Improved Column Generation for Highly Degenerate Master Problems

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Abstract

Column generation for solving linear programs with a huge number of variables alternately solves a (restricted) master problem and a pricing subproblem to add variables to the master problem as needed. The method is known to suffer from degeneracy, exposing what is known as tailing-off effect. Inspired by recent advances in coping with degeneracy in the primal simplex method, we propose an improved column generation (ICG) method that takes advantage of degenerate solutions. The idea is to reduce the number of constraints to the number of non-degenerate basic variables in the current master problem solution. The advantage of this row-reduction is a smaller basis, and thus a faster re-optimization of the master problem. This comes at the expense of a more involved pricing subproblem that needs to generate variables compatible with the row-reduction, if possible. Otherwise, incompatible variables may need to be added, and the row-reduction is dynamically updated. We show that, in either case, a strict improvement in the objective function value occurs. We further discuss extensions and some implementation issues.

Two special cases of ICG are the *improved primal simplex* method and the *dynamic constraints aggregation*. On highly degenerate linear programs, recent computational experiments with these algorithms show that the row-reduction of a problem might have a larger impact on the solution time than the column-reduction of standard column generation.

Key Words: Column generation, degeneracy, dynamic row-reduction.

1 Introduction

Column generation, invented to solve large-scale linear programs (LPs), is particularly successful in the context of branch-and-price (Barnhart et al. 1998, Lübbecke and Desrosiers 2005) for solving well-structured integer programs (IPs). Column generation is used to solve the LP relaxations at each node of the search tree, and often produces strong dual bounds. It involves alternately solving a restricted master problem (an LP) and one or several subproblems (usually IPs) in order to dynamically add new variables to the model. However, column generation has a bad reputation for its slow convergence, known as the *tailing-off effect*, a major reason for which being the degeneracy of the restricted master LP solutions. This defect is particularly visible when solving LP relaxations of combinatorial optimization problems—a main application area of branch-and-price.

In this paper we present an *improved column generation* (ICG) method which turns degeneracy into an advantage. We dynamically partition the restricted master problem constraints based on the numerical values of the current basic variables. The idea is to keep only those constraints in the restricted master problem that correspond to non-degenerate basic variables. This leads to a *row-reduced* restricted master problem which does not only discard a large number of variables from consideration in column generation, but also reduces the number of constraints, and in particular the size of the current basis. In linear algebra terms we work with a projection into the subspace spanned by the column-vectors of the non-degenerate variables. This is similar to the idea of a deficient basis in the simplex method (Pan 1998). The additional row reduction of the restricted master problem has a large impact on the solution time of degenerate master problems. Our work generalizes the *improved primal simplex* method (IPS) (Elhallaoui et al. 2010a, Raymond et al. 2010b) for solving highly degenerate linear programs, and the *dynamic constraints aggregation* method (DCA) (Elhallaoui et al. 2005, 2008, 2010b) used for solving LP relaxations of set partitioning problems (by column generation) stemming from vehicle routing and crew scheduling applications.

The paper is organized as follows. Section 2 recalls the column generation method, with the definitions of the master problem MP, its restricted version RMP in terms of variables, and its pricing subproblem SP. Section 3 presents ICG, the *improved column generation* approach. It essentially defines the row and column partitions of the master problem based on a current degenerate solution, introduces the *row-reduced* restricted master problem rMP and its associated pricing subproblem rSP, and finally brings in a specialized column generator cSP. An ICG algorithm and various of its properties are presented in Section 4. We discuss two extensions in Section 5, namely, upper bounded variables and inequality constraints, followed by implementation issues. In Section 6, applications based on set partitioning and multi-commodity flow formulations together with (external) computational experiments are reported.

2 Column Generation

Let us briefly recall the mechanism of column generation, see Lübbecke and Desrosiers (2005) for a general introduction. We would like to solve the following linear program, called the master problem (MP), with a prohibitively large number of variables $\boldsymbol{\lambda} \in \mathbb{R}^n_+$

$$\begin{array}{lrcll} \min & \mathbf{c}^\intercal \boldsymbol{\lambda} & & & \\ \text{s.t.} & A\boldsymbol{\lambda} & = & \mathbf{b} & [\boldsymbol{\pi}] \\ & \boldsymbol{\lambda} & \geq & \mathbf{0} & \end{array} \tag{1}$$

with $A \in \mathbb{R}^{m \times n}$ of full row rank, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, and the vector $\boldsymbol{\pi} \in \mathbb{R}^m$ of dual variables. In applications, every coefficient column $\mathbf{a}$ of A encodes a combinatorial object $\mathbf{x} \in X$ like a path, set, or permutation. To stress this fact, we write $\mathbf{a} = \mathbf{a}(\mathbf{x})$ and $c = c(\mathbf{x})$ for its cost coefficient. One works with a *restricted master problem* (RMP) which involves a small subset of variables only. In each iteration, the RMP is solved to optimality first. Then, like in the primal simplex algorithm, we look for a non-basic variable to price out and enter the current basis. That is, we either find a column $\mathbf{a}(\mathbf{x})$ of cost $c(\mathbf{x})$ with a negative reduced cost $\bar{c}(\mathbf{x})$ or need to prove that none such exists. This is accomplished by solving the *pricing subproblem* SP

$$\bar{c}^\star_{\text{SP}} := \min_{\mathbf{x}\in X} \{c(\mathbf{x}) - \boldsymbol{\pi}^\intercal \mathbf{a}(\mathbf{x})\} \quad . \tag{2}$$

If $\bar{c}^\star_{\text{SP}} \geq 0$, no negative reduced cost columns exist and the current solution $\boldsymbol{\lambda}$ of RMP (embedded into $\mathbb{R}^n_+$) optimally solves MP (1) as well. Otherwise, a minimizer of (2) gives rise to a variable to be added to the RMP, and we iterate.

Functions $c(\mathbf{x})$ and $\mathbf{a}(\mathbf{x})$ may be linear functions, as in a Dantzig-Wolfe reformulation of a linear problem (Dantzig and Wolfe 1960), but $c(\mathbf{x})$ is typically non-linear in many practical applications such as in rich vehicle routing and crew scheduling (Desaulniers et al. 1998). This may increase the difficulty in solving such non-linear subproblems, but it always ends up in a scalar cost c_j and a vector $\mathbf{a}_j$ of scalar coefficients for each variable λ_j in MP, $j = 1, \ldots, n$.

3 ICG: an Improved Column Generation

The RMP is a *column-reduced* MP and its variables are generated as needed by solving SP. The *improved column generation* ICG comes into play when the current solution of RMP is degenerate with $p < m$ positive variables. In what follows, we impose an additional row reduction, that is, we define a *row-reduced* RMP, denoted rMP, which decreases the size of the basis matrix to $p \times p$.

Notation. Let $\text{N} := \{1, \ldots, n\}$ and $\text{M} := \{1, \ldots, m\}$. We denote by I_k the $k \times k$ identity matrix, and by $\mathbf{0}$ a matrix or vector, respectively, with all zero entries of compatible dimensions in the respective context. A vector of all ones is denoted by $\mathbf{1}$. If B is an (ordered) index set, $\boldsymbol{\lambda}_{\text{B}}$ denotes the sub-vector of $\boldsymbol{\lambda}$ indexed by B. Similarly, we denote by A_{B} the $m \times |\text{B}|$ matrix whose columns are indexed by B. We use a superscript index on a vector or a matrix to refer to a subset of its rows. For compatibility reasons with standard vector notation we (ab-)use row-index P as a subscript to $\boldsymbol{\lambda}$. Even though one never actually computes the inverse of a basis matrix, our exposition will often rely on "tableau data," which is conceptually more convenient.

3.1 Row and Column Partitions

Let B denote the index set of the basic variables, i.e., the basis matrix for the current MP is given by A_{B}. Left-multiplication of $A\boldsymbol{\lambda} = \mathbf{b}$ by A_{B}^{-1} yields $\bar{A}\boldsymbol{\lambda} = \bar{\mathbf{b}}$. Let $\boldsymbol{\lambda}_{\text{B}} = \bar{\mathbf{b}}$ be a degenerate feasible solution with $p < m$ positive variables. Define $\text{P} := \{i \in \text{M} \mid \bar{b}_i > 0\}$ and $\text{Z} := \text{M} \setminus \text{P} = \{i \in \text{M} \mid \bar{b}_i = 0\}$, which induces a row-partition of A and $\mathbf{b}$ (after a suitable re-ordering of rows), denoted by

$$A_{\text{B}}^{-1}A = \bar{A} = \begin{bmatrix} \bar{A}^{\text{P}} \\ \bar{A}^{\text{Z}} \end{bmatrix} \quad \text{and} \quad A_{\text{B}}^{-1}\mathbf{b} = \bar{\mathbf{b}} = \begin{bmatrix} \bar{\mathbf{b}}^{\text{P}} \\ \bar{\mathbf{b}}^{\text{Z}} \end{bmatrix} \quad .$$

That is, $\boldsymbol{\lambda}_{\text{P}} = \bar{\mathbf{b}}^{\text{P}} > \mathbf{0}$ gives the values of the non-degenerate basic variables, whereas $\boldsymbol{\lambda}_{\text{Z}} = \bar{\mathbf{b}}^{\text{Z}} = \mathbf{0}$. With the corresponding partition of $\boldsymbol{\pi}$, MP (1) becomes

$$\begin{array}{llcll} \min & \mathbf{c}^\intercal\boldsymbol{\lambda} & & & \\ \text{s.t.} & A^{\text{P}}\boldsymbol{\lambda} & = & \mathbf{b}^{\text{P}} & [\boldsymbol{\pi}^{\text{P}}] \\ & A^{\text{Z}}\boldsymbol{\lambda} & = & \mathbf{b}^{\text{Z}} & [\boldsymbol{\pi}^{\text{Z}}] \\ & \boldsymbol{\lambda} & \geq & \mathbf{0} & \end{array}$$

and SP reads as $\bar{c}^{\star}_{\text{SP}} := \min_{\mathbf{x}\in X} \{c(\mathbf{x}) - (\boldsymbol{\pi}^{\text{P}})^{\intercal}\mathbf{a}^{\text{P}}(\mathbf{x}) - (\boldsymbol{\pi}^{\text{Z}})^{\intercal}\mathbf{a}^{\text{Z}}(\mathbf{x})\}$.

Definition 1. *Given the solution vector* $\boldsymbol{\lambda}_{\text{P}} > \mathbf{0}$ *of non-degenerate variables, vector* $\mathbf{a}$ *is* compatible with basis A_{B} *if and only if* $\bar{\mathbf{a}}^{\text{Z}} = \mathbf{0}$.

Right-hand side vector $\mathbf{b}$ is compatible because $\bar{\mathbf{b}}^{\text{Z}} = \mathbf{0}$, and so are the column-vectors A_{P} associated with the non-degenerate basic variables $\boldsymbol{\lambda}_{\text{P}} > \mathbf{0}$. When appropriate, we also say that a variable associated with a compatible column is compatible. Let $\text{C} \subset \text{N}$ and $\text{I} := \text{N} \setminus \text{C}$ denote the index sets of the compatible and incompatible variables, respectively. Observe that degenerate basic variables are incompatible. The interest in compatibility comes from the fact that compatible variables with negative reduced cost yield non-degenerate pivots when entered into the basis. Indeed, for non-basic compatible variables, the step size ρ_j, $j \in \text{C} \setminus \text{B}$, given by the *ratio-test* is computed only on the row-set P, that is:

$$\rho_j := \min_{i\in\text{M}} \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \mid \bar{a}_{ij} > 0 \right\} = \min_{i\in\text{P}} \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \mid \bar{a}_{ij} > 0 \right\} \quad . \tag{3}$$

Because $\bar{b}_i > 0$, $i \in \text{P}$, then $\rho_j > 0$ and the objective function strictly improves if $\bar{c}_j < 0$, unless $\bar{a}_{ij} \leq 0$, $i \in \text{P}$, in which case the objective function of the master problem MP is unbounded.

Partitioning $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\text{C}}, \boldsymbol{\lambda}_{\text{I}})$, $\mathbf{c} = (\mathbf{c}_{\text{C}}, \mathbf{c}_{\text{I}})$, and $A = (A_{\text{C}}, A_{\text{I}})$ and re-ordering variables accordingly, we can write MP as follows:

$$\begin{array}{rccccccl}
\min & \mathbf{c}_{\text{C}}^{\intercal}\boldsymbol{\lambda}_{\text{C}} & + & \mathbf{c}_{\text{I}}^{\intercal}\boldsymbol{\lambda}_{\text{I}} & & & & \\
\text{s.t.} & A_{\text{C}}^{\text{P}}\boldsymbol{\lambda}_{\text{C}} & + & A_{\text{I}}^{\text{P}}\boldsymbol{\lambda}_{\text{I}} & = & \mathbf{b}^{\text{P}} & & [\boldsymbol{\pi}^{\text{P}}] \\
 & A_{\text{C}}^{\text{Z}}\boldsymbol{\lambda}_{\text{C}} & + & A_{\text{I}}^{\text{Z}}\boldsymbol{\lambda}_{\text{I}} & = & \mathbf{b}^{\text{Z}} & & [\boldsymbol{\pi}^{\text{Z}}] \\
 & \boldsymbol{\lambda}_{\text{C}} & , & \boldsymbol{\lambda}_{\text{I}} & \geq & \mathbf{0} & . &
\end{array}$$

In "tableau data," via left-multiplication of $A\boldsymbol{\lambda} = \mathbf{b}$ by A_{B}^{-1}, we obtain $\bar{A}_{\text{C}}^{\text{Z}} = \mathbf{0}$ and $\bar{\mathbf{b}}^{\text{Z}} = \mathbf{0}$ according to Definition 1, and the following transformed but equivalent MP, with the associated transformed dual vector $\boldsymbol{\zeta}^{\intercal} = \boldsymbol{\pi}^{\intercal}A_{\text{B}}$, results:

$$\begin{array}{rccccccl}
\min & \mathbf{c}_{\text{C}}^{\intercal}\boldsymbol{\lambda}_{\text{C}} & + & \mathbf{c}_{\text{I}}^{\intercal}\boldsymbol{\lambda}_{\text{I}} & & & & \\
\text{s.t.} & \bar{A}_{\text{C}}^{\text{P}}\boldsymbol{\lambda}_{\text{C}} & + & \bar{A}_{\text{I}}^{\text{P}}\boldsymbol{\lambda}_{\text{I}} & = & \bar{\mathbf{b}}^{\text{P}} & & [\boldsymbol{\zeta}^{\text{P}}] \\
 & & & \bar{A}_{\text{I}}^{\text{Z}}\boldsymbol{\lambda}_{\text{I}} & = & \mathbf{0} & & [\boldsymbol{\zeta}^{\text{Z}}] \\
 & \boldsymbol{\lambda}_{\text{C}} & , & \boldsymbol{\lambda}_{\text{I}} & \geq & \mathbf{0} & . &
\end{array} \tag{4}$$

Pricing subproblem SP becomes $\bar{c}^{\star}_{\text{SP}} := \min_{\mathbf{x}\in X} \{c(\mathbf{x}) - (\boldsymbol{\zeta}^{\text{P}})^{\intercal}\bar{\mathbf{a}}^{\text{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\text{Z}})^{\intercal}\bar{\mathbf{a}}^{\text{Z}}(\mathbf{x})\}$.

