite class of facet-homogeneous simplicial complexes, which all have the same nerve.

EXAMPLE 2: For $n \geq 3$, the 2-fold multiple $2\,C_n$ is a chain of tetrahedra:

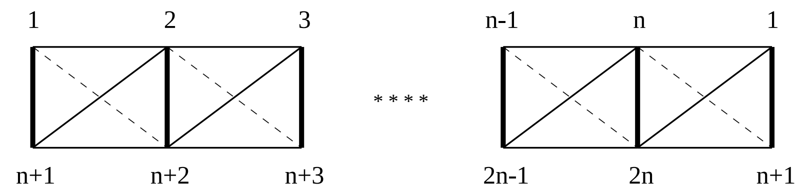


Figure 7.1: The 2-fold multiple $2\,C_n$.

With respect to the action of $\mathbb{Z}_n \times \mathbb{Z}_2$, which rotates the tetrahedra and flips the upper and lower vertices, the 2-fold multiple $2\,C_n$ is vertex-homogeneous and facet-homogeneous, with $C_n = \mathcal{N}(2\,C_n)$.

EXAMPLE 3: The 6-vertex triangulation $\mathbb{R}\mathbf{P}^2_6$ of the real projective plane is vertex-homogeneous and taut.

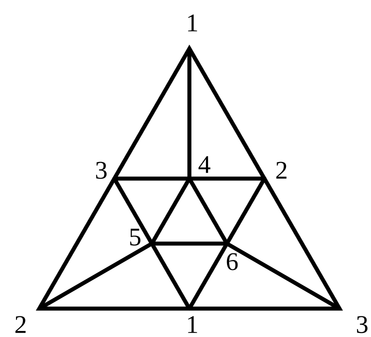


Figure 7.2: The 6-vertex triangulation of the real projective plane.

The symmetry group of $\mathbb{R}\mathbf{P}^2_6$ is $A_5(6) = \langle\,(1,2,3,4,6),(1,4)(5,6)\,\rangle$, with $A_5(6)$ acting transitively on the set of maximal faces:

1	$\{1,2,4\}$,	2	$\{1,2,5\}$,	3	$\{1,3,4\}$,	4	$\{1,3,6\}$,	5	$\{1,5,6\}$,
6	$\{2,3,5\}$,	7	$\{2,3,6\}$,	8	$\{2,4,6\}$,	9	$\{3,4,5\}$,	10	$\{4,5,6\}$.

The nerve of $\mathbb{R}\mathbf{P}^2_6$ is a taut 4-dimensional vertex-homogeneous and facet-homogeneous simplicial complex on 10 vertices:

1	$\{1,2,3,4,5\}$,	2	$\{1,2,6,7,8\}$,	3	$\{3,4,6,7,9\}$,
4	$\{1,3,8,9,10\}$,	5	$\{2,5,6,9,10\}$,	6	$\{4,5,7,8,10\}$.

If we compute the nerve of this complex, we get $\mathbb{R}\mathbf{P}^2_6$ again.

REMARK: We see by this example that the nerve of a vertex-homogeneous complex can have more vertices and can be of higher dimension than the original complex. On the other hand, it also can have less vertices and can be of lower dimension.

Example 4: $\mathbb{Q}$-Acyclic Vertex-Homogeneous Simplicial Complexes

We encountered the 6-vertex triangulation of the real projective plane in Chapter 6 before. It turned out that it is the smallest (non-trivial) example of a $\mathbb{Q}$-acyclic vertex-homogeneous simplicial complex. We will make use of Lemma 7.7 to derive further examples of vertex-homogeneous as well as facet-homogeneous $\mathbb{Q}$-acyclic simplicial complexes, which are homotopy equivalent to $\mathbb{R}\mathbf{P}^2$.

Besides that it is vertex-homogeneous, the 6-vertex triangulation of the projective plane **Aa** (cf. Figure 7.3) is facet-homogeneous as was mentioned above. Hence, the nerve complex **aA** of **Aa** (in Figure 7.3, vertex-homogeneous and facet-homogeneous simplicial complexes are labeled by capital letters and small letters respectively) is a facet-homogeneous and vertex-homogeneous simplicial complex on 10 vertices with the shaded pentagons depicting the six 4-dimensional facets glued together along the sides of the pentagons. Alternatively, this complex can be described as an ideal of the weighted A_5-poset (cf. Figure 6.5) on 10 vertices, with one orbit of six 4-dimensional facets.

