SIAM J. DISCRETE MATH.
Vol. 18, No. 1, pp. 103–109

SMALL EXAMPLES OF NONCONSTRUCTIBLE SIMPLICIAL BALLS AND SPHERES[*]

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Abstract. We construct nonconstructible simplicial d-spheres with $d+10$ vertices and nonconstructible, nonrealizable simplicial d-balls with $d+9$ vertices for $d \geq 3$.

Key words. simplicial balls and spheres, constructibility, shellability, vertex-decomposability, knots in 3-balls and 3-spheres

AMS subject classifications. 57Q15, 52B05, 52B22, 57M25

DOI. 10.1137/S0895480103430521

1. Introduction.

The concepts of *vertex-decomposability*, *shellability*, and *constructibility* describe three particular ways to assemble a simplicial complex from the collection of its facets (cf. Björner [5]). The following implications are strict for (pure) simplicial complexes:

$$\text{vertex decomposable} \implies \text{shellable} \implies \text{constructible}.$$

Shellability has its origin in Schläfli's computation from 1852 [33] of the Euler characteristics of convex polytopes, where he based his calculation on the assumption that the boundary complexes of polytopes are shellable. However, this property of polytopes was justified only much later in 1970 by Bruggesser and Mani [9] and then played a crucial role in McMullen's proof of the upper bound theorem in the same year [28]. Besides in polyhedral theory, shellability has found fruitful applications in topology, combinatorics, and computational geometry; see the surveys [4], [5], [12], [35, Ch. 8], [36], and the references contained therein.

The notion of constructibility was coined by Hochster in 1972 [19] but implicitly was used long before in combinatorial topology. In particular, it follows from Newman's and Alexander's fundamental works on the foundations of combinatorial and piecewise linear (PL) topology from 1926 [29] and 1930 [1] (cf. also Björner [5]) that a constructible d-dimensional simplicial complex in which every $(d-1)$-face is contained in exactly two or at most two d-dimensional facets is a PL d-sphere or a PL d-ball, respectively. For recent surveys on constructibility see [17] and [18].

The strongest concept, vertex-decomposability, was introduced by Provan and Billera in their proof from 1980 [31] that vertex decomposable simplicial complexes satisfy the simplicial form of the famous Hirsch conjecture (cf. [13, p. 168]) of linear programming.

Although boundary spheres of simplicial polytopes are shellable, Lockeberg [24] constructed a simplicial 4-polytope with 12 vertices which is not vertex-decomposable; and there even are not vertex-decomposable simplicial 4-polytopes with 10 vertices [21] and not vertex-decomposable, nonpolytopal simplicial 3-spheres with 9 vertices [8]. For two-dimensional balls and spheres it was proved by Bing [4] that they are shellable

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[*] Received by the editors June 26, 2003; accepted for publication (in revised form) February 10, 2004; published electronically July 20, 2004.
http://www.siam.org/journals/sidma/18-1/43052.html

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and by Provan and Billera [31] that they are vertex-decomposable. Klee and Kleinschmidt [21] also showed that all simplicial d-balls and all simplicial d-spheres with up to $d+3$, respectively, $d+4$ vertices, are vertex-decomposable. However, for $d \geq 3$ there are not vertex-decomposable simplicial d-balls with $d+4$ vertices and 10 facets as well as not vertex-decomposable simplicial d-spheres with $d+6$ vertices; see [8] and [27].

The first known example of a nonshellable cellular 3-ball is due to Furch and appeared in 1924 [15]. A nonshellable simplicial 3-ball with 30 vertices and 72 facets was provided by Newman in 1926 [30]. Newman's ball is *strongly nonshellable*; i.e., it has no *free* facet that can be removed from the triangulation without losing ballness. Much smaller strongly nonshellable simplicial 3-balls were obtained by Grünbaum (cf. [12]) with 14 vertices and 29 facets and by Ziegler [36] with 10 vertices and 21 facets. Rudin's 3-ball [32] with 14 vertices and 41 tetrahedra gives a strongly nonshellable rectilinear triangulation of a tetrahedron with all the vertices on the boundary; the vertices even can be moved slightly to yield a straight triangulation of a convex 3-polytope with 14 vertices [11]. Ziegler's ball is realizable as a straight yet nonconvex ball in 3-space. Coordinates for a rectilinear realization of Grünbaum's ball can be found in [17]. Vertex-minimal nonshellable 3-balls with 9 vertices are enumerated in [8]; see [26] for a geometric realization of one of these balls with 18 facets.

The existence of nonconstructible 3-balls was shown by Lickorish [22] in 1971, but it remained unclear whether there are nonshellable 3-spheres. Nonshellable cell partitions of S^3 were first constructed by Vince [34] in 1985 and then by Armentrout [3]. In 1991, Lickorish [23] described nonshellable triangulated 3-spheres that contain a knotted triangle made of the sum of (at least) three trefoil knots.