3.2 A Basis for the Master Problem MP

The definition of the row-index sets P and Z requires the knowledge of a basis A_{B} associated with the current solution of RMP. Conversely, set P induces a basis A_{B} for RMP, even though not uniquely. Knowing such a basis will be important in particular when working with a reduced set P of constraints only. In the following, we will see that we can construct a basis A_{B} of a particularly welcome form, that is, it is not necessary to work with "tableau data" in row-set P of the transformed MP (4).

Let $\text{L} := \text{N} \setminus \text{P}$ denote the index set of the variables at their lower bounds (of zero). A solution for MP is given as $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\text{P}}, \boldsymbol{\lambda}_{\text{L}})$. The proposed basis A_{B} is built as follows. Given the p-dimensional vector $\boldsymbol{\lambda}_{\text{P}}$ of positive basic variables with their associated columns in A_{P}, we add m artificial variables and apply a *phase I* of the primal simplex algorithm to this small $m \times (p+m)$ linear program. This identifies a basis, *not uniquely defined*, of the form

$$A_{\text{B}} = \begin{bmatrix} A_{\text{P}}^{\text{P}} & \mathbf{0} \\ A_{\text{P}}^{\text{Z}} & I_{m-p} \end{bmatrix} \quad , \tag{5}$$

where the p rows of $A_{\mathrm{P}}^{\mathrm{P}}$ are linearly independent, and identity matrix I_{m-p} corresponds to those artificial variables (at value zero) that "complete" the basis. A_{B} establishes the row-partition of the constraints, and the coefficient matrix of the positive basic variables $\boldsymbol{\lambda}_{\mathrm{P}}$ is given by

$$A_{\mathrm{P}} = \begin{bmatrix} A_{\mathrm{P}}^{\mathrm{P}} \\ A_{\mathrm{P}}^{\mathrm{Z}} \end{bmatrix} .$$

The inverse of A_{B} computes as

$$A_{\mathrm{B}}^{-1} = \begin{bmatrix} (A_{\mathrm{P}}^{\mathrm{P}})^{-1} & \mathbf{0} \\ -A_{\mathrm{P}}^{\mathrm{Z}}(A_{\mathrm{P}}^{\mathrm{P}})^{-1} & I_{m-p} \end{bmatrix} .$$

Left-multiplying the original system of constraints $A\boldsymbol{\lambda} = \mathbf{b}$ by the above inverse matrix A_{B}^{-1}, one obtains

$$\begin{array}{lrclcc} \min & \mathbf{c}_{\mathrm{C}}^{\intercal}\boldsymbol{\lambda}_{\mathrm{C}} & + & \mathbf{c}_{\mathrm{I}}^{\intercal}\boldsymbol{\lambda}_{\mathrm{I}} & & \\ \text{s.t.} & (A_{\mathrm{P}}^{\mathrm{P}})^{-1}A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & + & (A_{\mathrm{P}}^{\mathrm{P}})^{-1}A_{\mathrm{I}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{I}} & = & (A_{\mathrm{P}}^{\mathrm{P}})^{-1}\mathbf{b}^{\mathrm{P}} \\ & & & (A_{\mathrm{I}}^{\mathrm{Z}} - A_{\mathrm{P}}^{\mathrm{Z}}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}A_{\mathrm{I}}^{\mathrm{P}})\boldsymbol{\lambda}_{\mathrm{I}} & = & \mathbf{0} \\ & \boldsymbol{\lambda}_{\mathrm{C}} & , & \boldsymbol{\lambda}_{\mathrm{I}} & \geq & \mathbf{0} . \end{array}$$

Observe that the current solution $\boldsymbol{\lambda}_{\mathrm{P}} = \bar{\mathbf{b}}^{\mathrm{P}}$ is computed using the smaller $p \times p$ matrix $(A_{\mathrm{P}}^{\mathrm{P}})^{-1}$ rather than the larger $m \times m$ matrix A_{B}^{-1}. Actually, it is clearly not necessary to multiply the constraints in row-set P by $(A_{\mathrm{P}}^{\mathrm{P}})^{-1}$. Let

$$\bar{A}_{\mathrm{I}}^{\mathrm{Z}} := A_{\mathrm{I}}^{\mathrm{Z}} - A_{\mathrm{P}}^{\mathrm{Z}}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}A_{\mathrm{I}}^{\mathrm{P}} . \tag{6}$$

Therefore the transformed MP, with both row- and column-partitions and the dual vectors $\boldsymbol{\pi}^{\mathrm{P}} \in \mathbb{R}^p$ and $\boldsymbol{\zeta}^{\mathrm{Z}} \in \mathbb{R}^{m-p}$, becomes

$$\begin{array}{lrclccl} \min & \mathbf{c}_{\mathrm{C}}^{\intercal}\boldsymbol{\lambda}_{\mathrm{C}} & + & \mathbf{c}_{\mathrm{I}}^{\intercal}\boldsymbol{\lambda}_{\mathrm{I}} & & & \\ \text{s.t.} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & + & A_{\mathrm{I}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{I}} & = & \mathbf{b}^{\mathrm{P}} & [\boldsymbol{\pi}^{\mathrm{P}}] \\ & & & \bar{A}_{\mathrm{I}}^{\mathrm{Z}}\boldsymbol{\lambda}_{\mathrm{I}} & = & \mathbf{0} & [\boldsymbol{\zeta}^{\mathrm{Z}}] \\ & \boldsymbol{\lambda}_{\mathrm{C}} & , & \boldsymbol{\lambda}_{\mathrm{I}} & \geq & \mathbf{0} . & \end{array} \tag{7}$$

SP now equivalently writes as

$$\bar{c}_{\mathrm{SP}}^{\star} := \min_{\mathbf{x} \in X} \{c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathrm{P}})^{\intercal}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\mathrm{Z}})^{\intercal}\bar{\mathbf{a}}^{\mathrm{Z}}(\mathbf{x})\} . \tag{8}$$

3.3 Interpretation of Compatibility

A column $\mathbf{a}$ is compatible with basis A_{B} as constructed in (5) if and only if

$$\bar{\mathbf{a}}^{\mathrm{Z}} = \mathbf{a}^{\mathrm{Z}} - A_{\mathrm{P}}^{\mathrm{Z}}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{P}} = \mathbf{0} \ , \quad \text{i.e.,} \quad \mathbf{a}^{\mathrm{Z}} = A_{\mathrm{P}}^{\mathrm{Z}}\bar{\mathbf{a}}^{\mathrm{P}} \ ,$$

where

$$\bar{\mathbf{a}}^{\mathrm{P}} = (A_{\mathrm{P}}^{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{P}}, \quad \text{or equivalently,} \quad \mathbf{a}^{\mathrm{P}} = A_{\mathrm{P}}^{\mathrm{P}}\bar{\mathbf{a}}^{\mathrm{P}} \ .$$

Hence, $\mathbf{a}$ is compatible with A_{B} if and only if

$$\begin{bmatrix} \mathbf{a}^{\mathrm{P}} \\ \mathbf{a}^{\mathrm{Z}} \end{bmatrix} = \begin{bmatrix} A_{\mathrm{P}}^{\mathrm{P}} \\ A_{\mathrm{P}}^{\mathrm{Z}} \end{bmatrix} \bar{\mathbf{a}}^{\mathrm{P}}, \quad \text{or equivalently,} \quad \mathbf{a} = A_{\mathrm{P}}\bar{\mathbf{a}}^{\mathrm{P}} \ .$$

Since $A_{\mathrm{P}}^{\mathrm{P}}$ is invertible, the above condition can be written as $\mathbf{a} = A_{\mathrm{P}}\mathbf{y}$, the unique solution of it being $\mathbf{y} = (A_{\mathrm{P}}^{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{P}} = \bar{\mathbf{a}}^{\mathrm{P}}$. Therefore $\mathbf{a}$ is compatible with A_{B} if and only if it is a linear combination of the columns of A_{P} associated to the vector of non-degenerate variables $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$. Note that this is independent of the row-partition used and any reduced basis provides the same combination. The proof is analogous to that of Proposition 2 in Section 4.2. This gives us the following equivalent definition of compatibility.

Definition 2. *Given the solution vector* $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ *of non-degenerate variables, vector* $\mathbf{a}$ *is* compatible with basis A_{B} *if and only if* $\mathbf{a}$ *is a linear combination of the columns of* A_{P}*.*

This definition is related to the notion of a *deficient basis* (Pan 1998). We can now say that $\mathbf{a}$ is *compatible with basis* A_{B} if and only if it is *compatible with* $\boldsymbol{\lambda}_{\mathrm{P}}$.

3.4 rMP and its Pricing Subproblem rSP

To exploit the degeneracy of MP exhibited in (7) in column generation, we *row-reduce* the RMP, denoted by rMP. To this end, the vector of incompatible variables $\boldsymbol{\lambda}_{\mathrm{I}}$ is set to $\mathbf{0}$, the value it already takes in the current solution $(\boldsymbol{\lambda}_{\mathrm{P}}, \boldsymbol{\lambda}_{\mathrm{L}}) = (\bar{\mathbf{b}}^{\mathrm{P}}, \mathbf{0})$ of (7):

$$\begin{array}{llcll} \min & \mathbf{c}_{\mathrm{C}}^{\intercal}\boldsymbol{\lambda}_{\mathrm{C}} & & & \\ \text{s.t.} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & = & \mathbf{b}^{\mathrm{P}} & [\boldsymbol{\pi}^{\mathrm{P}}] \\ & \boldsymbol{\lambda}_{\mathrm{C}} & \geq & \mathbf{0} \ . & \end{array} \tag{9}$$

Restricting (9) to a subset $\mathrm{C}' \subseteq \mathrm{C}$ of the compatible variables as usual, we obtain the row-reduced restricted master problem rMP, on which we perform an adapted column generation. In the sequel, the distinction between C' and C is not essential, and we will refer to (9) as rMP.

The current solution to rMP, vector $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$, is composed of compatible variables. It is optimal for MP (1) if no negative reduced cost columns exist, that is, if $\bar{c}^{\star}_{\mathrm{SP}} \geq 0$, or equivalently, if $\bar{\mathbf{c}} \geq \mathbf{0}$. For the vector of non-degenerate basic variables $\boldsymbol{\lambda}_{\mathrm{P}}$, we already have from (9)

$$\bar{\mathbf{c}}_{\mathrm{P}}^{\intercal} = \mathbf{c}_{\mathrm{P}}^{\intercal} - (\boldsymbol{\pi}^{\mathrm{P}})^{\intercal} A_{\mathrm{P}}^{\mathrm{P}} = \mathbf{0}^{\intercal}$$

by complementary slackness, hence,

$$(\boldsymbol{\pi}^{\mathrm{P}})^{\intercal} = \mathbf{c}_{\mathrm{P}}^{\intercal}(A_{\mathrm{P}}^{\mathrm{P}})^{-1} \ .$$

However, dual vector $\boldsymbol{\zeta}^{\mathrm{Z}}$ in (7) is not known from the solution of rMP (9). Consequently, we cannot solve the pricing subproblem SP as expressed in (8), and we need to come up with an alternative.

Compute the partial reduced cost vector $\tilde{\mathbf{c}}^{\intercal} := \mathbf{c}^{\intercal} - (\boldsymbol{\pi}^{\mathrm{P}})^{\intercal} A^{\mathrm{P}}$ and, again from (9), write the current reduced cost vector $\bar{\mathbf{c}} \in \mathbb{R}^n$ in terms of the unknown vector $\boldsymbol{\zeta}^{\mathrm{Z}}$ of dual variables:

$$\bar{\mathbf{c}}^{\intercal} = \mathbf{c}^{\intercal} - (\boldsymbol{\pi}^{\mathrm{P}})^{\intercal} A^{\mathrm{P}} - (\boldsymbol{\zeta}^{\mathrm{Z}})^{\intercal} \bar{A}^{\mathrm{Z}} = \tilde{\mathbf{c}}^{\intercal} - (\boldsymbol{\zeta}^{\mathrm{Z}})^{\intercal} \bar{A}^{\mathrm{Z}} \ .$$

Let $\boldsymbol{\gamma}^{\intercal} := (\gamma, \ldots, \gamma) \in \mathbb{R}^n$. To verify the non-negativity of $\bar{\mathbf{c}}$, one can find the minimum value of its components by solving

$$\max\{\gamma \mid \mathbf{1}\gamma \leq \bar{\mathbf{c}}\} \ ,$$

i.e., by solving the following pricing subproblem rSP over γ and unknown vector $\boldsymbol{\zeta}^{\mathrm{Z}}$:

$$\begin{array}{rlcll} \bar{c}^{\star}_{\mathrm{rSP}} := & \max & \gamma & & \\ & \text{s.t.} & \mathbf{1}\gamma + (\bar{A}^{\mathrm{Z}})^{\intercal}\boldsymbol{\zeta}^{\mathrm{Z}} & \leq & \tilde{\mathbf{c}} \quad [\boldsymbol{\lambda}] \end{array} \tag{10}$$

where vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ acts as the dual variable vector for the constraint set. In other words, given the non-degenerate variables $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ from which we derive the row-partition and subsequently the partial reduced cost vector $\tilde{\mathbf{c}}$, we check whether any vector $\boldsymbol{\zeta}^{\mathrm{Z}} \in \mathbb{R}^{m-p}$ exists such that $\gamma < 0$ (to generate variables with negative reduced cost to be added to problem rMP) or to otherwise prove the optimality of $\boldsymbol{\lambda}_{\mathrm{P}}$ for rMP (and hence for MP). The dual of (10) defines rSP in terms of vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$:

$$\begin{array}{rlcll} \bar{c}^{\star}_{\mathrm{rSP}} := \min & \tilde{\mathbf{c}}^{\intercal}\boldsymbol{\lambda} & & & \\ \text{s.t.} & \mathbf{1}^{\intercal}\boldsymbol{\lambda} & = & 1 & [\gamma\,] \\ & \bar{A}^{\mathrm{Z}}\boldsymbol{\lambda} & = & \mathbf{0} & [\boldsymbol{\zeta}^{\mathrm{Z}}] \\ & \boldsymbol{\lambda} & \geq & \mathbf{0} \ . & \end{array} \tag{11}$$

Reduced cost value $\bar{c}^\star_{\text{rSP}}$ in pricing subproblem rSP (11) is bounded from above by zero: For the non-degenerate basic variables $\lambda_j, j \in \text{P}$, reduced cost $\bar{c}_j = \tilde{c}_j = 0$ and $\bar{\mathbf{a}}^{\text{Z}}_j = \mathbf{0}$. The optimum of rSP is either $\bar{c}^\star_{\text{rSP}} = 0$ establishing the optimality of rMP (and MP) with the current solution $\boldsymbol{\lambda}_{\text{P}} = \bar{\mathbf{b}}^{\text{P}}$, or $\bar{c}^\star_{\text{rSP}} < 0$, and further columns need to be added to rMP.