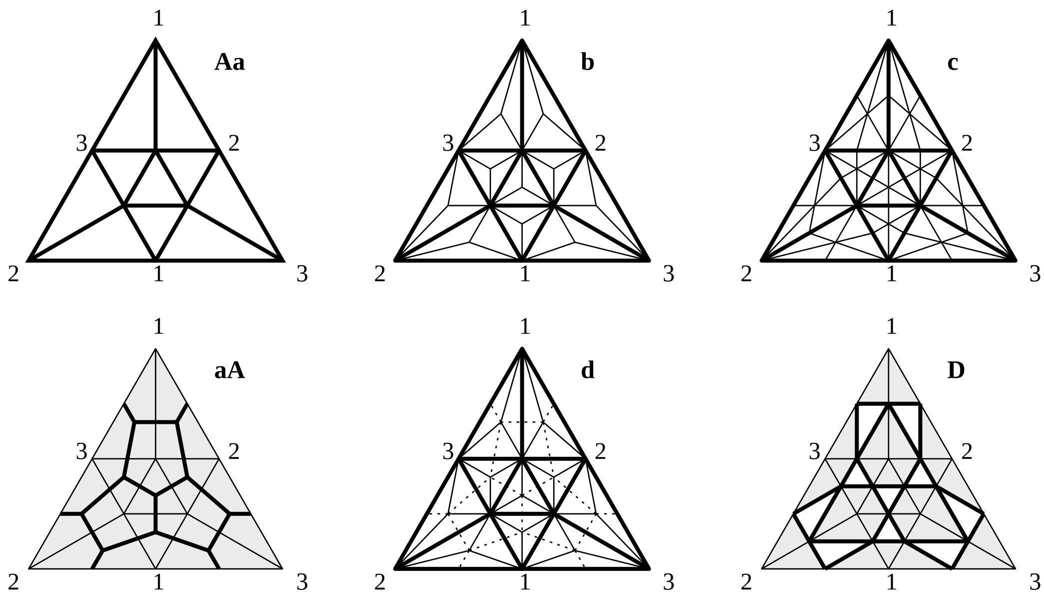


Figure 7.3: Vertex-/facet-homogeneous simplicial complexes homotopy equivalent to $\mathbb{R}\mathbf{P}^2$.

The nerve complex **B** (**C**) of the facet-homogeneous subdivision **b** (of the facet-homogeneous barycentric subdivision **c**) of **Aa** is vertex-homogeneous on 30 (60) vertices.

If we replace in **b** for any bold edge both neighboring triangles by a tetrahedron, then the resulting simplicial complex **d** is still facet-homogeneous. Its nerve complex **D** is on 15 vertices with two orbits of maximal faces, one consisting of six 4-simplices (shaded pentagons) and the other of 10 triangles. The same construction can be carried through for **c** leading to complexes **e** and **E** (the latter on 30 vertices).

A further example **F** of a vertex-homogeneous simplicial complex on 15 vertices, homotopy equivalent to the projective plane, can be obtained from **D** by gluing in 60 tetrahedra in the following way. For any white triangle of **D** we add six tetrahedra with vertex-sets the triangle and in addition one vertex of a neighboring white triangle respectively. The resulting space **F** is still vertex-homogeneous, and it can be worked out easily that it collapses to **D** and thus is homotopy equivalent to **D**.

7.2.2 Constructions with Vertex-Homogeneous Simplicial Complexes

After we now have seen some simple examples of vertex-homogeneous simplicial complexes, we will next discuss three constructions that allow us to derive further vertex-homogeneous simplicial complexes if we start with a given one.

Proposition 7.8 *Let (K, G) denote a pair of a simplicial complex K with vertex set E of cardinality m and a group $G < S_m$ that acts vertex-transitively on K. If F is a finite set with $n = |F|$ elements and $H < S_n$ is a transitive permutation group of degree n, which acts on F, then the simplicial complexes*

(i) $(K^{*n}, G \times H)$ (Oliver, cf. [80])

(ii) $(K \bowtie F, G \times H)$

(iii) $(nK, G \times H)$

are vertex-homogeneous for the obvious actions of $G \times H$.

Proof: (i) Let the direct product $G \times H$ act on the n-fold join product K^{*n}, with G acting transitively on every copy of K and with H permuting the n copies of K. The vertex set of K^{*n} is the union $E_{\cup} = \bigcup_{r=1}^{n} E^r$ of n copies of E, and the action of $G \times H$ is clearly transitive on $E_{\cup}$.