In fact, it suffices to use one single trefoil knot.

Theorem 1 (Hachimori and Ziegler [18]). *If a triangulated* 3*-ball or* 3*-sphere contains* any *knotted triangle, then it is nonconstructible (and thus nonshellable). Moreover, a* 3*-ball with a knotted spanning arc consisting of at most* 2 *edges is nonconstructible.*

A first explicit, but large, nonconstructible triangulated 3-sphere with f-vector $f = (381, 2309, 3856, 1928)$ based on Furch's 3-ball with a knotted spanning arc consisting of one edge was constructed by Hachimori [16]. Suspensions of such spheres produce nonconstructible simplicial PL d-spheres in dimensions $d \geq 3$. Examples of small non-PL (and hence nonconstructible) d-spheres of dimensions $d \geq 5$ with $d+13$ vertices can be found in [6]; see also [7]. Their construction makes use of the double suspension theorem of Edwards [14] (respectively, of its generalization by Cannon [10]) that double suspensions of nonspherical homology d-spheres give non-PL $(d+2)$-spheres.

2. The examples.

In the following, we employ the theorem of Hachimori and Ziegler to construct simplicial PL d-spheres in dimensions $d \geq 3$ with only $d+10$ vertices that are nonconstructible. From the enumeration in [8] it follows that all 3-spheres with $n \leq 10$ vertices are shellable. Hence, the nonconstructible 3-sphere $S^3_{13,56}$ with 13 vertices that we are going to obtain is, if not vertex-minimal, then close to vertex-minimality.

Theorem 2. *There is a nonconstructible* 3*-sphere* $S^3_{13,56}$ *with* 13 *vertices and* 56 *facets. Moreover, there are two strongly nonshellable, nonconstructible* 3*-balls* $B^3_{12,37,a}$ *and* $B^3_{12,37,b}$ *with* 12 *vertices and* 37 *facets that cannot be rectilinearly embedded into* $\mathbb{R}^3$.

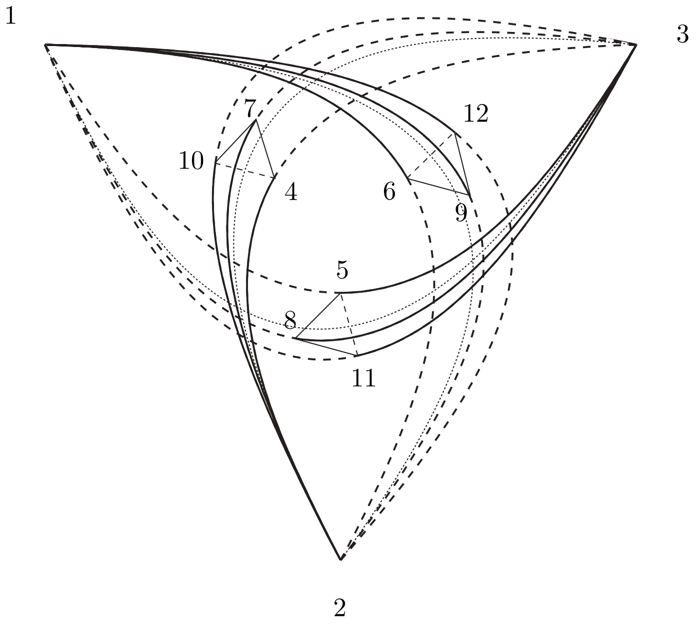

Fig. 1. *The trefoil knot with three protected edges.*

Proof. The examples are based on a trefoil knot consisting of three edges 1 2, 1 3, and 2 3 (the dotted lines in Figure 1) which we embed into $\mathbb{R}^3$. We shield off the edges by enclosing every edge with three tetrahedra, as listed in the first column of Table 1. We then close the holes of the knot by gluing in the following 16 triangles:

4 5 6	1 4 6	2 4 5	3 5 6
	1 4 7	2 5 8	3 6 9
	1 7 10	2 8 11	3 9 12
	1 5 10	2 6 11	3 4 12
	4 5 10	5 6 11	4 6 12.

Table 1
The ball $B^3_{16,46}$.