Pricing subproblem rSP (11) is a linear program with $m - p + 1$ constraints. It is similar to MP as it must also be solved by column generation *over the same set* of objects $\mathbf{x} \in X$. In the pricing step within the column generation process to solve rSP (11), we look for a column $[1, (\bar{\mathbf{a}}^{\text{Z}})^\intercal]^\intercal$ of cost $\tilde{c}$ with a reduced cost $\bar{c} = \tilde{c} - \gamma - (\boldsymbol{\zeta}^{\text{Z}})^\intercal \bar{\mathbf{a}}^{\text{Z}}$ of negative value. The cost $\tilde{c}$ and coefficients of $\bar{\mathbf{a}}^{\text{Z}}$ of such a column are computed according to an object $\mathbf{x} \in X$, that is,

$$\tilde{c}(\mathbf{x}) := c(\mathbf{x}) - (\boldsymbol{\pi}^{\text{P}})^\intercal \mathbf{a}^{\text{P}}(\mathbf{x}) \quad \text{and} \quad \bar{\mathbf{a}}^{\text{Z}}(\mathbf{x}) := \mathbf{a}^{\text{Z}}(\mathbf{x}) - A^{\text{Z}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}\mathbf{a}^{\text{P}}(\mathbf{x}) \ .$$

Given $\boldsymbol{\pi}^{\text{P}}$ retrieved from the solution of rMP (9) and γ and $\boldsymbol{\zeta}^{\text{Z}}$ retrieved from the current solution of (11), the pricing subproblem for generating variables as needed for solving rSP is given by

$$\bar{c}^\star := -\gamma + \min_{\mathbf{x} \in X} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\text{P}})^\intercal \mathbf{a}^{\text{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\text{Z}})^\intercal \bar{\mathbf{a}}^{\text{Z}}(\mathbf{x}) \ \middle| \ \bar{\mathbf{a}}^{\text{Z}}(\mathbf{x}) = \mathbf{a}^{\text{Z}}(\mathbf{x}) - A^{\text{Z}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}\mathbf{a}^{\text{P}}(\mathbf{x}) \right\} \ .$$

3.5 cSP: a Subproblem for Generating Compatible Columns

If we restrict attention to only generating columns which are compatible with the current basis A_{B} (which is a natural idea in our context), matters simplify considerably. Given the dual vector $\boldsymbol{\pi}^{\text{P}}$ retrieved from the solution of rMP (9), we define a specialized subproblem cSP which is the classical pricing subproblem SP (8) augmented with a set of linear constraints imposing compatibility with rMP for solution-column $\mathbf{a}$, that is,

$$\bar{c}^\star_{\text{cSP}} := \min_{\mathbf{x} \in X} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\text{P}})^\intercal \mathbf{a}^{\text{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\text{Z}})^\intercal \bar{\mathbf{a}}^{\text{Z}}(\mathbf{x}) \ \middle| \ \bar{\mathbf{a}}^{\text{Z}}(\mathbf{x}) = \mathbf{0} \right\} \ .$$

Therefore, there is no need to know dual vector $\boldsymbol{\zeta}^{\text{Z}}$ and the specialized pricing subproblem cSP can equivalently be written as follows:

$$\bar{c}^\star_{\text{cSP}} := \min_{\mathbf{x} \in X} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\text{P}})^\intercal \mathbf{a}^{\text{P}}(\mathbf{x}) \ \middle| \ \mathbf{a}^{\text{Z}}(\mathbf{x}) - A^{\text{Z}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}\mathbf{a}^{\text{P}}(\mathbf{x}) = \mathbf{0} \right\} . \tag{12}$$

If the current solution for rMP (9) is non-degenerate, i.e., $\bar{\mathbf{b}}^{\text{P}} > \mathbf{0}$, and a compatible variable $\boldsymbol{\lambda}_j, j \in \text{C}$, with negative reduced cost $\bar{c}_j = \bar{c}^\star_{\text{cSP}} > 0$ is generated from the solution of (12), it strictly decreases the objective function value of MP when it enters the basis of rMP because it yields a non-degenerate pivot on row-set P (unless all components of $\bar{\mathbf{a}}^{\text{P}}_j$ are non-positive, in which case MP is unbounded).

Only when $\bar{c}^\star_{\text{cSP}} = 0$, we solve the more involved rSP (11) to look for a convex combination of variables to improve the objective value of MP. In the next section, we show that this combination of columns is a vector compatible with A_{B} but made of incompatible columns. When all the variables of this combination enter the restricted master problem RMP, this also yields a strict improvement of the objective function (unless MP is unbounded). When incompatible variables are added to RMP, the row-partition defining rMP must be updated.

4 ICG in more Detail

4.1 Strictly Improving Pivots

Column generation generalizes the primal simplex algorithm. In the same spirit, the *improved primal simplex* method (IPS) used for solving highly degenerate linear programs (Elhallaoui et al.

2010a, Raymond et al. 2010b) can be seen as a special case of ICG. The main difference between ICG and IPS is that rSP (11) itself needs to be solved by column generation, whereas in IPS all columns are explicitly given in advance. Moreover, given A_{B}, all variables can be characterized *a priori* as either compatible or incompatible. Therefore, in IPS, the row-reduced restricted master problem rMP is defined on the compatible variables and a so-called *complementary subproblem* is solved over the incompatible variables only (in our context, it would be named iSP). Computational experiments for IPS show that iSP can be efficiently solved by the dual simplex algorithm in a very small number of iterations, see Elhallaoui et al. (2010a).

It is well known that if the current solution for MP is non-degenerate and non-optimal, the next pivot entails a strict improvement of the objective value. In ICG and IPS, this is also true for degenerate solutions. This important result is demonstrated next. Consider a degenerate solution for MP, i.e., $p < m$, and derive rMP (9) and rSP (11). When $\bar{c}^\star_{\text{rSP}} = 0$, this establishes the optimality of the current solution $\boldsymbol{\lambda}_{\text{P}} = \bar{\mathbf{b}}^{\text{P}}$ for rMP, and hence for MP. Otherwise, when $\bar{c}^\star_{\text{rSP}} < 0$, we have the following proposition.

Proposition 1. *Let $\boldsymbol{\lambda}^\star \in \mathbb{R}^n_+$ be an optimal solution to rSP (11). If $\bar{c}^\star_{\text{rSP}} < 0$, vector $A\boldsymbol{\lambda}^\star$ is compatible with A_{B} and it allows for a strict improvement of the objective function value of rMP (9).*

Proof. With the row-partition induced by basis A_{B}, $A\boldsymbol{\lambda} = \begin{bmatrix} A^{\text{P}} \\ A^{\text{Z}} \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} A^{\text{P}}\boldsymbol{\lambda} \\ A^{\text{Z}}\boldsymbol{\lambda} \end{bmatrix}$. Because $\bar{A}^{\text{Z}}\boldsymbol{\lambda} = \mathbf{0}$ in any solution to (11), $A\boldsymbol{\lambda}$ is compatible with A_{B} by definition. So is the optimal vector $A\boldsymbol{\lambda}^\star$.

When $\bar{c}^\star_{\text{rSP}} < 0$, an optimal solution $\boldsymbol{\lambda}^\star \in \mathbb{R}^n_+$ to rSP (11) identifies a convex combination of positive compatible and incompatible variables $0 < \lambda^\star_j \leq 1$, $j \in \text{C} \cup \text{I}$. If this optimal solution of rSP contains a single positive variable $\lambda^\star_j = 1$, then $\bar{\mathbf{a}}^{\text{Z}}_j = \mathbf{0}$, and vector $\mathbf{a}_j$ is compatible with A_{B} (such a column can alternatively be obtained by solving cSP (12)). Then, the p-dimensional column $\mathbf{a}^{\text{P}}_j, j \in \text{C}$, and the associated variable λ_j in MP enters the basis of rMP: we have $\rho_j > 0$ in the ratio-test (3) and a non-degenerate pivot occurs (unless $\bar{\mathbf{a}}^{\text{P}}_j \leq \mathbf{0}$ in which case MP is unbounded).

If an optimal solution of rSP (11) is composed of several positive variables, they can be considered as either all compatible or as all incompatible. Indeed, if there was a mix of types, all variables of any one type could be removed since no one type can be better than the other in terms of reduced cost (otherwise, it contradicts the optimality of the solution). Moreover, the case where all variables are compatible comes back to the case of a single compatible variable because, by optimality, all variables have the same reduced cost value.

Assume that the positive variables of the convex combination in an optimal solution of (11) are incompatible and let $J := \{j \in \text{I} \mid \lambda^\star_j > 0\}$. The p-dimensional vector $A^{\text{P}}\boldsymbol{\lambda}^\star$ can enter the basis of rMP as a single column. This is "a new type" of variable. Because $\bar{A}^{\text{Z}}\boldsymbol{\lambda}^\star = \mathbf{0}$, its reduced cost is computed as $\bar{\mathbf{c}}^\intercal\boldsymbol{\lambda}^\star = \tilde{\mathbf{c}}^\intercal\boldsymbol{\lambda}^\star = \bar{c}^\star_{\text{rSP}} < 0$. Unless all components of vector $A^{\text{P}}\boldsymbol{\lambda}^\star$ are non-positive, it strictly improves the objective function value when added to rMP. The value taken by the associated variable is established by the ratio-test (3). Note that the row-partition of rMP needs not be modified if column $A^{\text{P}}\boldsymbol{\lambda}^\star$ enters the basis of rMP as a single column. □

The incompatible variables $\lambda^\star_j, j \in J$, in MP can also be entered one by one in RMP, in any order. The last incompatible variable entered ensures a non-degenerate pivot: because we know that the convex combination $A^{\text{P}}\boldsymbol{\lambda}^\star$ of these variables is compatible with A_{B}, and moreover, it has negative reduced cost. If such a procedure is adopted, the row-partition defining rMP has to be updated (see also Elhallaoui et al. (2010a) in the context of IPS).

4.2 Independence of the Pricing Subproblems from a Basis for rMP

We already saw the construction of basis A_{B} for RMP such that the derived rMP can be written in terms of the original data in (9). Given the current solution $\boldsymbol{\lambda}_{\text{P}} = (A^{\text{P}}_{\text{P}})^{-1}\mathbf{b}^{\text{P}} = \bar{\mathbf{b}}^{\text{P}}$ to rMP, the associated $p \times p$ reduced basis A^{P}_{P} is a matrix containing p linearly independent rows of A_{P}. It is not uniquely defined by our primal simplex *phase I* approach to select these rows. Matrix A^{P}_{P} does not only characterize the row-partition of the constraints of rMP and rSP, but also their cost, partial reduced cost, and column components. In this subsection, we show by a linear algebra argument that, for any selected reduced basis, the pricing subproblems hold the same information based on $\boldsymbol{\lambda}_{\text{P}}$ and thus provide the same solution, hence the same generated column.

Let us express rSP (11) in terms of the variables $\boldsymbol{\lambda} \in \mathbb{R}^n_+$, and as a function of the reduced basis A^{P}_{P} and the original cost and data coefficients:

$$\begin{array}{rlrcl} \bar{c}^{\star}_{\text{rSP}} := & \min & (\mathbf{c}^{\intercal} - \mathbf{c}^{\intercal}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}})\boldsymbol{\lambda} & & \\ & \text{s.t.} & \mathbf{1}^{\intercal}\boldsymbol{\lambda} & = & 1 \\ & & (A^{\text{Z}} - A^{\text{Z}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}})\boldsymbol{\lambda} & = & \mathbf{0} \\ & & \boldsymbol{\lambda} & \geq & \mathbf{0}\ . \end{array}$$

Consider another reduced basis defined on row-set Q and denoted A^{Q}_{P}. It is again of dimension $p \times p$ and $\boldsymbol{\lambda}_{\text{P}} = (A^{\text{Q}}_{\text{P}})^{-1}\mathbf{b}^{\text{Q}} = \bar{\mathbf{b}}^{\text{Q}}$. Duplicate rows in set Q such that MP becomes

$$\begin{array}{rrcl} \min & \mathbf{c}^{\intercal}\boldsymbol{\lambda} & & \\ \text{s.t.} & A^{\text{P}}\boldsymbol{\lambda} & = & \mathbf{b}^{\text{P}} \\ & A^{\text{Z}}\boldsymbol{\lambda} & = & \mathbf{b}^{\text{Z}} \\ & A^{\text{Q}}\boldsymbol{\lambda} & = & \mathbf{b}^{\text{Q}} \\ & \boldsymbol{\lambda} & \geq & \mathbf{0}\ . \end{array}$$

Given the vector of non-degenerate variables $\boldsymbol{\lambda}_{\text{P}} > \mathbf{0}$, consider the following two bases, the first being defined according to the reduced basis A^{P}_{P} while the second is given according to the reduced basis A^{Q}_{P}:

$$\begin{bmatrix} A^{\text{P}}_{\text{P}} & \mathbf{0} & \mathbf{0} \\ A^{\text{Z}}_{\text{P}} & I_{m-p} & \mathbf{0} \\ A^{\text{Q}}_{\text{P}} & \mathbf{0} & I_p \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A^{\text{Q}}_{\text{P}} & \mathbf{0} & \mathbf{0} \\ A^{\text{Z}}_{\text{P}} & I_{m-p} & \mathbf{0} \\ A^{\text{P}}_{\text{P}} & \mathbf{0} & I_p \end{bmatrix} .$$

The corresponding inverses are given as follows:

$$\begin{bmatrix} (A^{\text{P}}_{\text{P}})^{-1} & \mathbf{0} & \mathbf{0} \\ -A^{\text{Z}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1} & I_{m-p} & \mathbf{0} \\ -A^{\text{Q}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1} & \mathbf{0} & I_p \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} (A^{\text{Q}}_{\text{P}})^{-1} & \mathbf{0} & \mathbf{0} \\ -A^{\text{Z}}_{\text{P}}(A^{\text{Q}}_{\text{P}})^{-1} & I_{m-p} & \mathbf{0} \\ -A^{\text{P}}_{\text{P}}(A^{\text{Q}}_{\text{P}})^{-1} & \mathbf{0} & I_p \end{bmatrix} .$$