(ii) Since (K, G) and (F, H) are vertex-homogeneous complexes, their nerve complexes $(\mathcal{N}(K), G)$ and $(\mathcal{N}(F), H)$ are facet-homogeneous, with $\mathcal{N}(F) = F$. The join product $\mathcal{N}(K) * F$ is facet-homogeneous for the diagonal action of $G \times H$, and thus $(K \bowtie F, G \times H)$ is vertex-homogeneous.

(iii) As in (i), $G \times H$ acts transitively on $E_{\cup} = \bigcup_{r=1}^{n} E^r$. □

Corollary 7.9 *If K is a $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex, then the n-fold multiples nK form an infinite series of $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes. Moreover, the series K^{*n} and $K \bowtie F$ provide examples of contractible vertex-homogeneous simplicial complexes.*

Proof: It remains to show that K^{*n} and $K \bowtie F$ are contractible for $n \geq 2$. If K is $\mathbb{Z}$-acyclic, then it is connected. Now, the join product of a k-connected complex with an l-connected complex is $(k + l + 2)$-connected. In particular, K^{*n} and $K \bowtie F$ are at least $(0 - 1 + 2)$-connected, that is, simply connected. But since a simplicial complex is contractible if and only if it is simply connected and $\mathbb{Z}$-acyclic (cf. [22]), the result follows. □

7.3 The Identified Dodecahedron and Seven Related $\mathbb{Z}$-Acyclic Vertex-Homogeneous Simplicial Complexes

Let us consider the boundary complex of the dodecahedron with 12 pentagonal facets, 30 edges, and 20 vertices. If we identify opposite pentagons by a coherent twist of $\pi/5$ radians, then the resulting cell complex Q is $\mathbb{Z}$-acyclic (see [36, p. 57] and Chapter 1). The symmetry group of the *identified dodecahedron* Q is the alternating group A_5.

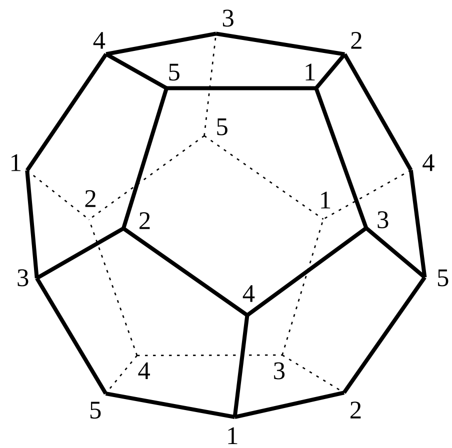


Figure 7.4: The $\mathbb{Z}$-acyclic identified dodecahedron.

Lemma 7.10 *There are exactly two A_5-invariant facet-transitive triangulations of the $\mathbb{Z}$-acyclic complex Q.*

Proof: Every edge of the cell complex Q is the intersection of *three* pentagonal cells, which implies that the 1-skeleton of Q is necessarily included in the 1-skeleton of any A_5-invariant triangulation of Q. This is also the case for the five vertices of Q. The action of A_5 on Q is transitive on the pentagons, and any of the pentagons has the dihedral group $D_5 < A_5$ as its isotropy group. It is therefore sufficient to determine D_5-invariant facet-transitive triangulations of a pentagon. There are exactly two such triangulations, one with 5 and the other with 10 triangles. We denote the corresponding triangulations of Q with 30 and 60 triangles by N_I and N_O respectively (see Figures 7.6 and 7.5). □

Corollary 7.11 *The nerve complexes $K_1 := \mathcal{N}(N_I)$ and $K_O := \mathcal{N}(N_O)$ are examples of $\mathbb{Z}$-acyclic but not contractible vertex-homogeneous simplicial complexes on* 30 *and* 60 *vertices respectively.*

Oliver's Example K_O

The complex K_O was first found by Oliver. His construction is algebraic and was mentioned in [80] and in a paper by Segev [147]. In fact, Segev presented an explicit proof that the nerve complex $N_O = \mathcal{N}(K_O)$, and hence that K_O, is $\mathbb{Z}$-acyclic. Moreover, it was conjectured in [147] that $\mathcal{N}(K_O)$ is homeomorphic to Q. We will show that this is indeed the case.

Let A_5 be the alternating group of even permutations of the set $\{1, 2, 3, 4, 5\}$. Define the subgroups

$$\begin{array}{lclcl} U & := & \{\, g \in A_5 : g \cdot 2 = 2\,\} & \cong & A_4, \\ V & := & N_{A_5}(\,\langle\,(1,2,3,4,5)\,\rangle\,) & \cong & D_5, \\ W & := & N_{A_5}(\,\langle\,(1,3,5)\,\rangle\,) & \cong & D_3, \end{array}$$

where $N_{A_5}(H)$ denotes the normalizer in A_5 of a subgroup H of A_5. The stabilizer U of the point 2 is isomorphic to the alternating group A_4 and has 12 elements. The subgroups V and W are isomorphic to the dihedral groups D_5 and D_3 with 10 and 6 elements respectively.