1 2 6 9	1 4 6 12	1 4 7 13	2 5 8 14	3 6 9 15	4 5 6 16
1 2 6 12	2 4 5 10	2 4 7 13	3 5 8 14	1 6 9 15	1 4 6 16
1 2 9 12	3 5 6 11	1 7 10 13	2 8 11 14	3 9 12 15	1 4 13 16
		2 7 10 13	3 8 11 14	1 9 12 15	2 4 13 16
1 3 5 8		1 5 10 13	2 6 11 14	3 4 12 15	2 4 5 16
1 3 5 11		2 5 10 13	3 6 11 14	1 4 12 15	2 5 14 16
1 3 8 11		1 5 8 13	2 6 9 14	3 4 7 15	3 5 14 16
		2 5 8 13	3 6 9 14	1 4 7 15	3 5 6 16
2 3 4 7					3 6 15 16
2 3 4 10					1 6 15 16
2 3 7 10					

The resulting simplicial complex C is contractible. By adding the 37 tetrahedra in the columns 2–6 of Table 1 we thicken C to a ball $B^3_{16,46}$ with 16 vertices, 46 facets, and f-vector $f = (16, 75, 106, 46)$. Since $B^3_{16,46}$ contains a trefoil knot composed of three edges, it follows from Theorem 1 of Hachimori and Ziegler that $B^3_{16,46}$ is not

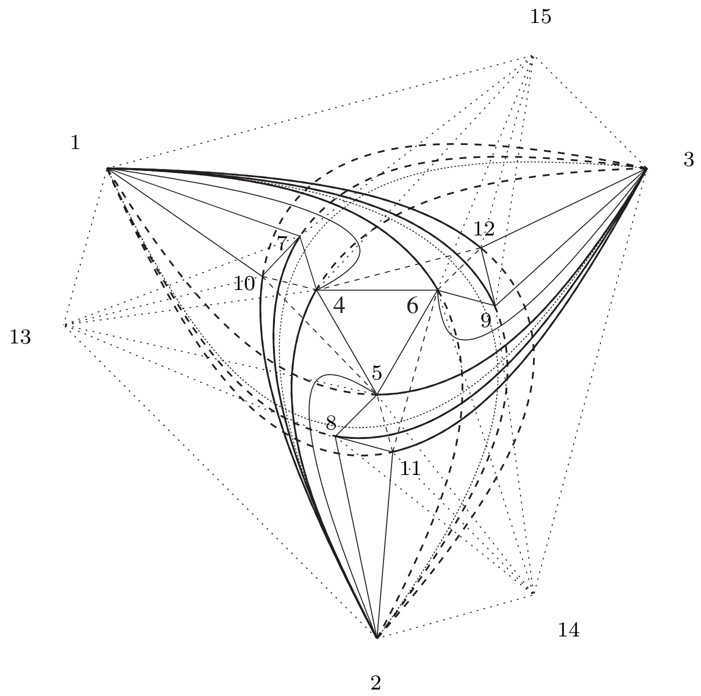

FIG. 2. *The contractible complex C with three cones.*

constructible and thus not shellable. In fact, $B^3_{16,46}$ is strongly nonshellable, as the removal of any of its facets destroys the ballness. Moreover, the presence of the 3-edge knot prevents $B^3_{16,46}$ from having a straight embedding into $\mathbb{R}^3$.

In Figure 2 we display the complex C. We also indicate the cones with respect to the vertices 13, 14, and 15 over eight of the triangles of C each, as listed in columns 3–5 of Table 1. The cone with respect to vertex 16 is then placed “above” the drawing.

The boundary of $B^3_{16,46}$ consists of 28 triangles:

1 13 16
2 13 16
2 14 16
3 14 16
3 15 16
1 15 16

4 5 6

4 5 10
1 5 10
1 5 11
1 8 11
2 8 11
2 8 13
1 8 13

5 6 11
2 6 11
2 6 12
2 9 12
3 9 12
3 9 14
2 9 14

4 6 12
3 4 12
3 4 10
3 7 10
1 7 10
1 7 15
3 7 15.

If we add to $B^3_{16,46}$ the cone over these 28 triangles with respect to a new vertex 17, then we get a 3-sphere $S^3_{17,74}$ with $f = (17, 91, 148, 74)$. This 3-sphere still contains the complex C and with it the trefoil knot composed of the three edges 1 2, 1 3, and 2 3. Hence, $S^3_{17,74}$ is a nonconstructible, nonshellable sphere. By construction, $B^3_{16,46}$ and $S^3_{17,74}$ have a $\mathbb{Z}_3$-symmetry.

Since all 3-spheres with $n \leq 10$ vertices are shellable [8], 17 vertices is close to the minimal number of vertices that are needed for a nonshellable 3-sphere. In order to still improve on the number of vertices, we applied the bistellar flip program BISTELLAR [25] to $S^3_{17,74}$, under the additional restriction that the edges of the knot should not be touched. (The objective of BISTELLAR is to decrease the size of a triangulation of a manifold by performing bistellar flips that locally modify the triangulation without changing the topological type; see [6] for an explicit description.) As result, we obtained a simplicial 3-sphere $S^3_{13,56}$ with $f = (13, 69, 112, 56)$ that has no nontrivial symmetry. The removal of the star of vertex 13

1 7 9 13
1 7 11 13
1 9 10 13
1 10 11 13
2 5 7 13
2 5 8 13
2 6 11 13
2 6 12 13
2 7 11 13
2 8 12 13
3 5 8 13
3 5 9 13
3 6 10 13
3 6 12 13
3 8 12 13
3 9 10 13
5 7 9 13
6 10 11 13

from this complex yields a 12-vertex 3-ball $B^3_{12,38}$ with 38 facets, as listed in Table 2.