Pricing subproblem rSP can now be written according to the selected basis, namely rSP^{P} and rSP^{Q}:

$$\begin{array}{rlrcl} \bar{c}^{\star}_{\text{rSP}^{\text{P}}} := & \min & (\mathbf{c}^{\intercal} - \mathbf{c}^{\intercal}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}})\boldsymbol{\lambda} & & \\ & \text{s.t.} & \mathbf{1}^{\intercal}\boldsymbol{\lambda} & = & 1 \\ & & (A^{\text{Z}} - A^{\text{Z}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}})\boldsymbol{\lambda} & = & \mathbf{0} \\ & & (A^{\text{Q}} - A^{\text{Q}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}})\boldsymbol{\lambda} & = & \mathbf{0} \\ & & \boldsymbol{\lambda} & \geq & \mathbf{0} \end{array} \tag{13}$$

and

$$\begin{array}{rlrcl} \bar{c}^{\star}_{\text{rSP}^{\text{Q}}} := & \min & (\mathbf{c}^{\intercal} - \mathbf{c}^{\intercal}_{\text{P}}(A^{\text{Q}}_{\text{P}})^{-1}A^{\text{Q}})\boldsymbol{\lambda} & & \\ & \text{s.t.} & \mathbf{1}^{\intercal}\boldsymbol{\lambda} & = & 1 \\ & & (A^{\text{Z}} - A^{\text{Z}}_{\text{P}}(A^{\text{Q}}_{\text{P}})^{-1}A^{\text{Q}})\boldsymbol{\lambda} & = & \mathbf{0} \\ & & (A^{\text{P}} - A^{\text{P}}_{\text{P}}(A^{\text{Q}}_{\text{P}})^{-1}A^{\text{Q}})\boldsymbol{\lambda} & = & \mathbf{0} \\ & & \boldsymbol{\lambda} & \geq & \mathbf{0}\ . \end{array} \tag{14}$$

Proposition 2. *Given two reduced bases $A^{\mathrm{P}}_{\mathrm{P}}$ and $A^{\mathrm{Q}}_{\mathrm{P}}$, pricing subproblems* $\mathrm{rSP}^{\mathrm{P}}$ *and* $\mathrm{rSP}^{\mathrm{Q}}$ *are equivalent. An optimal solution for* rSP *is therefore independent of the selected reduced basis derived from the current degenerate solution $\boldsymbol{\lambda}_{\mathrm{P}}$.*

Proof. Left-multiplying the third set of equations of (13) by $-A^{\mathrm{P}}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}$, one obtains the third set of equations of (14):

$$-A^{\mathrm{P}}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}(A^{\mathrm{Q}} - A^{\mathrm{Q}}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}A^{\mathrm{P}})\boldsymbol{\lambda} = (A^{\mathrm{P}} - A^{\mathrm{P}}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}A^{\mathrm{Q}})\boldsymbol{\lambda} = \mathbf{0} \ .$$

From the above equation, we have the identity $A^{\mathrm{P}}\boldsymbol{\lambda} = A^{\mathrm{P}}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}A^{\mathrm{Q}}\boldsymbol{\lambda}$. Substituting for $A^{\mathrm{P}}\boldsymbol{\lambda}$ in the objective function and in the second constraint set of (13), one completes the linear transformation of $\mathrm{rSP}^{\mathrm{P}}$ into $\mathrm{rSP}^{\mathrm{Q}}$:

$$(\mathbf{c}^{\intercal} - \mathbf{c}^{\intercal}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}A^{\mathrm{P}})\boldsymbol{\lambda} = \mathbf{c}^{\intercal}\boldsymbol{\lambda} - \mathbf{c}^{\intercal}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}A^{\mathrm{P}}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}A^{\mathrm{Q}}\boldsymbol{\lambda} = (\mathbf{c}^{\intercal} - \mathbf{c}^{\intercal}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}A^{\mathrm{Q}})\boldsymbol{\lambda}$$

and

$$(A^{\mathrm{Z}} - A^{\mathrm{Z}}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}A^{\mathrm{P}})\boldsymbol{\lambda} = A^{\mathrm{Z}}\boldsymbol{\lambda} - A^{\mathrm{Z}}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}A^{\mathrm{P}}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}A^{\mathrm{Q}}\boldsymbol{\lambda} = (A^{\mathrm{Z}} - A^{\mathrm{Z}}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}A^{\mathrm{Q}})\boldsymbol{\lambda} \ . \quad \square$$

The reduced costs of the non-degenerate variables are zero by complementary slackness, hence independent of the selected reduced basis. However, their column coefficients in rMP (9) are different since they are given by the column coefficients of the selected reduced basis, say, either $A^{\mathrm{P}}_{\mathrm{P}}$ or $A^{\mathrm{Q}}_{\mathrm{P}}$. We next show that the reduced cost and the coefficient components of the compatible variables are indeed identical in the two versions of cSP, namely $\mathrm{cSP}^{\mathrm{P}}$ and $\mathrm{cSP}^{\mathrm{Q}}$. Therefore cSP, the specialized pricing subproblem for generating compatible variables, is independent of the selected reduced basis. We first provide these two versions of cSP, adapted from (12):

$$\begin{array}{rrcl} \bar{c}^{\star}_{\mathrm{cSP^P}} := \min_{\mathbf{x}\in X} & c(\mathbf{x}) - \mathbf{c}^{\intercal}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) & & \\ \text{s.t.} & \mathbf{a}^{\mathrm{Z}}(\mathbf{x}) - A^{\mathrm{Z}}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) & = & \mathbf{0} \\ & \mathbf{a}^{\mathrm{Q}}(\mathbf{x}) - A^{\mathrm{Q}}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) & = & \mathbf{0} \end{array} \tag{15}$$

and

$$\begin{array}{rrcl} \bar{c}^{\star}_{\mathrm{cSP^Q}} := \min_{\mathbf{x}\in X} & c(\mathbf{x}) - \mathbf{c}^{\intercal}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{Q}}(\mathbf{x}) & & \\ \text{s.t.} & \mathbf{a}^{\mathrm{Z}}(\mathbf{x}) - A^{\mathrm{Z}}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{Q}}(\mathbf{x}) & = & \mathbf{0} \\ & \mathbf{a}^{\mathrm{P}}(\mathbf{x}) - A^{\mathrm{P}}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{Q}}(\mathbf{x}) & = & \mathbf{0} \ . \end{array} \tag{16}$$

Proposition 3. *Given two reduced bases $A^{\mathrm{P}}_{\mathrm{P}}$ and $A^{\mathrm{Q}}_{\mathrm{P}}$, the pricing subproblems* $\mathrm{cSP}^{\mathrm{P}}$ *and* $\mathrm{cSP}^{\mathrm{Q}}$ *are identical. Therefore, the solution of* cSP *is independent of the selected reduced basis derived from the current degenerate solution $\boldsymbol{\lambda}_{\mathrm{P}}$.*

Proof. The proof is similar to that of Proposition 2 except that vector $\boldsymbol{\lambda}$ is not involved. Multiplying the last set of constraints of (15) by $-A^{\mathrm{P}}_{\mathrm{P}}(A^{\mathrm{Q}}_{\mathrm{P}})^{-1}$, one obtains the corresponding constraint set of (16). Since $\mathbf{a}^{\mathrm{P}}(\mathbf{x}) = A^{\mathrm{P}}_{\mathrm{P}}(A^{\mathrm{P}}_{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{Q}}(\mathbf{x})$, it can be substituted in the objective function and the first set of constraints of (15). This shows that $\mathrm{cSP}^{\mathrm{P}}$ and $\mathrm{cSP}^{\mathrm{Q}}$ are identical. $\square$

Combining Propositions 2 and 3, we have the following. Although the reduced cost coefficients and column components of incompatible variables in rSP depend on the selected reduced basis, the optimal convex combination $\boldsymbol{\lambda}^{\star}$ remains the same, hence the entering column $A^{\mathrm{P}}\boldsymbol{\lambda}^{\star}$ in rMP. The generated columns, either from the solution of rSP or cSP, do not depend on the choice of the reduced basis derived from the current degenerate solution $\boldsymbol{\lambda}_{\mathrm{P}}$.

4.3 An ICG Algorithm

We summarize our discussion with a pseudo-code of our ICG algorithm for degenerate master problems. As long as the solution is non-degenerate, we have the classical alternation between the restricted master problem RMP and the pricing subproblem SP (until line 8). When a degenerate solution is identified in line 9 ($p < m$), the row-reduced rMP benefits from this (lines 10 to 18).

After solving RMP (line 4) or rMP (line 12), the optimality test for MP is via the solution of a pricing subproblem, either the classical SP (line 7) if the current solution is non-degenerate or rSP which is solved by column generation (line 16). In both situations, new columns are added to RMP if $\bar{c}^\star_{\text{SP}} < 0$ or $\bar{c}^\star_{\text{rSP}} < 0$, otherwise MP is optimal.

If the current RMP solution is degenerate in line 9, row-index sets P and Z are defined/updated (line 10), and the rMP is built and solved (line 12). In that case, priority can be given to the specialized pricing subproblem cSP (line 13) and, if $\bar{c}^\star_{\text{cSP}} < 0$, compatible columns are added to rMP (line 14). Otherwise, the pricing problem rSP needs to be solved by column generation (line 16). If $\bar{c}^\star_{\text{rSP}} < 0$ in line 17, we add incompatible columns to RMP, and iterate (again from line 4). The algorithm stops when rMP and hence MP are optimal (line 18).

1 **algorithm** improved column generation ICG;
2 initialize RMP (1);
3 **repeat**
4   solve RMP (1);
5   identify a basis A_{B} with $p \leq m$ non-degenerate basic variables;
6   **if** $p = m$ **then** `// non-degenerate solution`
7   solve SP (2);
8   **if** $\bar{c}^\star_{\text{SP}} < 0$ **then** add column(s) to RMP (1);
9   **else** `//` $p < m$`, degenerate solution`
10   if necessary, update P $:= \{i \mid \bar{b}_i > 0\}$, Z $:= \{i \mid \bar{b}_i = 0\}$;
11   **repeat**
12   solve rMP (9), the *row-reduced* RMP;
13   solve cSP (12);
14   **if** $\bar{c}^\star_{\text{cSP}} < 0$ **then** add compatible column(s) to rMP (9);
15   **until** *no more compatible columns added*;
16   solve rSP (11) by column generation;
17   **if** $\bar{c}^\star_{\text{rSP}} < 0$ **then** add incompatible column(s) to RMP (1);
18 **until** *no more columns added*;

5 ICG Extensions and Implementation

In this section, we consider two important extensions to ICG: Master problems with upper bounded variables, and inequality constraints. For the first extension we examine the special case of implicit bounds. For the second, we analyze the impact of two different partitions. These two extensions demonstrate the flexibility of the compatibility concept. Therefore, we introduce the definition of compatibility with a set of rows, and discuss some implementation issues regarding the choice of the partition and of the pricing subproblem.

5.1 ICG with Upper Bounded Variables

In this subsection, we examine the impact of upper bounded variables on ICG. For example, such variables appear in applications formulated as set partitioning or set covering models, see Section 6 on various binary clustering problems. Degenerate variables can then appear in the basis either at their lower bound or at their upper bound. This may increase the number of degenerate variables and allows for a further reduction of the rMP row-size.

Consider the following master problem with upper bounded variables:

$$\min \quad \mathbf{c}^\intercal \boldsymbol{\lambda} \quad \text{s.t.} \quad A\boldsymbol{\lambda} \geq \mathbf{b}, \quad \mathbf{0} \leq \boldsymbol{\lambda} \leq \mathbf{u} \ . \tag{17}$$

Let the current degenerate basic solution be given by $(\boldsymbol{\lambda}_{\mathrm{P}}, \boldsymbol{\lambda}_{\mathrm{U}}, \boldsymbol{\lambda}_{\mathrm{L}})$ where $\mathbf{0} < \boldsymbol{\lambda}_{\mathrm{P}} < \mathbf{u}_{\mathrm{P}}$, $\mathrm{P} \subseteq \mathrm{B}$, is the vector of basic variables with slack, $\boldsymbol{\lambda}_{\mathrm{U}} = \mathbf{u}_{\mathrm{U}}$, $\mathrm{U} \subseteq \mathrm{B}$ is the vector of basic variables at their upper bounds, and $\boldsymbol{\lambda}_{\mathrm{L}} = \mathbf{0}$, where the index set of the variables at their zero lower bounds is defined as $\mathrm{L} := \mathrm{N} \setminus (\mathrm{P} \cup \mathrm{U})$. As before, $|\mathrm{P}| = p$, and let $|\mathrm{U}| = u$. We assume that all non-basic variables are at their lower bounds. If this was not the case for a $\lambda_j = u_j$, we performed the standard transformation with a complementary variable $\lambda_j^- := u_j - \lambda_j$, and this variable would appear in set L.

There are $p+u$ positive variables in the current solution. The mentioned standard transformation $\boldsymbol{\lambda}_{\mathrm{U}}^- := \mathbf{u}_{\mathrm{U}} - \boldsymbol{\lambda}_{\mathrm{U}}$, $\mathbf{0} \leq \boldsymbol{\lambda}_{\mathrm{U}}^- \leq \mathbf{u}_{\mathrm{U}}$, allows us to benefit from the degenerate variables at their upper bounds. Substituting in (17), the master problem with upper bounded variables becomes:

$$\begin{array}{rccccl} \mathbf{c}_{\mathrm{U}}^\intercal \mathbf{u}_{\mathrm{U}} + \min & \mathbf{c}_{\mathrm{P}}^\intercal \boldsymbol{\lambda}_{\mathrm{P}} & - \mathbf{c}_{\mathrm{U}}^\intercal \boldsymbol{\lambda}_{\mathrm{U}}^- & + \mathbf{c}_{\mathrm{L}}^\intercal \boldsymbol{\lambda}_{\mathrm{L}} & & \\ \text{s.t.} & A_{\mathrm{P}} \boldsymbol{\lambda}_{\mathrm{P}} & - A_{\mathrm{U}} \boldsymbol{\lambda}_{\mathrm{U}}^- & + A_{\mathrm{L}} \boldsymbol{\lambda}_{\mathrm{L}} & = & \mathbf{b} - A_{\mathrm{U}} \mathbf{u}_{\mathrm{U}} \\ & \mathbf{0} \leq \boldsymbol{\lambda}_{\mathrm{P}} \leq \mathbf{u}_{\mathrm{P}}, & \mathbf{0} \leq \boldsymbol{\lambda}_{\mathrm{U}}^- \leq \mathbf{u}_{\mathrm{U}}, & \mathbf{0} \leq \boldsymbol{\lambda}_{\mathrm{L}} \leq \mathbf{u}_{\mathrm{L}}, & & \end{array} \tag{18}$$

where the current solution is now given by $\mathbf{0} < \boldsymbol{\lambda}_{\mathrm{P}} < \mathbf{u}_{\mathrm{P}}$, $\boldsymbol{\lambda}_{\mathrm{U}}^- = \mathbf{0}$, and $\boldsymbol{\lambda}_{\mathrm{L}} = \mathbf{0}$. Keeping only $\boldsymbol{\lambda}_{\mathrm{P}}$ in the basis as in Section 4.2, one can apply a *phase I* procedure to derive A_{B} and the corresponding row-partition for MP, where $\boldsymbol{\lambda}_{\mathrm{P}}$ are basic in row-set P. Hence, rMP comprises only p rows whereas the row-size of the pricing subproblem rSP is $m - p + 1$.