OLIVER takes as vertex set E for the simplicial complex K_O the 60 elements of A_5 and lets A_5 act transitively on E by left multiplication. He defines K_O to be the simplicial complex that has the left cosets of U, V, and W as its (orbits of) maximal faces:

$$K_O := \bigcup_{g \in A_5} 2^{\,g \cdot U} \cup \bigcup_{g \in A_5} 2^{\,g \cdot V} \cup \bigcup_{g \in A_5} 2^{\,g \cdot W}.$$

Theorem 7.12 (OLIVER, cf. [80]) *The 11-dimensional simplicial complex K_O is vertex-homogeneous and ℤ-acyclic.*

Proof: By construction, K_O is 11-dimensional and vertex-homogeneous. To see that K_O is ℤ-acyclic, we compute the nerve $\mathcal{N}(K_O)$ of K_O. As maximal faces of the nerve we get 60 triangles. By a suitable labeling of the vertices, $\mathcal{N}(K_O)$ turns out to be the triangulation N_O of the ℤ-acyclic identified dodecahedron Q (see Figure 7.5). Thus K_O is ℤ-acyclic by the Nerve Theorem 7.1. □

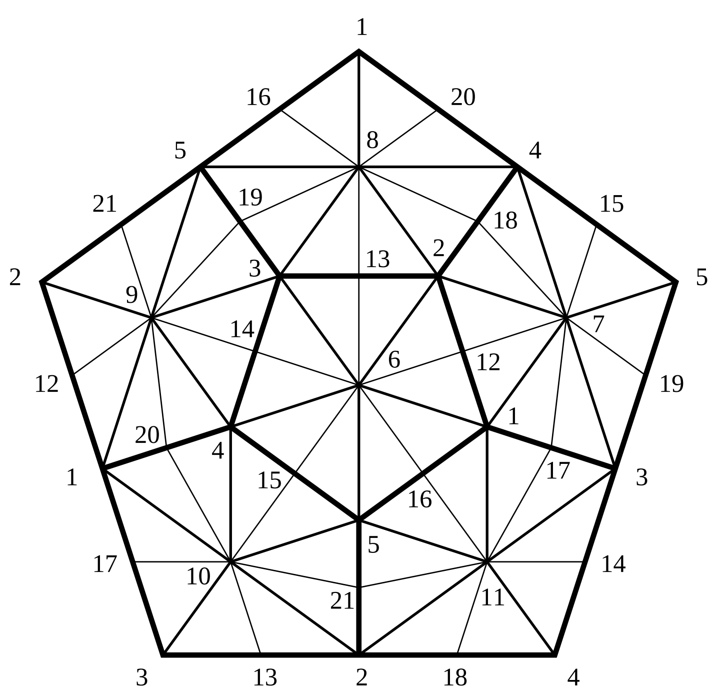


Figure 7.5: Triangulation N_O of the identified dodecahedron Q with 60 triangles.

New $\mathbb{Z}$-Acyclic Vertex-Homogeneous Simplicial Complexes

In Chapter 6, we enumerated all vertex-homogeneous simplicial complexes with reduced Euler characteristic $\tilde{\chi} = 0$ corresponding to a given group action on few vertices. For the A_5-action on 60 vertices it is hopeless to generate all vertex-homogeneous simplicial complexes with $\tilde{\chi} = 0$ and then compute their homology in order to find $\mathbb{Z}$-acyclic examples. But if we restrict our computer search to complexes that have only few orbits of maximal faces with orbit size less than 30, then, in particular, we obtain the above example K_O. Recall that it follows from Proposition 6.4 that an A_5-orbit of k-sets on 60 vertices can have size less than 30 if and only if $\gcd(k, 60) > 2$. We formed combinations of at most six orbits with at most two orbits of maximal faces of the same dimension. For every simplicial complex K corresponding to one of these collections of orbits of facets, we computed the reduced Euler characteristic $\tilde{\chi}(\mathcal{N}(K))$ of the nerve complex of K. Whenever $\tilde{\chi}$ was zero, we computed the homology of $\mathcal{N}(K)$ with the program HOMOLOGY by Heckenbach [71]. Including K_O, we found five $\mathbb{Z}$-acyclic A_5-invariant complexes on 60 vertices that we denote by K_O, K_2, K_4, K_5, and K_6. The examples K_2 and K_4 are not taut, and it turns out that $K_1 := \mathcal{N}(\mathcal{N}(K_2))$ and $K_3 := \mathcal{N}(\mathcal{N}(K_4))$ are taut A_5-invariant $\mathbb{Z}$-acyclic simplicial complexes on 30 vertices. We believe that if we extended our search, then further complexes would appear.

