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CONSTRUCTIVE APPROXIMATION

Faber Polynomials Corresponding to Rational Exterior Mapping Functions

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Abstract. Faber polynomials corresponding to rational exterior mapping functions of degree $(m, m-1)$ are studied. It is shown that these polynomials always satisfy an $(m+1)$-term recurrence. For the special case $m = 2$, it is shown that the Faber polynomials can be expressed in terms of the classical Chebyshev polynomials of the first kind. In this case, explicit formulas for the Faber polynomials are derived.

1. Introduction

Suppose that $\Omega \subset \mathbf{C}$ is a compact set containing more than one point. Further, suppose that its complement $\Omega^C := \hat{\mathbf{C}} \backslash \Omega$ is simply connected in the extended complex plane $\hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$. Let $\mathbf{E} := \{z : |z| \leq 1\}$ denote the closed unit disk. Then the Riemann mapping theorem guarantees the existence of a conformal map

$$z = \Psi(w), \qquad \Psi : \mathbf{E}^C \to \Omega^C, \tag{1}$$

which is made unique by the normalization

$$\Psi(\infty) = \infty \qquad \text{and} \qquad \Psi'(\infty) =: t > 0. \tag{2}$$

We call Ψ the *exterior mapping function of* Ω.

In a neighborhood of infinity, Ψ can be expanded as

$$\Psi(w) = t\left(w + \alpha_0 + \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + \cdots\right). \tag{3}$$

For $R > 1$, we define

$$\mathcal{L}_R := \{\Psi(w) : |w| = R\}. \tag{4}$$

Then the *nth Faber polynomial* $F_n(z)$ *for* Ω is defined by the following expansion:

$$\frac{\Psi'(w)}{\Psi(w) - z} =: \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}, \qquad |w| > R, \quad z \in \text{int}(\mathcal{L}_R), \tag{5}$$

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where $\text{int}(\mathcal{L}_R)$ denotes the interior of the Jordan curve $\mathcal{L}_R$. It is easy to show that $F_n(z)$ is of exact degree n with leading term $(z/t)^n$. Using (3), the following well-known recurrence relation can be derived by comparing equal powers of w in (5):

$$(6) \quad F_0(z) \equiv 1, \qquad F_n(z) = \frac{z}{t} F_{n-1}(z) - \sum_{j=0}^{n-2} \alpha_j F_{n-1-j}(z) - n\alpha_{n-1}, \qquad n \geq 1.$$

Faber introduced these polynomials in 1903 in the context of polynomial approximation of analytic functions in the complex plane [6]. Since then his work has found applications in many areas of mathematics and a large number of papers on Faber polynomials have been published. Examples of applications and further properties can be found in [3], [15], [16]. Suetin's recent book [16] additionally contains a comprehensive bibliography of the literature on Faber polynomials (188 references).

Recently, Faber polynomials for particular regions in the complex plane have been the subject of much research. For example, He studied Faber polynomials for circular arcs [8] and circular lunes [9]. Coleman and Smith [2], as well as Gatermann, Hoffmann, and Opfer [7], considered circular sectors, while Coleman and Myers [1] worked on annular sectors. Eiermann and Varga [5], as well as He [10] and He and Saff [11], considered hypocycloidal domains.

Here we study Faber polynomials for sets that have *rational exterior mapping functions*. Examples for such sets include some convex sets (circles, ellipses), some non-convex but starlike sets (hypocycloids), and non-starlike sets (circular arcs, and the "bratwurst" shape sets we introduced in [12]). Because of this generality such sets have many applications, in particular in numerical linear algebra, where they are used as inclusion sets for the eigenvalues of a given matrix (see, e.g., [13], and the references therein for more details).

In Section 2, we show that the Faber polynomials for a set Ω with rational exterior mapping function Ψ always satisfy a short recurrence, even if the Laurent expansion (3) of Ψ has infinitely many terms and thus (6) does not yield a short recurrence relation. In Section 3, we show that if Ψ has degree $(2, 1)$, the Faber polynomials can be expressed in terms of the classical Chebyshev polynomials of the first kind. In this case, we also give explicit formulas for the Faber polynomials.

2. General Results

In this paper we consider Faber polynomials for sets Ω that have rational exterior mapping functions, i.e., we assume that the conformal map Ψ satisfies (1), (2), and

$$(7) \qquad \Psi(w) = \frac{P(w)}{Q(w)} = \frac{w^m + \mu_{m-1} w^{m-1} + \cdots + \mu_0}{\nu_{m-1} w^{m-1} + \nu_{m-2} w^{m-2} + \cdots + \nu_0}, \qquad \nu_{m-1} > 0,$$

for some positive integer m. The polynomials $P(w)$ and $Q(w)$ are assumed to have no common zeros. We point out that because Ψ is bijective in $\mathbf{E}^C$, the zeros w_j of $Q(w)$ satisfy $|w_j| < 1$. Similarly, for $z \in \text{int}(\mathcal{L}_R)$, the zeros $w_j(z)$ of the polynomial $P(w) - zQ(w)$ satisfy $|w_j(z)| < R$.