Implicit Upper Bounds. Upper bounds on variables may be only implicitly imposed, e.g., in set partitioning models, see Section 6. In these cases, is there any advantage in explicitly imposing the upper bounds on the variables? In the following paragraphs, we derive the transformed master problems and pricing subproblems for both situations, the one exploiting the degeneracy of variables at their upper bounds and the other for which *such degeneracy is not recognized*.

Consider the same basic solution to MP in (1) and (17) given by $(\boldsymbol{\lambda}_{\mathrm{P}}, \boldsymbol{\lambda}_{\mathrm{U}}, \boldsymbol{\lambda}_{\mathrm{L}})$ where $\mathbf{0} < \boldsymbol{\lambda}_{\mathrm{P}} < \mathbf{u}_{\mathrm{P}}$, $\mathrm{P} \subseteq \mathrm{B}$, $\boldsymbol{\lambda}_{\mathrm{U}} = \mathbf{u}_{\mathrm{U}}$, $\mathrm{U} \subseteq \mathrm{B}$, and $\boldsymbol{\lambda}_{\mathrm{L}} = \mathbf{0}$, where $\mathrm{L} := \mathrm{N} \setminus (\mathrm{P} \cup \mathrm{U})$. Recall that in the master problem (1), upper bounds are not explicitly given. Let us define a row-partition used for both modeling situations. Keep sets of positive variables $\boldsymbol{\lambda}_{\mathrm{P}}$ and $\boldsymbol{\lambda}_{\mathrm{U}}$ in the basis of formulation (1) and apply a *phase I* procedure to derive A_{B} and the corresponding row-partition for MP, where $\boldsymbol{\lambda}_{\mathrm{P}}$ and $\boldsymbol{\lambda}_{\mathrm{U}}$ are basic in row-sets P and U, respectively. The row-partition is completed by set $\mathrm{Z} := \mathrm{M} \setminus (\mathrm{P} \cup \mathrm{U})$. Therefore, a basis for (1) is

$$A_{\mathrm{B}} = \begin{bmatrix} A_{\mathrm{P}}^{\mathrm{P}} & A_{\mathrm{U}}^{\mathrm{P}} & \mathbf{0} \\ A_{\mathrm{P}}^{\mathrm{U}} & A_{\mathrm{U}}^{\mathrm{U}} & \mathbf{0} \\ A_{\mathrm{P}}^{\mathrm{Z}} & A_{\mathrm{U}}^{\mathrm{Z}} & I_{m-p-u} \end{bmatrix}, \tag{19}$$

whereas, when only keeping $\boldsymbol{\lambda}_{\mathrm{P}}$ in a basis for (17), it becomes:

$$A_{\mathrm{B}} = \begin{bmatrix} A_{\mathrm{P}}^{\mathrm{P}} & \mathbf{0} & \mathbf{0} \\ A_{\mathrm{P}}^{\mathrm{U}} & I_u & \mathbf{0} \\ A_{\mathrm{P}}^{\mathrm{Z}} & \mathbf{0} & I_{m-p-u} \end{bmatrix} . \tag{20}$$

Is there any advantage of taking one or the other version of rMP or rSP? In the reminder of this subsection, we derive the transformed master problems MP^{P} and $\text{MP}^{\text{P}\cup\text{U}}$ identified by the row-set kept in rMP, that is, by either P or $\text{P} \cup \text{U}$. Proposition 4 shows that the corresponding pricing subproblems rSP^{P} and $\text{rSP}^{\text{P}\cup\text{U}}$ are equivalent although they are clearly not of the same size.

Similarly to the transformed matrix defined in (6), let $\bar{A}^{\text{U}}_{\text{U}} = A^{\text{U}}_{\text{U}} - A^{\text{U}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{U}}$, $\bar{A}^{\text{U}}_{\text{L}} = A^{\text{U}}_{\text{L}} - A^{\text{U}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{L}}$, $\bar{A}^{\text{Z}}_{\text{U}} = A^{\text{Z}}_{\text{U}} - A^{\text{Z}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{U}}$, and $\bar{A}^{\text{Z}}_{\text{L}} = A^{\text{Z}}_{\text{L}} - A^{\text{Z}}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{L}}$.

On the one hand, when the MP (18) if left-multiplied by the inverse of basis (20), the transformed master problem MP^{P} is as follows:

$$
\begin{array}{llllllll}
\mathbf{c}^{\intercal}_{\text{U}}\mathbf{u}_{\text{U}} + \min & \mathbf{c}^{\intercal}_{\text{P}}\boldsymbol{\lambda}_{\text{P}} & - \mathbf{c}^{\intercal}_{\text{U}}\boldsymbol{\lambda}^{-}_{\text{U}} & + \mathbf{c}^{\intercal}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & & \\
\text{s.t.} & \boldsymbol{\lambda}_{\text{P}} & - (A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{U}}\boldsymbol{\lambda}_{\text{U}} & + (A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & = & (A^{\text{P}}_{\text{P}})^{-1}(\mathbf{b}^{\text{P}} - A^{\text{P}}_{\text{U}}\mathbf{u}_{\text{U}}) \\
 & & - \bar{A}^{\text{U}}_{\text{U}}\boldsymbol{\lambda}^{-}_{\text{U}} & + \bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & = & \mathbf{0} \\
 & & - \bar{A}^{\text{Z}}_{\text{U}}\boldsymbol{\lambda}^{-}_{\text{U}} & + \bar{A}^{\text{Z}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & = & \mathbf{0} \\
 & \mathbf{0} \leq \boldsymbol{\lambda}_{\text{P}} \leq \mathbf{u}_{\text{P}}, & \mathbf{0} \leq \boldsymbol{\lambda}^{-}_{\text{U}} \leq \mathbf{u}_{\text{U}}, & \mathbf{0} \leq \boldsymbol{\lambda}_{\text{L}} \leq \mathbf{u}_{\text{L}} \ . & &
\end{array}
\tag{21}
$$

The pricing subproblem rSP^{P} with $m - p + 1$ constraints is therefore given by

$$
\begin{array}{llllllll}
\bar{c}^{\star}_{\text{rSP}^{\text{P}}} := \min & & - \tilde{\mathbf{c}}^{\intercal}_{\text{U}}\boldsymbol{\lambda}^{-}_{\text{U}} & + \tilde{\mathbf{c}}^{\intercal}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & & \\
\text{s.t.} & \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{P}} & + \mathbf{1}^{\intercal}\boldsymbol{\lambda}^{-}_{\text{U}} & + \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{L}} & = & 1 \\
 & & - \bar{A}^{\text{U}}_{\text{U}}\boldsymbol{\lambda}^{-}_{\text{U}} & + \bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & = & \mathbf{0} \\
 & & - \bar{A}^{\text{Z}}_{\text{U}}\boldsymbol{\lambda}^{-}_{\text{U}} & + \bar{A}^{\text{Z}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & = & \mathbf{0} \\
 & \boldsymbol{\lambda}_{\text{P}}, & \boldsymbol{\lambda}^{-}_{\text{U}}, & \boldsymbol{\lambda}_{\text{L}} & \geq & \mathbf{0} \ ,
\end{array}
\tag{22}
$$

where $\tilde{\mathbf{c}}^{\intercal}_{\text{P}} = \mathbf{c}^{\intercal}_{\text{P}} - \mathbf{c}^{\intercal}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{P}} = \mathbf{0}$, $\tilde{\mathbf{c}}^{\intercal}_{\text{U}} = \mathbf{c}^{\intercal}_{\text{U}} - \mathbf{c}^{\intercal}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{U}}$, and $\tilde{\mathbf{c}}^{\intercal}_{\text{L}} = \mathbf{c}^{\intercal}_{\text{L}} - \mathbf{c}^{\intercal}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{L}}$.

On the other hand, applying the inverse of basis (19) to the MP (1) (for which no explicit upper bounds appear on the variables although $\boldsymbol{\lambda}_{\text{U}} \leq \mathbf{u}_{\text{U}}$ is implicitly known), the transformed master problem $\text{MP}^{\text{P}\cup\text{U}}$ is as follows:

$$
\begin{array}{llllll}
\min & \mathbf{c}^{\intercal}_{\text{P}}\boldsymbol{\lambda}_{\text{P}} & + \mathbf{c}^{\intercal}_{\text{U}}\boldsymbol{\lambda}_{\text{U}} & + \mathbf{c}^{\intercal}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & & \\
\text{s.t.} & \boldsymbol{\lambda}_{\text{P}} & & + \hat{A}^{\text{P}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & = & (A^{\text{P}}_{\text{P}})^{-1}\mathbf{b}^{\text{P}} \\
 & & \boldsymbol{\lambda}_{\text{U}} & + (\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & = & \mathbf{u}_{\text{U}} \\
 & & & (\bar{A}^{\text{Z}}_{\text{L}} - \bar{A}^{\text{Z}}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}})\boldsymbol{\lambda}_{\text{L}} & = & \mathbf{0} \\
 & \boldsymbol{\lambda}_{\text{P}}, & \boldsymbol{\lambda}_{\text{U}}, & \boldsymbol{\lambda}_{\text{L}} & \geq & \mathbf{0} \ ,
\end{array}
\tag{23}
$$

where $\hat{A}^{\text{P}}_{\text{L}} = (A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{L}} - (A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}$. Note that from implicit modeling considerations in the second set of constraints, $(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} = \mathbf{u}_{\text{U}} - \boldsymbol{\lambda}_{\text{U}} \geq \mathbf{0}$. In subproblem $\text{rSP}^{\text{P}\cup\text{U}}$, cost coefficients are zero for $\boldsymbol{\lambda}_{\text{P}}$ and $\boldsymbol{\lambda}^{-}_{\text{U}}$ but those of $\boldsymbol{\lambda}_{\text{L}}$ are computed as follows:

$$
\begin{array}{lll}
\mathbf{c}^{\intercal}_{\text{L}} - \mathbf{c}^{\intercal}_{\text{P}}\hat{A}^{\text{P}}_{\text{L}} - \mathbf{c}^{\intercal}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}} & = & \mathbf{c}^{\intercal}_{\text{L}} - \mathbf{c}^{\intercal}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{L}} + \mathbf{c}^{\intercal}_{\text{P}}(A^{\text{P}}_{\text{P}})^{-1}A^{\text{P}}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}} - \mathbf{c}^{\intercal}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}} \\
 & = & \tilde{\mathbf{c}}^{\intercal}_{\text{L}} - \tilde{\mathbf{c}}^{\intercal}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}} \ .
\end{array}
$$

Hence, the pricing subproblem $\text{rSP}^{\text{P}\cup\text{U}}$ with only $m - p - u + 1$ constraints is given by:

$$
\begin{array}{llllll}
\bar{c}^{\star}_{\text{rSP}^{\text{P}\cup\text{U}}} := & \min & & & (\tilde{\mathbf{c}}^{\intercal}_{\text{L}} - \tilde{\mathbf{c}}^{\intercal}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}})\boldsymbol{\lambda}_{\text{L}} & & \\
 & \text{s.t.} & \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{P}} & + \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{U}} & + \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{L}} & = & 1 \\
 & & & & (\bar{A}^{\text{Z}}_{\text{L}} - \bar{A}^{\text{Z}}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}})\boldsymbol{\lambda}_{\text{L}} & = & \mathbf{0} \\
 & & \boldsymbol{\lambda}_{\text{P}}, & \boldsymbol{\lambda}_{\text{U}}, & \boldsymbol{\lambda}_{\text{L}} & \geq & \mathbf{0} \ .
\end{array}
\tag{24}
$$

Proposition 4. *Pricing subproblems* rSP^{P} (22) *and* $\text{rSP}^{\text{P}\cup\text{U}}$ (24) *are equivalent. At any iteration, they either prove the optimality of the current solution of the master problem or provide the same improving direction.*

Proof. The proof is in two steps. Firstly, we show that any solution to rSP^{P} is a solution to $\text{rSP}^{\text{P}\cup\text{U}}$. Secondly, we show the converse.

In any solution to rSP^{P} in (22), $\bar{A}^{\text{U}}_{\text{U}}\boldsymbol{\lambda}^{-}_{\text{U}} = \bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}}$, hence $\boldsymbol{\lambda}^{-}_{\text{U}} = (\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} \geq \mathbf{0}$. By substitution in rSP^{P}, we almost obtain the second subproblem $\text{rSP}^{\text{P}\cup\text{U}}$:

$$
\begin{array}{rllllll}
\bar{c}^{\star}_{\text{SP}^{\text{P}}} := & \min & & & (\tilde{\mathbf{c}}^{\intercal}_{\text{L}} - \tilde{\mathbf{c}}^{\intercal}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}})\boldsymbol{\lambda}_{\text{L}} & & \\
& \text{s.t.} & \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{P}} & + \mathbf{1}^{\intercal}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} & + \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{L}} & = & 1 \\
& & & & (\bar{A}^{\text{Z}}_{\text{L}} - \bar{A}^{\text{Z}}_{\text{U}}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}})\boldsymbol{\lambda}_{\text{L}} & = & \mathbf{0} \\
& & \boldsymbol{\lambda}_{\text{P}}, & (\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}}, & \boldsymbol{\lambda}_{\text{L}} & \geq & \mathbf{0}\ .
\end{array}
\tag{25}
$$

When $(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} = \mathbf{0}$, $\bar{c}^{\star}_{\text{rSP}^{\text{P}}} = \bar{c}^{\star}_{\text{rSP}^{\text{P}\cup\text{U}}}$ and both subproblems are identical since variables $\boldsymbol{\lambda}_{\text{P}}$ and $\boldsymbol{\lambda}_{\text{U}}$ in the second subproblem can be merged as they act as slack variables. When $(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} \neq \mathbf{0}$, the current solution to rMP^{P} is not optimal and $\boldsymbol{\lambda}_{\text{P}} = \mathbf{0}$. Therefore $\mathbf{1}^{\intercal}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} > 0$ and $\boldsymbol{\lambda}_{\text{L}} \neq \mathbf{0}$. Consequently, $0 < \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{L}} < 1$ to satisfy the convexity constraint of (25). Let $\mathbf{1}^{\intercal}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} = \alpha$; then $\mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{L}} = 1 - \alpha$. The rSP^{P}-solution is feasible for the second subproblem by fixing $\boldsymbol{\lambda}_{\text{P}}$ and $\boldsymbol{\lambda}_{\text{U}}$ to zero and by applying a scaling on the $\boldsymbol{\lambda}_{\text{L}}$-variables, i.e., $\frac{\boldsymbol{\lambda}_{\text{L}}}{(1-\alpha)}$. Observe that $\bar{c}^{\star}_{\text{rSP}^{\text{P}}} = (1-\alpha)\ \bar{c}^{\star}_{\text{rSP}^{\text{P}\cup\text{U}}}$.