Theorem 7.13 *There are at least seven non-contractible $\mathbb{Z}$-acyclic simplicial complexes with a vertex-transitive A_5-action that are homotopy equivalent to the identified dodecahedron Q.*

Table 7.1 gives an overview of the examples. All seven complexes can be characterized algebraicly, and this we will do for K_1 to K_6 in the following. Moreover, we give geometric descriptions of the corresponding facet-homogeneous nerve complexes N_I to N_{IV}.

Complex	# vertices	dim
$K_O = \mathcal{N}(N_O)$	60	11
$K_1 = \mathcal{N}(N_I)$	30	11
$K_2 = 2\,K_1$	60	23
$K_3 = \mathcal{N}(N_{II})$	30	5
$K_4 = 2\,K_3$	60	11
$K_5 = \mathcal{N}(N_{III})$	60	11
$K_6 = \mathcal{N}(N_{IV})$	60	11

Table 7.1: $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes with A_5-action.

Remark: Although the examples K_O and K_1 to K_6 are not contractible, by Proposition 7.8 there exist infinite series of contractible vertex-homogeneous simplicial complexes associated with K_O and K_1 to K_6.

The $\mathbb{Z}$-Acyclic Complexes K_1 and K_2

Consider the subgroups of A_5,

$$\begin{array}{rcccl} U & := & \{\, g \in A_5 : g \cdot 2 = 2\,\} & \cong & A_4, \\ V & := & N_{A_5}(\langle\,(1,2,3,4,5)\,\rangle) & \cong & D_5. \end{array}$$

Then the 24-element set

$$A \quad := \quad U \cup U \cdot (2,5,3)$$

determines an A_5-orbit of size 5. Define

$$K_2 := \bigcup_{g \in A_5} 2^{\,g \cdot A} \cup \bigcup_{g \in A_5} 2^{\,g \cdot V},$$

and

$$K_1 := \mathcal{N}(\mathcal{N}(K_2)).$$

Theorem 7.14 *The examples K_1 and K_2 are $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes on* 30 *and* 60 *vertices respectively, with $K_2 = 2K_1$.*

Proof: The nerve complex $N_I = \mathcal{N}(K_1) = \mathcal{N}(K_2)$ of K_1 and K_2 is the facet-homogeneous triangulation of the identified dodecahedron Q with 30 triangles (see Figure 7.6). $\square$

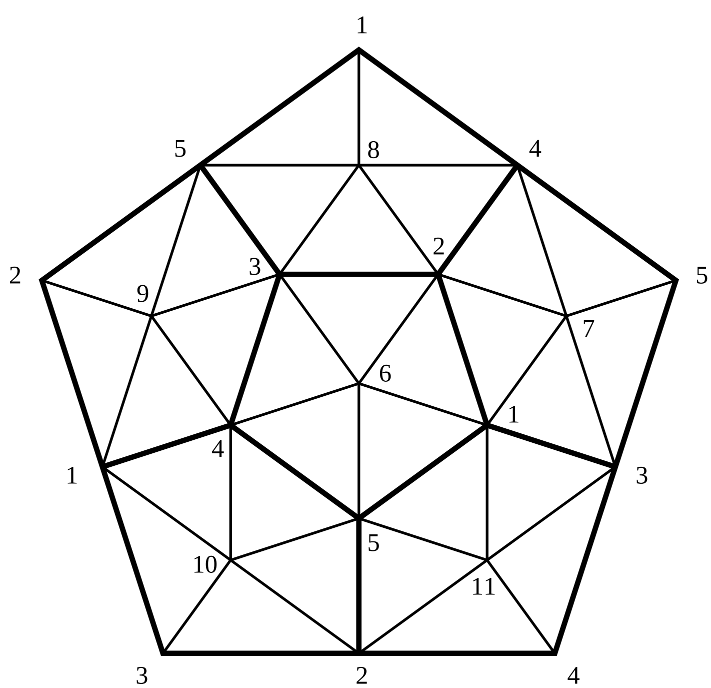


Figure 7.6: Triangulation N_I of the identified dodecahedron Q with 30 triangles.