Lemma 2.1. *Suppose that* Ω *has an exterior mapping function of the form* (7). *Furthermore, suppose that we have the factorizations*

$$Q(w) = \nu_{m-1} \prod_{j=1}^{l} (w - w_j)^{m_j}, \quad m_j \in \mathbf{N}, \qquad \sum_{j=1}^{l} m_j = m - 1, \tag{8}$$

and, for $z \in \mathbf{C}$:

$$P(w) - zQ(w) = \prod_{j=1}^{l(z)} (w - w_j(z))^{m_j(z)}, \quad m_j(z) \in \mathbf{N}, \qquad \sum_{j=1}^{l(z)} m_j(z) = m. \tag{9}$$

Then the n*th Faber polynomial for* Ω *is given by*

$$F_n(z) = \sum_{j=1}^{l(z)} m_j(z) w_j(z)^n - \sum_{j=1}^{l} m_j w_j^n, \qquad n \geq 1. \tag{10}$$

Proof. Let $\mathcal{L}_R$ be as in (4). Suppose that $|w| > R$ and $z \in \operatorname{int}(\mathcal{L}_R)$. Using the factorizations (8) and (9), we get

$$\begin{aligned}
\frac{\Psi'(w)}{\Psi(w) - z} &= \frac{d}{dw}[\operatorname{Log}(\Psi(w) - z)] = \frac{d}{dw}\left[\operatorname{Log}\left(\frac{P(w) - zQ(w)}{Q(w)}\right)\right] \\
&= \frac{d}{dw}[\operatorname{Log}(P(w) - zQ(w))] - \frac{d}{dw}[\operatorname{Log}(Q(w)/\nu_{m-1})] \\
&= \frac{d}{dw}\left[\sum_{j=1}^{l(z)} \operatorname{Log}(w - w_j(z))^{m_j(z)}\right] - \frac{d}{dw}\left[\sum_{j=1}^{l} \operatorname{Log}(w - w_j)^{m_j}\right] \\
&= \sum_{j=1}^{l(z)} \frac{m_j(z)}{w - w_j(z)} - \sum_{j=1}^{l} \frac{m_j}{w - w_j} \\
&= \frac{1}{w}\left[\sum_{j=1}^{l(z)} \frac{m_j(z)}{1 - w_j(z)/w} - \sum_{j=1}^{l} \frac{m_j}{1 - w_j/w}\right].
\end{aligned}$$

As noted above, $|w_j| < 1$ and, since we assume $z \in \operatorname{int}(\mathcal{L}_R)$, $|w_j(z)| < R$. Thus, for $|w| > R > 1$:

$$\begin{aligned}
\frac{\Psi'(w)}{\Psi(w) - z} &= \frac{1}{w}\left[\sum_{j=1}^{l(z)} \left(m_j(z) \sum_{n=0}^{\infty} \frac{w_j(z)^n}{w^n}\right) - \sum_{j=1}^{l} \left(m_j \sum_{n=0}^{\infty} \frac{w_j^n}{w^n}\right)\right] \\
&= \sum_{n=0}^{\infty} \left[\left(\sum_{j=1}^{l(z)} m_j(z) w_j(z)^n - \sum_{j=1}^{l} m_j w_j^n\right) \Big/ w^{n+1}\right].
\end{aligned}$$

A comparison with (5) shows that (10) holds for all $z \in \operatorname{int}(\mathcal{L}_R)$. But since $R > 1$ can be chosen arbitrarily, (10) holds for all $z \in \mathbf{C}$. ■

Remark. Considering Faber polynomials for m-cusped hypocycloids, He and Saff [11, Prop. 2.3] derive a special case of (10) for exterior mapping functions of the form $\Psi(w) = w + w^{1-m}/(m-1)$, $m \geq 2$.

For $F_n(z)$ as in (10), we introduce the corresponding *shifted Faber polynomial*

$$\hat{F}_n(z) := F_n(z) + \sum_{j=1}^{l} m_j w_j^n = \sum_{j=1}^{l(z)} m_j(z) w_j(z)^n, \qquad n \geq 1. \tag{11}$$

We also define $\hat{F}_0(z) := m$. This separate definition is necessary, because (11) for $n = 0$ potentially requires forming 0^0.

Note that

$$|\hat{F}_n(z) - F_n(z)| \leq \sum_{j=1}^{l} |m_j w_j^n| < m - 1 \qquad \text{for all} \quad n \geq 1 \quad \text{and} \quad z \in \mathbf{C}.$$

In particular, the difference between $F_n(z)$ and $\hat{F}_n(z)$ approaches zero as n approaches infinity. We next show that the shifted Faber polynomials satisfy a short recurrence relation.