Any solution $\boldsymbol{\lambda}_{\text{L}} \neq \mathbf{0}$ to $\text{rSP}^{\text{P}\cup\text{U}}$ in (24) implies that the current solution to $\text{rMP}^{\text{P}\cup\text{U}}$ is not optimal and hence, $\mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{P}} + \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{U}} = 0$. Let $\boldsymbol{\lambda}^{-}_{\text{U}} = (\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}}$. The implicit modeling considerations of (23) imply that $\mathbf{u}_{\text{U}} - \boldsymbol{\lambda}_{\text{U}} = (\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} \geq \mathbf{0}$. Hence, the second and third constraint sets of rSP^{P} are satisfied. Our last concern is the convexity constraint in (22). Compute $\beta = \mathbf{1}^{\intercal}(\bar{A}^{\text{U}}_{\text{U}})^{-1}\bar{A}^{\text{U}}_{\text{L}}\boldsymbol{\lambda}_{\text{L}} = \mathbf{1}^{\intercal}\boldsymbol{\lambda}^{-}_{\text{U}}$. Therefore, $\mathbf{1}^{\intercal}\boldsymbol{\lambda}^{-}_{\text{U}} + \mathbf{1}^{\intercal}\boldsymbol{\lambda}_{\text{L}} = 1 + \beta$ and a scaling of these variables (i.e., $\frac{\boldsymbol{\lambda}^{-}_{\text{U}}}{(1+\beta)}$ and $\frac{\boldsymbol{\lambda}_{\text{L}}}{(1+\beta)}$) shows that a solution $\boldsymbol{\lambda}_{\text{L}} \neq \mathbf{0}$ to $\text{rSP}^{\text{P}\cup\text{U}}$ can be transformed into a solution for rSP^{P}. When $\boldsymbol{\lambda}_{\text{L}} = \mathbf{0}$ the equivalence is trivial since the current solution to the master is optimal. Observe that $\bar{c}^{\star}_{\text{rSP}^{\text{P}}} = (\frac{1}{1+\beta})\ \bar{c}^{\star}_{\text{rSP}^{\text{P}\cup\text{U}}}$. □

In case of implicit upper bounds on variables, both subproblems are equivalent and a scaling of the variables is sufficient to show that. They conclude to the optimality of the current rMP solution or provide the same improving direction. The values of the objective functions might be different but are such that $\bar{c}^{\star}_{\text{rSP}^{\text{P}}} = (1-\alpha)\bar{c}^{\star}_{\text{rSP}^{\text{P}\cup\text{U}}} = (\frac{1}{1+\beta})\bar{c}^{\star}_{\text{rSP}^{\text{P}\cup\text{U}}}$, hence $\beta = (\frac{\alpha}{1-\alpha})$ for $0 \leq \alpha < 1$.

5.2 ICG with Inequality Constraints

The *cutting stock problem* (CSP) is a classical application of column generation, see Gilmore and Gomory (1961, 1963). It calls for determining the minimum number of rolls of width W to cut in order to satisfy the demand of m clients with orders of b_i items of width w_i, $i \in \text{M} := \{1, \ldots, m\}$. All parameters are assumed to be positive integers. A roll can be cut in various ways, called *cutting patterns*, where a_{ij} represents the number of items of width w_i cut in pattern j. Let N denote the set of all n feasible cutting patterns, that is $\sum_{i\in\text{M}} a_{ij}w_i \leq W$, for $j \in \text{N}$. Variable λ_j represents the number of rolls cut according to pattern $j \in \text{N}$. The formulation is as follows:

$$
\min \quad \mathbf{1}^{\intercal}\boldsymbol{\lambda} \quad \text{s.t.} \quad A\boldsymbol{\lambda} \geq \mathbf{b}, \quad \boldsymbol{\lambda} \in \mathbb{Z}^{n}_{+}\ .
$$

Surplus variables transform the above inequalities into equalities. Any basic solution to the linear relaxation is given by a number of patterns represented by the $\boldsymbol{\lambda}$-variables and a certain number of surplus variables. Some non-degenerate surplus variables may be in the current basis. In this subsection, we show that the rows associated with these need not be considered in the row-reduced master problem rMP, which results in an even more reduced problem. We also discuss the impact on the subproblem formulation rSP and its specialized version cSP. Finally, we analyze the impact of keeping the positive surplus variables in the basis.

Consider the following linear master problem MP with greater-or-equal inequality constraints:

$$\begin{array}{lrcl} \min & \mathbf{c}^\intercal \boldsymbol{\lambda} & & \\ \text{s.t.} & A\boldsymbol{\lambda} & \geq & \mathbf{b} \\ & \boldsymbol{\lambda} & \geq & \mathbf{0}\ . \end{array} \tag{26}$$

Introducing a vector of surplus variables $\boldsymbol{\delta} \in \mathbb{R}_+^m$, one obtains MP in standard form:

$$\begin{array}{lrcl} \min & \mathbf{c}^\intercal \boldsymbol{\lambda} & & \\ \text{s.t.} & A\boldsymbol{\lambda} \ - \boldsymbol{\delta} & = & \mathbf{b} \\ & \boldsymbol{\lambda} & \geq & \mathbf{0} \\ & \boldsymbol{\delta} & \geq & \mathbf{0}\ . \end{array} \tag{27}$$

The case with less-or-equal inequality constraints $A\boldsymbol{\lambda} \leq \mathbf{b}$ can be treated in a similar way with the addition of slack variables, or by considering the transformation $-A\boldsymbol{\lambda} \geq -\mathbf{b}$.

When the current basic solution to (27) with non-degenerate variables $\boldsymbol{\lambda}_{\text{P}} > \mathbf{0}$, $\text{P} \subseteq \text{B}$, is such that $\mathbf{b}^{'} := A\boldsymbol{\lambda}_{\text{P}} \neq \mathbf{b}$, the basis also contains non-degenerate surplus variables $\boldsymbol{\delta}_{\text{S}} > \mathbf{0}$, for $\text{S} \subseteq \text{B}$. As for index set P used as a column-set or a row-set when appropriate, let $\text{S} \subseteq \text{M}$ also denote the index set of rows where vector $\boldsymbol{\delta}_{\text{S}} > \mathbf{0}$ is basic. Therefore, $\boldsymbol{\delta}_{\text{S}} = \mathbf{b}^{'}_{\text{S}} - \mathbf{b}_{\text{S}} > \mathbf{0}$. Let $|\text{P}| = p$ and $|\text{S}| = s$. Hence, according to Section 3, the row-size of rMP should be $p + s$, the number of non-degenerate variables.

To show that rows in S can be discarded from rMP, make the following change of variables regarding the surplus variables, i.e, redefine vector $\boldsymbol{\delta} \in \mathbb{R}_+^m$ as

$$\boldsymbol{\delta} := \mathbf{b}^{'} - \mathbf{b} + \boldsymbol{\delta}^+ - \boldsymbol{\delta}^- \ ,$$

where $\boldsymbol{\delta}^+ \geq \mathbf{0}$ and $\mathbf{0} \leq \boldsymbol{\delta}^- \leq \mathbf{b}^{'} - \mathbf{b}$. For $i \in \text{M}$, δ_i^+ allows for increasing the current value of δ_i whereas δ_i^- allows for a decrease of it by at most $b_i^{'} - b_i$ units. As a function of the current solution $\boldsymbol{\lambda}_{\text{P}} > \mathbf{0}$ such that $A_{\text{P}}\boldsymbol{\lambda}_{\text{P}} = \mathbf{b}^{'} \geq \mathbf{b}$, MP with inequalities in (26) can be written as follows:

$$\begin{array}{lrcl} \min & \mathbf{c}^\intercal \boldsymbol{\lambda} & & \\ \text{s.t.} & A\boldsymbol{\lambda} \ - \boldsymbol{\delta}^+ + \boldsymbol{\delta}^- & = & \mathbf{b}^{'} \\ & \boldsymbol{\lambda} & \geq & \mathbf{0} \\ & \boldsymbol{\delta}^+ & \geq & \mathbf{0} \\ & \boldsymbol{\delta}^- & \in & [\mathbf{0}, \mathbf{b}^{'} - \mathbf{b}]\ . \end{array} \tag{28}$$

Given that $A_{\text{P}}\boldsymbol{\lambda}_{\text{P}} = \mathbf{b}^{'}$, then $\boldsymbol{\delta}^+ = \boldsymbol{\delta}^- = \mathbf{0}$ and these variables are degenerate in (28). Therefore, the row-size of rMP can be reduced from $p + s$ to only p constraints and basis A_{B} can be taken as in (5). Similarly to the definition of $\bar{A}_{\text{I}}^{\text{Z}}$ in (6), let $\bar{I}_{\text{P}}^{\text{Z}} := I_{m-p} - A_{\text{P}}^{\text{Z}}(A_{\text{P}}^{\text{P}})^{-1}I_p$. While keeping the first set of constraints unchanged, the transformed MP, according to the row-partition P and Z and in terms of the column-partition with respect to compatible and incompatible variables in sets C and I, becomes:

$$\begin{array}{llllllllll} \min & \mathbf{c}_{\text{C}}^\intercal \boldsymbol{\lambda}_{\text{C}} & + & \mathbf{c}_{\text{I}}^\intercal \boldsymbol{\lambda}_{\text{I}} & & & & & & \\ \text{s.t.} & A_{\text{C}}^{\text{P}} \boldsymbol{\lambda}_{\text{C}} & + & A_{\text{I}}^{\text{P}} \boldsymbol{\lambda}_{\text{I}} & - & \boldsymbol{\delta}_{\text{P}}^+ & & + \ \boldsymbol{\delta}_{\text{P}}^- & = & (\mathbf{b}^{'})^{\text{P}} \quad [\boldsymbol{\pi}^{\text{P}}] \\ & & & \bar{A}_{\text{I}}^{\text{Z}} \boldsymbol{\lambda}_{\text{I}} & - & \bar{I}_{\text{P}}^{\text{Z}} \boldsymbol{\delta}_{\text{P}}^+ & - \ \boldsymbol{\delta}_{\text{Z}}^+ & + \ \bar{I}_{\text{P}}^{\text{Z}} \boldsymbol{\delta}_{\text{P}}^- + \boldsymbol{\delta}_{\text{Z}}^- & = & \mathbf{0} \quad [\boldsymbol{\zeta}^{\text{Z}}] \\ & & & & & & & \boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\text{C}}, \boldsymbol{\lambda}_{\text{I}}) & \geq & \mathbf{0} \\ & & & & & & & \boldsymbol{\delta}^+ = (\boldsymbol{\delta}_{\text{P}}^+, \boldsymbol{\delta}_{\text{Z}}^+) & \geq & \mathbf{0} \\ & & & & & & & \boldsymbol{\delta}^- = (\boldsymbol{\delta}_{\text{P}}^-, \boldsymbol{\delta}_{\text{Z}}^-) & \in & [\mathbf{0}, \bar{\mathbf{b}} - \mathbf{b}]\ . \end{array} \tag{29}$$

Observe the modified right-hand side $(\mathbf{b}^{'})^{\text{P}} := A_{\text{P}}^{\text{P}}\boldsymbol{\lambda}_{\text{P}}$ computed according to the current solution and the fact that variables $\boldsymbol{\delta}^+$ and $\boldsymbol{\delta}^-$ are incompatible. Additionally, row-index Z is also used as a row-and-column index for the $\boldsymbol{\delta}^+$ and $\boldsymbol{\delta}^-$ variables. From (29), we can derive rMP, rSP, and cSP in the case of inequality constraints. The *row-reduced* master problem with p constraints is obtained by fixing all variables to zero except the compatible ones

$$\begin{array}{rrcll} \min & \mathbf{c}_{\mathrm{C}}^{\intercal}\boldsymbol{\lambda}_{\mathrm{C}} & & & \\ \text{s.t.} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & = & (\mathbf{b}')^{\mathrm{P}} & [\boldsymbol{\pi}^{\mathrm{P}}] \\ & \boldsymbol{\lambda}_{\mathrm{C}} & \geq & \mathbf{0}\ . & \end{array} \tag{30}$$

and rMP is defined on the current generated subset of C. The specialized subproblem cSP for generating compatible variables remains as in (12):

$$\bar{c}^{\star}_{\mathrm{cSP}} := \min_{\mathbf{x}\in X} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathrm{P}})^{\intercal}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) \ \mid\ \mathbf{a}^{\mathrm{Z}}(\mathbf{x}) - A_{\mathrm{P}}^{\mathrm{Z}}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) = \mathbf{0} \right\} \ ,$$

whereas, given $(\boldsymbol{\pi}^{\mathrm{P}})^{\intercal} = \mathbf{c}_{\mathrm{P}}^{\intercal}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}$, the pricing subproblem rSP becomes

$$\begin{array}{rrcrcrcll} \bar{c}^{\star}_{\mathrm{rSP}} := \min & \tilde{\mathbf{c}}^{\intercal}\boldsymbol{\lambda} & + & \mathbf{c}_{\mathrm{P}}^{\intercal}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}\boldsymbol{\delta}_{\mathrm{P}}^{+} & - & \mathbf{c}_{\mathrm{P}}^{\intercal}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}\boldsymbol{\delta}_{\mathrm{P}}^{-} & & & \\ \text{s.t.} & \mathbf{1}^{\intercal}\boldsymbol{\lambda} & + & \mathbf{1}^{\intercal}(\boldsymbol{\delta}_{\mathrm{P}}^{+} + \boldsymbol{\delta}_{\mathrm{Z}}^{+}) & + & \mathbf{1}^{\intercal}(\boldsymbol{\delta}_{\mathrm{P}}^{-} + \boldsymbol{\delta}_{\mathrm{Z}}^{-}) & = & 1 & [\gamma] \\ & \bar{A}^{\mathrm{Z}}\boldsymbol{\lambda} & - & \bar{I}_{\mathrm{P}}^{\mathrm{Z}}\boldsymbol{\delta}_{\mathrm{P}}^{+} - \boldsymbol{\delta}_{\mathrm{Z}}^{+} & + & \bar{I}_{\mathrm{P}}^{\mathrm{Z}}\boldsymbol{\delta}_{\mathrm{P}}^{-} + \boldsymbol{\delta}_{\mathrm{Z}}^{-} & = & \mathbf{0} & [\boldsymbol{\zeta}^{\mathrm{Z}}] \\ & & & & & \boldsymbol{\lambda} & \geq & \mathbf{0} & \\ & & & & & \boldsymbol{\delta}^{+} = (\boldsymbol{\delta}_{\mathrm{P}}^{+}, \boldsymbol{\delta}_{\mathrm{Z}}^{+}) & \geq & \mathbf{0} & \\ & & & & & \boldsymbol{\delta}^{-} = (\boldsymbol{\delta}_{\mathrm{P}}^{-}, \boldsymbol{\delta}_{\mathrm{Z}}^{-}) & \geq & \mathbf{0}\ . & \end{array} \tag{31}$$

Problem rSP in (31) is solved by column generation except that $\boldsymbol{\delta}$-variables need not be generated. Moreover, variables $(\boldsymbol{\delta}_{\mathrm{Z}}^{+}, \boldsymbol{\delta}_{\mathrm{Z}}^{-})$ can be eliminated from the formulation allowing for a greater-or-equal inequality in the second set of constraints. The convexity constraint remains an equality at one upon a scaling of the variables $\boldsymbol{\lambda}$ and $(\boldsymbol{\delta}_{\mathrm{P}}^{+}, \boldsymbol{\delta}_{\mathrm{P}}^{-})$. When $\bar{c}^{\star}_{\mathrm{rSP}} < 0$, the selected variables are introduced in the restricted master problem which is re-optimized. Special attention is given to δ_i^- variables, $i \in \mathrm{M}$: if δ_i^- is chosen to enter the basis of RMP, it is limited by its upper bound $b_i' - b_i$ and the dynamic change of variable process is repeated for defining the new row-partition of rMP.