The $\mathbb{Z}$-Acyclic Complexes K_3 and K_4

Take the subgroups of A_5,

$$\begin{array}{lcl} U & := & \{\, g \in A_5 : g \cdot 2 = 2 \,\} & \cong & A_4, \\ V & := & N_{A_5}(\langle (1,2,3,4,5) \rangle) & \cong & D_5, \\ W & := & N_{A_5}(\langle (1,3,5) \rangle) & \cong & D_3, \end{array}$$

and consider the 12-element set

$$B \;:=\; W \cup W \cdot (3,4,5).$$

Define

$$K_4 := \bigcup_{g \in A_5} 2^{\,g \cdot U} \cup \bigcup_{g \in A_5} 2^{\,g \cdot B} \cup \bigcup_{g \in A_5} 2^{\,g \cdot V},$$

and set

$$K_3 := \mathcal{N}(\mathcal{N}(K_4)).$$

The nerve $N_{II} = \mathcal{N}(K_3) = \mathcal{N}(K_4)$ of K_3 and K_4 is a 3-dimensional facet-homogeneous simplicial complex with 30 tetrahedra (see Figure 7.7). For every pentagon of N_O, 5 tetrahedra are glued in as indicated by the dashed lines. Since N_{II} collapses to N_O, the complex N_{II} is homotopy equivalent to Q.

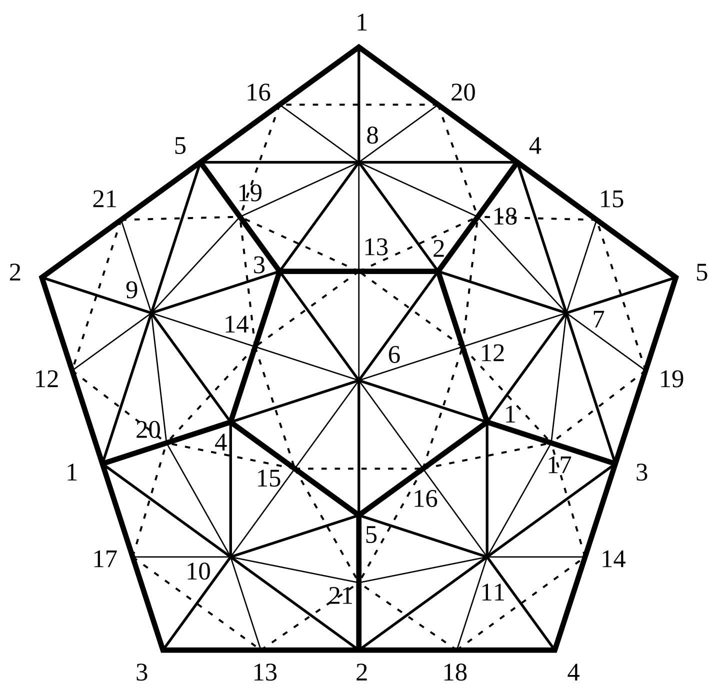


Figure 7.7: Triangulation N_{II} with 30 tetrahedra replacing the 60 triangles.

Theorem 7.15 *The example K_3 provides a 5-dimensional non-contractible $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex on 30 vertices.*

We saw in Chapter 6 that there are no (non-trivial) 2- and 3-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes, and that if there were a 4-dimensional example, then it would have 15, 20, 30, or 60 vertices. Our attempts failed to find a 4-dimensional example.

Conjecture 7.16 *The complex K_3 with f-vector $f = (1, 30, 195, 340, 255, 96, 15)$ is the smallest example of a non-contractible $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex, with respect to dimension, the number of vertices, and the total number of faces. The join $K_3 * K_3$ of dimension 11 with 60 vertices is, apart from a simplex, the smallest contractible vertex-homogeneous simplicial complexes.*