Theorem 2.1. *Suppose that Ω has an exterior mapping function of the form* (7). *Then the nth shifted Faber polynomial $\hat{F}_n(z)$ for Ω as defined in* (11) *satisfies*

$$\hat{F}_n(z) = \sum_{h=0}^{m-1} (\nu_h z - \mu_h) \hat{F}_{n-(m-h)}(z), \qquad n \geq m. \tag{12}$$

Proof. First note that $P(w) - zQ(w) = w^m + \sum_{h=0}^{m-1} (\mu_h - \nu_h z) w^h$. Hence, if $w_j(z)$ is a zero of $P(w) - zQ(w)$, then

$$w_j(z)^m = \sum_{h=0}^{m-1} (\nu_h z - \mu_h) w_j(z)^h.$$

Thus, for $n \geq m$:

$$\begin{aligned}
\hat{F}_n(z) &= \sum_{j=1}^{l(z)} m_j(z) w_j(z)^n = \sum_{j=1}^{l(z)} m_j(z) \sum_{h=0}^{m-1} (\nu_h z - \mu_h) w_j(z)^{n-(m-h)} \\
&= \sum_{h=0}^{m-1} (\nu_h z - \mu_h) \sum_{j=1}^{l(z)} m_j(z) w_j(z)^{n-(m-h)} \\
&= \sum_{h=0}^{m-1} (\nu_h z - \mu_h) \hat{F}_{n-(m-h)}(z).
\end{aligned}$$

■

Remark. For $0 \leq n \leq m - 1$, the shifted Faber polynomials can be efficiently computed using the recurrence (6).

If $Q(w) = \nu_{m-1} w^{m-1}$, then for $n \geq 1$, the Faber polynomials and the corresponding shifted Faber polynomials coincide. In this case, (12) reduces to the familiar recurrence (6). However, if $Q(w) \neq \nu_{m-1} w^{m-1}$, then the Laurent series (3) of Ψ generally has infinitely many nonzero coefficients α_j. Thus, the direct approach for computing $F_n(z)$ by (6) in these cases requires storing all previous values $F_j(z)$. The key point of Theorem 2.1 is that the shifted Faber polynomials—and hence the Faber polynomials—corresponding to an $(m, m-1)$-degree rational exterior mapping function *in general* satisfy an $(m+1)$-term recurrence.

If the factorization (8) is not known explicitly, $F_n(z)$ can be computed from $\hat{F}_n(z)$ by using the well-known Newton identities for the power sums $s_n := \sum_{j=1}^{l} m_j w_j^n$. With $p_j := \nu_{m-1-j}/\nu_{m-1}$, $1 \le j \le m-1$:

$$s_n = \begin{cases} -np_n - \displaystyle\sum_{j=1}^{n-1} p_j s_{n-j} & \text{for} \quad 1 \le n \le m-1, \\ -\displaystyle\sum_{j=1}^{m-1} p_j s_{n-j} & \text{for} \quad n > m-1. \end{cases}$$

The recurrence (6) has been frequently used in the construction of iterative methods based on Faber polynomials. To make such methods feasible, Eiermann [4] as well as Manteuffel, Starke, and Varga [14] consider only finite Laurent series with k terms, i.e., $(k, k-1)$-degree exterior mapping functions with $Q(w) = w^{k-1}$. The resulting methods are called non-stationary k-step methods. However, when computing the Faber polynomials corresponding to a rational exterior mapping function Ψ as suggested by Theorem 2.1, iterative methods with short recurrences can be constructed *although* the Laurent series of Ψ has infinitely many nonzero terms. Based on a family of (2, 1)-degree rational exterior mapping functions, we proposed such a method in [13]. We will study this iterative method in more detail in a forthcoming paper.

We finally point out that a nonrational exterior mapping function for a given set Ω might be approximated by a rational function Ψ, for example, by using the Carathéorory–Fejér method [17]. Using Theorem 2.1, the Faber polynomials for the set $(\Psi(\mathbf{E}^C))^C$, the approximation of Ω, can then be generated by a short-term recurrence.