An Alternative Partition. Even though the above development is mathematically correct, alternative row and column partitions might be computationally more adequate. For example, there are many situations where a set partitioning formulation can be replaced by an equivalent set covering formulation. Therefore, the current solution $A\boldsymbol{\lambda}_{\mathrm{P}}$ may exceed the right-hand side of $\mathbf{1}$ on a number of rows. Since all the surplus variables eventually take a zero value, subproblem rSP in (31) has to manage the introduction of a mix of incompatible $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$ variables in RMP, at the price of an update of the row-partition of rMP. Alternatively, positive surplus variables $\boldsymbol{\delta}_{\mathrm{S}} > \mathbf{0}$ can remain in the basis and hence are allowed to vary within the solution process of rMP. If the density of the compatible columns is not very high, the extra work done in the master problem rather than in the pricing subproblem might be worthwhile. Let us examine the impact of such a choice on the rMP, rSP, and cSP formulations.

Keeping both $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ and $\boldsymbol{\delta}_{\mathrm{S}} > \mathbf{0}$ in the basis, the row-partition is given by sets P, S, and $\mathrm{Z} := \mathrm{M} \setminus (\mathrm{P} \cup \mathrm{S})$, and basis A_{B} writes as

$$A_{\mathrm{B}} = \begin{bmatrix} A_{\mathrm{P}}^{\mathrm{P}} & \mathbf{0} & \mathbf{0} \\ A_{\mathrm{P}}^{\mathrm{S}} & -I_s & \mathbf{0} \\ A_{\mathrm{P}}^{\mathrm{Z}} & \mathbf{0} & I_{m-p-s} \end{bmatrix} \ .$$

The transformed master problem, where only the Z-constraints are modified, becomes:

$$\begin{array}{llllllll}
\min & \mathbf{c}_{\mathrm{C}}^{\intercal}\boldsymbol{\lambda}_{\mathrm{C}} & + & \mathbf{c}_{\mathrm{I}}^{\intercal}\boldsymbol{\lambda}_{\mathrm{I}} & & & & \\
\text{s.t.} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & + & A_{\mathrm{I}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{I}} & - & \boldsymbol{\delta}_{\mathrm{P}} & = & \mathbf{b}^{\mathrm{P}} \quad [\boldsymbol{\pi}^{\mathrm{P}}] \\
& A_{\mathrm{C}}^{\mathrm{S}}\boldsymbol{\lambda}_{\mathrm{C}} - \boldsymbol{\delta}_{\mathrm{S}} & + & A_{\mathrm{I}}^{\mathrm{S}}\boldsymbol{\lambda}_{\mathrm{I}} & & & = & \mathbf{b}^{\mathrm{S}} \quad [\boldsymbol{\pi}^{\mathrm{S}}] \\
& & & \bar{A}_{\mathrm{I}}^{\mathrm{Z}}\boldsymbol{\lambda}_{\mathrm{I}} & - & A_{\mathrm{P}}^{\mathrm{Z}}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}\boldsymbol{\delta}_{\mathrm{P}} - \boldsymbol{\delta}_{\mathrm{Z}} & = & \mathbf{0} \quad [\boldsymbol{\zeta}^{\mathrm{Z}}] \\
& & & & & \boldsymbol{\lambda}_{\mathrm{C}}, \boldsymbol{\lambda}_{\mathrm{I}} & \geq & \mathbf{0} \\
& & & & & \boldsymbol{\delta}_{\mathrm{P}}, \boldsymbol{\delta}_{\mathrm{S}}, \boldsymbol{\delta}_{\mathrm{Z}} & \geq & \mathbf{0}\ .
\end{array} \tag{32}$$

From (32), we can derive the alternative rMP, rSP, and cSP. The row-reduced master problem rMP with $p+s$ constraints is obtained by fixing all variables to zero except for compatible ones $\boldsymbol{\lambda}_{\mathrm{C}}$ while surplus ones $\boldsymbol{\delta}_{\mathrm{S}}$ are removed from the formulation:

$$\begin{array}{llll}
\min & \mathbf{c}_{\mathrm{C}}^{\intercal}\boldsymbol{\lambda}_{\mathrm{C}} & & \\
\text{s.t.} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & = & \mathbf{b}^{\mathrm{P}} \quad [\boldsymbol{\pi}^{\mathrm{P}}] \\
& A_{\mathrm{C}}^{\mathrm{S}}\boldsymbol{\lambda}_{\mathrm{C}} & \geq & \mathbf{b}^{\mathrm{S}} \quad [\boldsymbol{\pi}^{\mathrm{S}}] \\
& \boldsymbol{\lambda}_{\mathrm{C}} & \geq & \mathbf{0}\ .
\end{array} \tag{33}$$

Row-restricted master problem rMP is defined on the current generated subset of C. Specialized subproblem cSP for generating compatible variables is slightly modified to account for the additional S-set of constraints in (33) for which the dual variables are $\boldsymbol{\pi}^{\mathrm{S}} \geq \mathbf{0}$:

$$\bar{c}_{\mathrm{cSP}}^{\star} := \min_{\mathbf{x}\in X} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathrm{P}})^{\intercal}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) - (\boldsymbol{\pi}^{\mathrm{S}})^{\intercal}\mathbf{a}^{\mathrm{S}}(\mathbf{x}) \ \middle|\ \mathbf{a}^{\mathrm{Z}}(\mathbf{x}) - A_{\mathrm{P}}^{\mathrm{Z}}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) = \mathbf{0} \right\} \ ,$$

whereas the pricing subproblem rSP, for which the $\boldsymbol{\delta}_{\mathrm{Z}}$ have been removed, becomes

$$\begin{array}{llllllll}
\bar{c}_{\mathrm{rSP}}^{\star} := \min & \tilde{\mathbf{c}}^{\intercal}\boldsymbol{\lambda} & + & \mathbf{c}_{\mathrm{P}}^{\intercal}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}\boldsymbol{\delta}_{\mathrm{P}} & & & \\
\text{s.t.} & \mathbf{1}^{\intercal}\boldsymbol{\lambda} & + & \mathbf{1}^{\intercal}\boldsymbol{\delta}_{\mathrm{P}} & = & 1 & [\gamma] \\
& \bar{A}^{\mathrm{Z}}\boldsymbol{\lambda} & + & A_{\mathrm{P}}^{\mathrm{Z}}(A_{\mathrm{P}}^{\mathrm{P}})^{-1}\boldsymbol{\delta}_{\mathrm{P}} & \geq & \mathbf{0} & [\boldsymbol{\zeta}^{\mathrm{Z}}] \\
& \boldsymbol{\lambda} & , & \boldsymbol{\delta}_{\mathrm{P}} & \geq & \mathbf{0}\ . &
\end{array} \tag{34}$$

Problem rSP in (34) is solved by column generation except that $\boldsymbol{\delta}_{\mathrm{P}}$-variables need not be generated during the solution process. When $\bar{c}_{\mathrm{rSP}}^{\star} < 0$, the selected variables are introduced in the restricted master problem which is re-optimized; otherwise, the current solution $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ is optimal for rMP and for MP.

5.3 Implementation Issues

In the previous Section 5.2 for master problems with inequalities, the row-reduced master problem rMP could be defined in terms of the $p+s$ rows associated with all the non-degenerate variables $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ and $\boldsymbol{\delta}_{\mathrm{S}} > \mathbf{0}$, $\mathrm{P} \cup \mathrm{S} \subseteq \mathrm{M}$. However, we showed that its row-size can be reduced to the p constraints in P only. A similar situation occurred in the case of implicit upper bounds on the variables, see Section 5.1. This indicates that the compatibility notion is somewhat flexible and in the next definition, we bring in compatibility with a subset of rows.

Definition 3. *Given the solution vector* $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ *of non-degenerate variables, vector* $\mathbf{a}$ *is* compatible with row-set $\mathrm{Q} \subseteq \mathrm{P}$ *if and only if* $\bar{\mathbf{a}}^{\mathrm{Z}} = \mathbf{0}$ *for* $\mathrm{Z} := \mathrm{M} \setminus \mathrm{Q}$.

Indeed, row-compatibility induces a two-step pivot procedure with respect to a strict improvement of the current solution. Firstly, a selected row-set $\mathrm{Q} \subseteq \mathrm{P}$ for rMP (still of row-size $p \leq m$) enforces the set of possible exiting variables from the current basis, the ratio-test being computed only for those $|Q| := q \leq p$ constraints. Secondly, a specialized pricing subproblem cSP selects an entering variable λ_j, $j \notin \mathrm{B}$, such that $\bar{c}_j < 0$ and $\bar{a}_j^{\mathrm{Z}} = \mathbf{0}$, this zero vector being defined on $\mathrm{Z} := \mathrm{M} \setminus \mathrm{Q}$ according to the row-compatibility in use. This supports various implementation strategies. For example, one could temporarily restrain the search for entering variables to those for which Q is a strict subset of P. This should accelerate the solution of rMP since the values of the basic variables in $\mathrm{P} \setminus \mathrm{Q}$ remain unchanged.

Alternatively, when the entering variable is compatible with the solution $\boldsymbol{\lambda}_{\mathrm{P}}$, the pivot is non-degenerate but the new basic solution may become degenerate, say $\boldsymbol{\lambda}_{\mathrm{Q}} > \mathbf{0}$, $\boldsymbol{\lambda}_{\mathrm{N}\setminus\mathrm{Q}} = \mathbf{0}$, with $\mathrm{Q} \subseteq \mathrm{P}$. Then one can update or not the actual row-partition of rMP, namely $\mathrm{rMP}^{\mathrm{P}}$. If it is updated, it becomes $\mathrm{rMP}^{\mathrm{Q}}$ with only q rows and $\mathrm{Z} := \mathrm{M} \setminus \mathrm{Q}$. If not, it should be pointed out that this also results in an exact algorithm as it still solves the transformed MP formulation in (7). In that case, the current compatibility rule with row-set P is simply maintained. No overhead computations are needed for an update of $\mathrm{rMP}^{\mathrm{P}}$ and the entering variables are still selected such that $\bar{\mathbf{a}}_j^{\mathrm{Z}} = \mathbf{0}$, where $\mathrm{Z} := \mathrm{M} \setminus \mathrm{P}$ is a strictly smaller set of rows than what would be required in an updated pricing subproblem. In that case, degenerate pivots may occur because of the degeneracy of some $\mathrm{rMP}^{\mathrm{P}}$ solutions.

In the case of implicit upper bounds, Proposition 4 shows that the pricing subproblem does not depend on the fact that the rows corresponding to basic variables at their implicit upper bounds are kept in rMP or not. Then one can solve the smallest row-size rMP with p rows corresponding to $\mathbf{0} < \boldsymbol{\lambda}_{\mathrm{P}} < \mathbf{u}_{\mathrm{P}}$ and also use the smallest row-size rSP with $m - p - u + 1$ rows. Therefore, both rMP and SP are reduced in row-size.

Another example of implementation strategies to be explored is the following one. Given the solution vector $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ of non-degenerate variables, it might be interesting to relax the compatibility requirements $\bar{\mathbf{a}}_j^{\mathrm{Z}} = \mathbf{0}$ of an entering variable λ_j, $j \notin \mathrm{B}$, to the weaker condition $\bar{\mathbf{a}}_j^{\mathrm{Z}} \leq \mathbf{0}$. This would also give rise to a non-degenerate pivot. However, when $\bar{\mathbf{a}}_j^{\mathrm{Z}} \neq \mathbf{0}$, an update of rMP would be necessary, as the entering variable is incompatible with the current row-partition.

6 Large Scale Applications

Column generation is widely used to solve large scale integer programs. We already introduced the *cutting stock problem* in Section 5.2 for which a large number of publications can be found in the literature regarding its solution by column generation, see Ben Amor and Valério de Carvalho (2005). More than 20 areas of applications are reported in Lübbecke and Desrosiers (2005). In this section, we focus on set partitioning problems.

6.1 Set Partitioning Models

A set partitioning formulation is standard in many large scale applications solved by branch-and-price. In general, we are given elements of a ground set and possible subsets of elements. We have to select a set of subsets at minimum total cost, such that each element appears in a selected subset exactly once. Some examples are *binary clustering problems* (Jans and Desrosiers 2010), the *vertex coloring problem* (Mehrotra and Trick 1996, Nemhauser and Park 1991), the *bin packing problem* (Valério de Carvalho 1999, Vanderbeck 1999) and some of its variants (Peeters and Degraeve 2006), *node capacitated graph partitioning* (Ferreira et al. 1996), and the *vehicle routing problem with time windows* (VRPTW) for which recent developments can be found in Baldacci et al. (2008, 2010) and Jepsen et al. (2008).

Research on methods for solving highly degenerate master problems more efficiently originated from vehicle routing and crew scheduling applications, see Desrosiers et al. (1995) and Desaulniers et al. (1998). In these models, each row of the set partitioning formulation represents a task to perform whereas each column represents a feasible vehicle itinerary or crew schedule. These itineraries and schedules can be represented as paths on adequate time-space networks and generated by solving a variety of resource-constrained shortest path pricing subproblems (Irnich and Desaulniers 2005). Binary coefficient a_{ij} of a column $\mathbf{a}_j$, $j \in \mathrm{N}$, equals 1 if task $i \in \mathrm{M}$ takes place on itinerary or schedule j, and 0 otherwise. In some applications, it is natural that several tasks are aggregated to form a multi-task activity. For example, a pilot usually follows its aircraft so that, if one already knows the aircraft itineraries, some of the consecutive flight legs assigned to an aircraft can tentatively be grouped together, hence reducing the number of (multi-)tasks to choose from for each pilot in a row-reduced formulation. More formally, a set partitioning problem is

$$\min \quad \mathbf{c}^\intercal \boldsymbol{\lambda} \quad \text{s.t.} \quad A\boldsymbol{\lambda} = \mathbf{1}, \quad \boldsymbol{\lambda} \in \{0,1\}^n \ , \tag{35}$$

with $A \in \{0,1\}^{m\times n}$. Such a formulation naturally results from applying a Dantzig-Wolfe decomposition to a multi-commodity flow model in which each vehicle or crew member is represented by a separate commodity, see Desaulniers et al. (1998).