The $\mathbb{Z}$-Acyclic Complexes K_5 and K_6

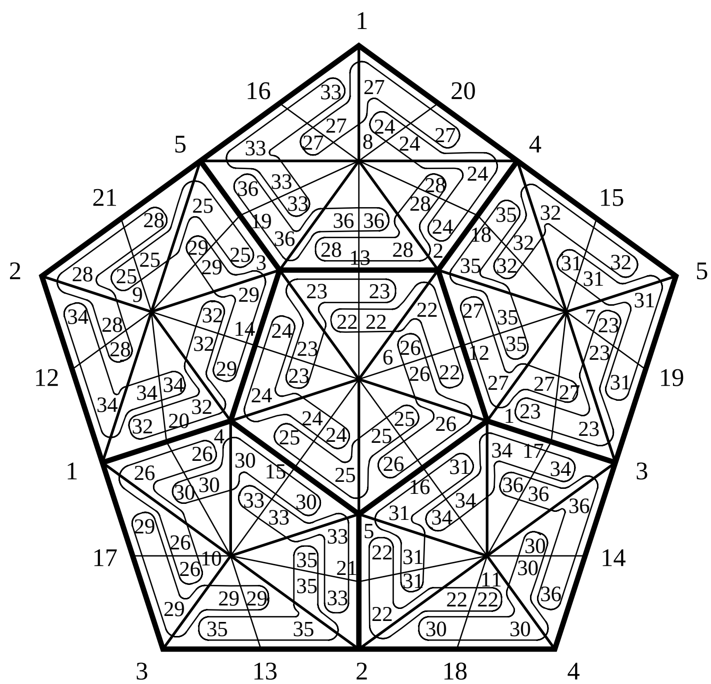


Figure 7.8: Triangulation N_{III} with 60 4-simplices.

Let U, V, W, and R be subgroups of A_5 with

$$\begin{array}{rcllcl} U & := & \{\, g \in A_5 : g \cdot 2 = 2 \,\} & \cong & A_4, \\ V & := & N_{A_5}(\langle (1,2,3,4,5) \rangle) & \cong & D_5, \\ W & := & N_{A_5}(\langle (1,3,5) \rangle) & \cong & D_3, \\ R & := & \langle (1,2)(3,5), (1,3)(2,5) \rangle & \cong & \mathbb{Z}_2 \times \mathbb{Z}_2, \end{array}$$

and consider the 8-element set

$$C \quad := \quad R \cup R \cdot (2,3,4).$$

Define

$$K_5 := \bigcup_{g \in A_5} 2^{\,g \cdot U} \cup \bigcup_{g \in A_5} 2^{\,g \cdot V} \cup \bigcup_{g \in A_5} 2^{\,g \cdot C} \cup \bigcup_{g \in A_5} 2^{\,g \cdot W}.$$

The nerve $N_{III} = \mathcal{N}(K_5)$, composed of 60 4-simplices, is a facet-homogeneous complex homotopy equivalent to Q. Figure 7.8 gives an illustration of N_{III}. To every of the 60 triangles of the triangulation N_O of Q there uniquely corresponds a 4-simplex that has as vertices the three vertices of the triangle and in addition the two vertices that are placed within the triangle. It can easily be verified that N_{III} collapses to Q.

Let once more U, V, W, and S be subgroups of A_5 with

$$\begin{array}{rcllcl} U & := & \{\, g \in A_5 : g \cdot 2 = 2 \,\} & \cong & A_4, \\ V & := & N_{A_5}(\langle (1,2,3,4,5) \rangle) & \cong & D_5, \\ W & := & N_{A_5}(\langle (1,3,5) \rangle) & \cong & D_3, \\ S & := & \langle (1,3)(4,5), (1,4)(3,5) \rangle & \cong & \mathbb{Z}_2 \times \mathbb{Z}_2, \end{array}$$

and consider the 8-element set

$$D \;:=\; S \cup S \cdot (2,3,5).$$

Define

$$K_6 := \bigcup_{g \in A_5} 2^{\,g \cdot U} \cup \bigcup_{g \in A_5} 2^{\,g \cdot V} \cup \bigcup_{g \in A_5} 2^{\,g \cdot D} \cup \bigcup_{g \in A_5} 2^{\,g \cdot W}.$$

The nerve complex $N_{IV} = \mathcal{N}(K_6)$ is again 4-dimensional but combinatorially distinct from N_{III}, and provides another example of a facet-homogeneous simplicial complex homotopy equivalent to Q. The 60 4-simplices of N_{IV} are drawn in Figure 7.9.