TABLE 2
The ball $B^3_{12,38}$.

1 2 6 9	1 5 8 10	*2 4 5 7*	3 4 6 7	4 5 6 7
1 2 6 12	1 5 10 11	2 4 5 10	*3 4 6 10*	4 5 6 10
1 2 9 12	1 6 7 9	2 5 8 10	3 5 9 11	5 6 7 9
	1 6 7 12	2 6 9 11	3 6 7 12	5 6 9 11
1 3 5 8	1 7 8 10	2 7 8 10	3 7 10 12	5 6 10 11
1 3 5 11	1 7 8 11	2 7 8 11	3 8 9 11	
1 3 8 11	1 7 10 12	2 8 9 11	3 8 9 12	
	1 9 10 12	2 8 9 12	3 9 10 12	
2 3 4 7				
2 3 4 10				
2 3 7 10				

This ball has two free facets, 2 4 5 7 and 3 4 6 10, so is not strongly nonshellable. However, when we remove either of the two tetrahedra, we get strongly nonshellable, nonconstructible 3-balls $B^3_{12,37,a}$ and $B^3_{12,37,b}$ with 37 facets and $f = (12, 58, 84, 37)$, respectively. These two balls are not isomorphic, although they have isomorphic boundaries. (The permutation $(2, 3)(5, 6)(7, 10)(8, 12)(9, 11)$ maps the boundary spheres onto each other, but, if we add to each ball the cone over its boundary with respect to a new vertex, then the resulting 3-spheres have different Altshuler–Steinberg determinants [2].) Both balls (and also the sphere $S^3_{13,56}$) still contain the original 3-edge trefoil knot for which, this time, the triangles

4 5 6
4 6 7
1 6 7
1 7 10
1 5 10
4 5 10
2 4 5
2 5 8
2 8 11
2 6 11
5 6 11
5 6 9
3 5 9
3 9 12
3 6 12
3 4 6

are glued in to close the holes of the knot; see Figure 3. □

COROLLARY 3. *For $d \geq 3$ there are nonconstructible d-spheres with $d+10$ vertices. Also there are nonconstructible d-balls, $d \geq 3$, with $d + 9$ vertices and 37 facets that do not have a straight embedding into $\mathbb{R}^d$.*

Proof. The cone over a nonconstructible, nonrealizable d-ball is a nonconstructible, nonrealizable $(d + 1)$-ball with the same number of facets. Similarly, the one-point suspension of a nonconstructible d-sphere is a nonconstructible $(d + 1)$-sphere; see [20]. □

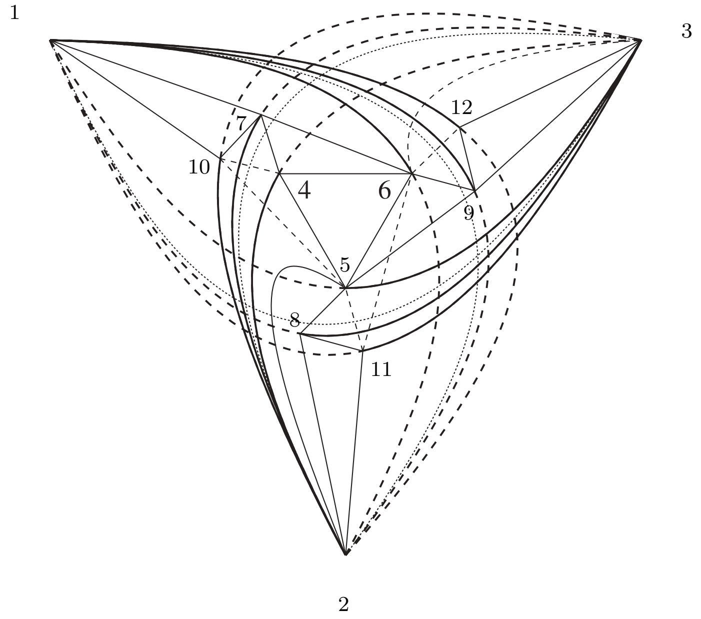

FIG. 3. *The 3-edge trefoil knot lying in the nonshellable sphere* $S^3_{13,56}$.

Acknowledgment. The author is grateful to Günter M. Ziegler for helpful remarks.

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