Assume that the current solution $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ to the linear relaxation of (35) is integral (i.e., binary). The corresponding (subsets represented by the) columns of A_{P} are disjoint and induce a partition of the row-set M into p groups. Observe that such a solution usually is highly degenerate: in applications, a group may typically cover, say, ten tasks, which implies that only about one tenth of basic variables assume a positive value. From A_{P}, it is easy to construct a reduced basis: take a single row from each group and therefore, $A^{\mathrm{P}}_{\mathrm{P}} = I_p$, is an identity matrix, and the computation of an inverse matrix is not required. A compatible column $\mathbf{a}_j$, $j \in \mathrm{N}$, is such that all coefficients of a group are identical, that is, either all at zero or all at one. In the context of routing and scheduling, *compatibility* thus means generating itineraries or schedules which precisely respect the current aggregation/grouping into multi-task activities. Of course, this is in perfect harmony with our Definition 2.

If the solution $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ is degenerate and fractional, the partition is derived from the groups of identical rows of A_{P}, and the reduced basis is again an identity matrix. Note that a partition of the rows can also be derived from a heuristic solution, and moreover, it need not be feasible. In any case, a single row per group is kept in the construction of the reduced basis and a column is compatible with this row-partition if, for each group, it covers all its tasks or none of them. The same rules can be applied to all applications formulated as set partitioning problems.

6.2 Computational Results from the Literature

Our aim was to lay the theoretical groundwork for solving highly degenerate master problems by column generation. Generic computational experiments with this framework are beyond our paper's scope. Yet, the literature reports on very encouraging results for four special cases of ICG. In all these, degeneracy is no longer a drawback when solving linear programs.

IPS. A comparison between the *improved primal simplex* method IPS and the primal simplex algorithm of `CPLEX` was performed as follows. At a given iteration of IPS, all columns compatible with the current degenerate solution $\boldsymbol{\lambda}_{\mathrm{P}} > \mathbf{0}$ are identified by an external procedure. The master problem rMP is solved by the primal simplex method over $\boldsymbol{\lambda}_{\mathrm{C}}$. Then, the complementary subproblem iSP is solved by the dual simplex method over all incompatible variables $\boldsymbol{\lambda}_{\mathrm{I}}$ to select a subset with negative reduced costs. The master problem is re-optimized and the row-partition is updated according to the new degenerate solution.

On ten instances involving 2000 constraints and up to 10 000 variables for *simultaneous vehicle and crew scheduling problems in urban mass transit systems*, IPS reduces CPU times by a factor of 4.1 compared to CPLEX (Elhallaoui et al. 2010a, Raymond et al. 2010a). Degeneracy is about 50% for these problems. IPS was also tested on 14 instances of *aircraft fleet assignment*. These consist in maximizing the profits of assigning a type of aircraft to each flight segment over a horizon of one week. The multi-commodity flow formulation comprises 5000 constraints and 25 000 variables; degeneracy is about 65%. Compared to the primal simplex algorithm of CPLEX, IPS reduces CPU times by a factor of 15.8 on average (Raymond et al. 2010a,b).

ubIPS. In the above fleet assignment problems, an upper bound of one was explicitly imposed on arc flow variables (Raymond et al. 2008). When these upper bounds are taken into account in IPS, called ubIPS, CPU times were reduced by a factor of 32.1 on average compared to the primal simplex algorithm of CPLEX. However, a careful look at the computational results shows that ubIPS provides results that are almost identical to those obtained by IPS, that is, without explicitly taking upper bounds into account. A statistical t-test on the paired data shows that there is no significant difference between IPS and ubIPS algorithms on these instances. What happened is that CPLEX considerably slows down when upper bounds are explicitly imposed whereas ubIPS does not. This observation motivated our investigations on ICG with implicit upper bounds on variables in Section 5.1.

DCA. The third special case of ICG is the *dynamic constraints aggregation* method. Indeed, DCA is the first implementation of ICG. It has been specifically designed and used for solving set partitioning formulations of vehicle routing and crew scheduling problems by branch-and-price. As explained in Subsection 6.1, all but one constraint for each row-group are removed from the set partitioning problem. Each such row-group identifies the tasks that are aggregated to form a multi-task activity. The selected rows provide the row-size of rMP which is solved over generated compatible variables only, that is, those itineraries or schedules that respect the current task aggregation. As pricing subproblems in the mentioned applications are solved on network representations of itineraries and schedules, compatibility requirements can easily be imposed within the pricing subproblems by removing unnecessary arcs or by creating multi-task arcs. Once rMP has been solved to optimality, the unknown dual variable values are recovered by solving shortest path problems, and incompatible variables are generated next. For problems with 2000 constraints and 25 000 variables, with 50% to 60% degenerate basic variables, the implementation of DCA in the GENCOL software system (Elhallaoui et al. 2005, 2008, 2010b) allows a reduction of solution times by factors of 50 to 100.

The CPU reduction obtained by DCA over the classical branch-and-price implementation is a combination of many factors: a smaller rMP dynamically adjusted in terms of the number of rows, reduced number of degenerate pivots, smaller cSP (the task aggregation is also done within the underlying time-space networks for the generation of compatible columns), restricted versions of rSP to generate incompatible columns with so-called low-rank incompatibilities, smaller number of column generation iterations, less fractional linear programming relaxations, and smaller branch-and-price search trees.

SDCA. To overcome degeneracy, Benchimol et al. (2011) propose a *stabilized* DCA, called SDCA, that combines the above mentioned DCA and the dual variable stabilization (DVS) method of Oukil et al. (2007). The rationale behind this combination is that the first method reduces the primal space whereas the second acts in the dual space. Thus, combining both allows to fight degeneracy from primal and dual perspectives simultaneously. This method is again designed for solving the linear relaxation of set partitioning type models only. The computational results obtained on randomly generated instances of the multi-depot vehicle scheduling problem show that SDCA is not affected by degeneracy and that it can reduce the average computational time of the master problem by a factor of up to 7 with respect to DVS.

7 Summary and Conclusions

Classical column generation works with a restricted master problem (RMP), that is, a subset of a model's variables that are dynamically added via a pricing subproblem SP. Like the simplex method, column generation is known to suffer from degeneracy. Inspired by recent successes in coping with degeneracy in the primal simplex method, we propose an *improved column generation* method (ICG). ICG *exploits* degeneracy and operates with a reduced restricted master problem (rMP), that is, it additionally reduces the row-size of RMP: The rMP only has a many rows a there are non-zero basic variables.

Columns/variables are characterized as compatible or incompatible with respect to rMP. Compatible columns allow for a strict decrease of the objective function when entered into the basis, that is, a non-degenerate pivot. Two types of subproblems are proposed to generate variables: a specialized subproblem cSP for compatible variables, and rSP to price out incompatible variables. The latter needs to be solved by column generation itself.

Pricing subproblem cSP simply is the original subproblem SP augmented with a set of linear constraints imposing compatibility requirements. It selects compatible columns as long as they are useful for non-degenerate pivots in rMP. When the reduced costs of all compatible columns are zero (or larger than a specified threshold), the pricing subproblem rSP is solved. It selects a convex combination of incompatible columns such that, again, the objective value strictly improves when they all enter into the current basis. In this case, the row-size of rMP is dynamically modified.

As for any simplex or column generation algorithm, future work is needed on ICG, mainly on implementation strategies. Amongst these are the moment for an update of the row-partition of rMP, efficient solvers for cSP and rSP, which pricing subproblem to call, whether to add many incompatible columns or their (single column) convex combination as returned from rSP, and much more on relaxed compatibility and restricted incompatibility.

Decomposition in column generation for integer programs is based on the modeling structure of a compact formulation. It exploits the pricing subproblem for the selection of objects $\mathbf{x} \in X$. Column generation solves linear programs with a huge number of variables but uses only a very small subset of them. One of its main drawbacks, as for the primal simplex algorithm, is the effect of degenerate iterations encountered during the solution process. Row-partition in ICG, indeed an additional decomposition applied on the master problem, takes advantage of degenerate solutions. This approach reduces the row-size of the restricted master problem and its associated basis, and thus, the computational effort for re-optimization.

Combining column generation and dynamic row-partition during the solution of a problem allows for exploiting both the modeling structure of its formulation and the algebraic structure of its solutions. Two special cases of ICG are the *improved primal simplex* method and the *dynamic constraints aggregation*. On highly degenerate instances, recent computational experiments with these algorithms have shown that the row-reduction of a problem might have a larger impact on the solution time than the column-reduction. This opens the door to a large amount of research within that field.

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References

R. Baldacci, N. Christofides, and A. Mingozzi. An exact algorithm for the vehicle routing problem based on the set partitioning formulation with additional cuts. *Mathematical Programming*, 115(2):351–385, 2008. doi: 10.1007/s10107-007-0178-5.

R. Baldacci, E. Bartolini, A. Mingozzi, and R. Roberti. An exact solution framework for a broad class of vehicle routing problems. *Computational Management Science*, 7(3):229–268, 2010. doi: 10.1007/s10287-009-0118-3.

C. Barnhart, E.L. Johnson, G.L. Nemhauser, M.W.P. Savelsbergh, and P.H. Vance. Branch-and-price: Column generation for solving huge integer programs. *Operations Research*, 46(3):316–329, 1998. doi: 10.1287/opre.46.3.316.

H. Ben Amor and J.M. Valério de Carvalho. Cutting stock problems. In G. Desaulniers, J. Desrosiers, and M.M. Solomon, editors, *Column Generation*, pages 131–161. Springer, 2005. doi: 10.1007/0-387-25486-2_5.

P. Benchimol, G. Desaulniers, and J. Desrosiers. Stabilized dynamic constraint aggregation for solving set partitioning problems. Les Cahiers du GERAD G-2011-56, HEC Montréal, Canada, 2011.

G.B. Dantzig and P. Wolfe. Decomposition principle for linear programs. *Operations Research*, 8(1):101–111, 1960. doi: 10.1287/opre.8.1.101.

G. Desaulniers, J. Desrosiers, I. Ioachim, M.M. Solomon, F. Soumis, and D. Villeneuve. A unified framework for deterministic time constrained vehicle routing and crew scheduling problems. In T.G. Crainic and G. Laporte, editors, *Fleet Management and Logistics*, pages 57–93. Kluwer, 1998.

J. Desrosiers, Y. Dumas, M.M. Solomon, and F. Soumis. Time constrained routing and scheduling. In M. Ball, T. Magnanti, C. Monma, and G. Nemhauser, editors, *Handbooks in Operations Research and Management Science, Vol. 8: Network Routing*, pages 35–139. Elsevier B.V, 1995. doi: 10.1016/S0927-0507(05)80106-9.

I. Elhallaoui, D. Villeneuve, F. Soumis, and G. Desaulniers. Dynamic aggregation of set partitioning constraints in column generation. *Operations Research*, 53(4):632–645, 2005. doi: 10.1287/opre.1050.0222.

I. Elhallaoui, G. Desaulniers, A. Metrane, and F. Soumis. Bi-dynamic constraint aggregation and subproblem reduction. *Computers & Operations Research*, 35(5):1713–1724, 2008. doi: 10.1016/j.cor.2006.10.007.

I. Elhallaoui, A. Metrane, G. Desaulniers, and F. Soumis. An Improved Primal Simplex algorithm for degenerate linear programs. *INFORMS Journal on Computing*, 2010a. doi: 10.1287/ijoc.1100.0425.

I. Elhallaoui, A. Metrane, F. Soumis, and G. Desaulniers. Multi-phase dynamic constraint aggregation for set partitioning type problems. *Mathematical Programming*, 123(2):345–370, 2010b. doi: 10.1007/s10107-008-0254-5.

C.E. Ferreira, Martin, C.C. de Souza, R. Weismantel, and L.A. Wolsey. Formulations and valid inequalities for the node capacitated graph partitioning problem. *Mathematical Programming*, 74(3):247–266, 1996. doi: 10.1007/BF02592198.

P.C. Gilmore and R.E. Gomory. A linear programming approach to the cutting-stock problem. *Operations Research*, 9(6):849–859, 1961. doi: 10.1287/opre.9.6.849.

P.C. Gilmore and R.E. Gomory. A linear programming approach to the cutting-stock problem. Part II. *Operations Research*, 11(6):863–888, 1963. doi: 10.1287/opre.11.6.863.

S. Irnich and G. Desaulniers. Shortest path problems with resource constraints. In G. Desaulniers, J. Desrosiers, and M.M. Solomon, editors, *Column Generation*, pages 33–66. Springer, 2005. doi: 10.1007/0-387-25486-2_2.

R. Jans and J. Desrosiers. Binary clustering problems: Symmetric, asymmetric and decomposition formulations. Les Cahiers du GERAD G-2010-44, HEC Montréal, Canada, 2010.

M. Jepsen, B. Petersen, S. Spoorendonk, and D. Pisinger. Subset-row inequalities applied to the vehicle-routing problem with time windows. *Operations Research*, 56(2):497–511, 2008. doi: 10.1287/opre.1070.0449.

M.E. Lübbecke and J. Desrosiers. Selected topics in column generation. *Operations Research*, 53(6):1007–1023, 2005. doi: 10.1287/opre.1050.0234.

A. Mehrotra and M.A. Trick. A column generation approach for graph coloring. *INFORMS Journal on Computing*, 8(4):344–354, 1996. doi: 10.1287/ijoc.8.4.344.

G.L. Nemhauser and S. Park. A polyhedral approach to edge coloring. *Operations Research Letters*, 10(6): 315–322, 1991. doi: 10.1016/0167-6377(91)90003-8.

A. Oukil, H. Ben Amor, J. Desrosiers, and H. Elgueddari. Stabilized column generation for highly degenerate multiple-depot vehicle scheduling problems. *Computers & Operations Research*, 34(3):817–834, 2007. doi: 10.1016/j.cor.2005.05.011.

P.-Q. Pan. A basis deficiency-allowing variation of the simplex method for linear programming. *Computers & Mathematics with Applications*, 36(3):33–53, 1998. doi: 10.1016/S0898-1221(98)00127-8.

M. Peeters and Z. Degraeve. Branch-and-price algorithms for the dual bin packing and maximum cardinality bin packing problem. *European Journal of Operational Research*, 170(2):416–439, 2006. doi: 10.1016/j.ejor.2004.06.034.

V. Raymond, F. Soumis, and A. Metrane. Improved primal simplex version 3: Cold start, generalization for bounded variable problems and a new implementation. Les Cahiers du GERAD G-2009-15, HEC Montréal, Canada, 2008.

V. Raymond, F. Soumis, A. Metrane, and J. Desrosiers. Positive edge: A pricing criterion for the identification of non-degenerate simplex pivots. Les Cahiers du GERAD G-2010-61, HEC Montréal, Canada, 2010a.

V. Raymond, F. Soumis, and D. Orban. A new version of the Improved Primal Simplex for degenerate linear programs. *Computers & Operations Research*, 37(1):91–98, 2010b. doi: 10.1016/j.cor.2009.03.020.

J.M. Valério de Carvalho. Exact solution of bin-packing problems using column generation and branch-and-bound. *Annals of Operations Research*, 86(0):629–659, 1999. doi: 10.1023/A:1018952112615.

F. Vanderbeck. Computational study of a column generation algorithm for bin packing and cutting stock problems. *Mathematical Programming*, 86(3):565–594, 1999. doi: 10.1007/s101070050105.