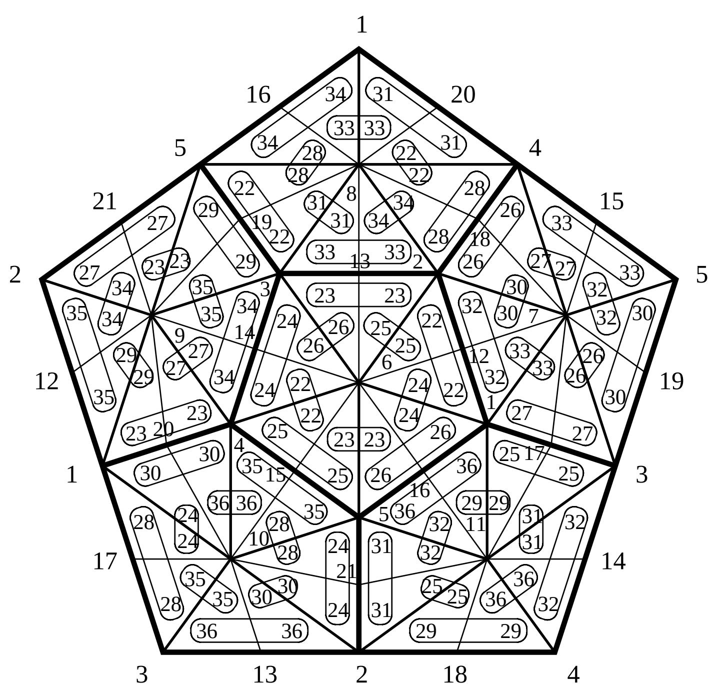


Figure 7.9: Triangulation N_{IV} with 60 4-simplices.

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Zusammenfassung

In den Anfangszeiten der Topologie wurden Mannigfaltigkeiten vielfach anhand ihrer Triangulierungen untersucht. Insbesondere die Berechnung von Invarianten nutzte die zugrundeliegende kombinatorische Struktur, während hierfür später zunehmend algebraische Methoden Einzug fanden. Obgleich es sich herausstellte, daß nicht jede Mannigfaltigkeit notwendigerweise triangulierbar ist, ist seit dem Aufkommen von Computern das Interesse an kombinatorischen Aspekten von Mannigfaltigkeiten und ihren Triangulierungen erneut gewachsen. Nötig für viele Untersuchungen ist dabei, auf Triangulierungen von handhabbarem Format zugreifen zu können.

In dieser Arbeit werden zwei Methoden vorgestellt, die es erlauben, Triangulierungen von Mannigfaltigkeiten mit wenigen Ecken zu gewinnen:

- Mit Hilfe von sogenannten *bistellaren Operationen* lassen sich Triangulierungen von Mannigfaltigkeiten lokal, unter Erhaltung des PL-Homöomorphietyps, modifizieren. Betrachtet man die Summe aller Seiten einer Triangulierung als zu minimierende Zielfunktion, dann erlaubt der Einsatz von Simulated annealing, lokale und sogar globale Minima der Zielfunktion aufzusuchen. Auf diese Weise ließen sich minimale Triangulierungen der Mannigfaltigkeiten $S^2 \times S^2$ mit 11, $S^3 \times S^2$ mit 12, $S^3 \times S^3$ mit 13, $(S^2 \times S^2)\#(S^2 \times S^2)$ mit 12 und $\mathbb{R}\mathbf{P}^4$ mit 16 Ecken gewinnen.
- Eine systematischere Herangehensweise ist, alle triangulierten Mannigfaltigkeiten mit bestimmten Eigenschaften für festgelegte Parameterbereiche vollständig zu enumerieren. Dies wurde beispielsweise für eckentransitive Triangulierungen mit bis zu 13 Ecken durchgeführt, wobei viele bekannte, aber auch einige interessante neue Beispiele gefunden wurden. Es zeigte sich zudem, daß es eckentransitive, minimale Triangulierungen von $S^2 \times S^2$, $S^3 \times S^2$ und $S^3 \times S^3$ nicht gibt.

Für beide angeführten Methoden lassen sich schnell weitere Anwendungen finden. So kann auf bistellaren Flips basierendes Simulated annealing als Heuristik dazu verwendet werden, den Homöomorphietyp einer Mannigfaltigkeit zu bestimmen. Hierzu werden auf ein Testobjekt so lange bistellare Operationen angewandt, bis es letztendlich kombinatorisch isomorph zu einer bekannten "Referenztriangulierung" einer Mannigfaltigkeit mit übereinstimmenden Invarianten ist. Dieses Vorgehen konnte in der Arbeit mit großem Erfolg eingesetzt werden.

Neben Mannigfaltigkeiten lassen sich die enumerativen Verfahren auch generell für Simplizialkomplexe mit speziellen Merkmalen anwenden. Gesucht und gefunden wurden z.B. nicht-kontrahierbare, $\mathbb{Z}$-azyklische Simplizialkomplexe mit einer eckentransitiven Gruppenwirkung; das kleinste solche Beispiel in Dimension 5 auf 30 Ecken.

Insgesamt konnten in der vorliegenden Arbeit eine Vielzahl zusätzlicher Ergebnisse erzielt und weitere nützliche "Werkzeuge" und "Tricks" für die Konstruktion von triangulierten Mannigfaltigkeiten entwickelt werden.

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