3. The Special Case *m* = 2

We now consider the special case of sets Ω with (2, 1)-degree rational exterior mapping functions. Our goal is to relate the Faber polynomials for such sets to the classical Chebyshev polynomials of the first kind, which are given by

$$C_n(z) = \frac{w^n + w^{-n}}{2}, \qquad z = \frac{w + w^{-1}}{2}, \qquad n \ge 1. \tag{13}$$

It is well known that the Chebyshev polynomials satisfy a three-term recurrence of the form

$$C_0(z) \equiv 1, \qquad C_1(z) = z, \qquad \text{and} \qquad C_n(z) = 2zC_{n-1}(z) - C_{n-2}(z), \qquad n \ge 2.$$

Suppose that Ω has an exterior mapping function of the form

$$\Psi(w) = \frac{P(w)}{Q(w)} = \frac{w^2 + \mu_1 w + \mu_0}{\nu_1 w + \nu_0}. \tag{14}$$

In this case, the Faber polynomials and the corresponding shifted Faber polynomials for Ω are related by

$$\hat{F}_n(z) - F_n(z) = \left(-\frac{\nu_0}{\nu_1}\right)^n, \quad n \geq 0; \qquad \hat{F}_0(z) - F_0(z) = 1 \quad \text{in case } \nu_0 = 0.$$

From Lemma 2.1 and the definition of the shifted Faber polynomials (11), it follows that for $z \in \mathbf{C}$ and $n \geq 1$:

$$\hat{F}_n(z) = w_1(z)^n + w_2(z)^n, \tag{15}$$

where $w_1(z)$ and $w_2(z)$ are the zeros of the polynomial $P(w) - zQ(w)$. These are implicitly defined by

$$(w - w_1(z))(w - w_2(z)) = w^2 + (\mu_1 - \nu_1 z)w + (\mu_0 - \nu_0 z). \tag{16}$$

We define

$$2W(z) := w_1(z) + w_2(z) = \nu_1 z - \mu_1, \tag{17}$$

and

$$V(z) := w_1(z)w_2(z) = \mu_0 - \nu_0 z. \tag{18}$$

Suppose that $V(z) \neq 0$, and define $\zeta_j(z) := V(z)^{-1/2} w_j(z)$, $j = 1, 2$. This yields $\zeta_1(z)\zeta_2(z) = 1$, i.e., $\zeta_2(z) = \zeta_1(z)^{-1}$, and thus

$$\hat{F}_n(z) = V(z)^{n/2}(\zeta_1(z)^n + \zeta_2(z)^n).$$

We use (13) and get, for $n \geq 1$:

$$\begin{aligned} \hat{F}_n(z) &= 2V(z)^{n/2} C_n\left(\frac{\zeta_1(z) + \zeta_1(z)^{-1}}{2}\right) \\ &= 2V(z)^{n/2} C_n(V(z)^{-1/2} W(z)). \end{aligned}$$

We next determine the value of $\hat{F}_n(z_0)$ for the zero z_0 of $V(z)$. First suppose that $\nu_0 = 0$. Then $V(z) = 0$ if, and only if, $\mu_0 = 0$. But this implies that Ψ is only of degree $(1, 0)$, i.e., $m = 1$. Thus, in the case $m = 2$, we either have $V(z) \neq 0$ for all $z \in \mathbf{C}$, or $\nu_0 \neq 0$, and the unique zero of $V(z)$ is $z_0 = \mu_0/\nu_0$. In the latter case, $\hat{F}_n(z_0) = (\mu_0\nu_1 - \mu_1\nu_0)/\nu_0$ can be easily computed from (16)–(18).

We summarize our results in the following theorem:

Theorem 3.1. *Suppose that Ω has an exterior mapping function of the form* (14). *Let $C_n(z)$ denote the nth Chebyshev polynomial* (13) *and let $W(z)$ and $V(z)$ be defined as in* (17) *and* (18), *respectively.*

If $\nu_0 = 0$, the shifted Faber polynomials (11) *for Ω are given by*

$$\hat{F}_n(z) = 2V(z)^{n/2} C_n(V(z)^{-1/2} W(z)), \qquad n \geq 1. \tag{19}$$

If $\nu_0 \neq 0$, (19) *holds for all* $z \in \mathbf{C}\backslash\{\mu_0/\nu_0\}$ *and*

$$\hat{F}_n(\mu_0/\nu_0) = \frac{\mu_0\nu_1 - \mu_1\nu_0}{\nu_0}. \tag{20}$$

Furthermore, the following three-term recurrence holds:

$$\hat{F}_0(z) \equiv 2, \qquad \hat{F}_1(z) = 2W(z), \tag{21}$$

and

$$\hat{F}_n(z) = 2W(z)\hat{F}_{n-1}(z) - V(z)\hat{F}_{n-2}(z), \qquad n \geq 2. \tag{22}$$

A special case of (19) is the well-known relation between Faber polynomials for ellipses and the Chebyshev polynomials (13): Suppose that Ω is an ellipse with foci ± 1 and semiaxes $r \pm r^{-1}$ for some $r \geq 1$. Then the exterior mapping function of Ω is the Joukowsky map $J(w) = (rw + (rw)^{-1})/2$. Hence, $V(z) = 1/r^2$, $W(z) = z/r$, and (19) yields

$$F_n(z) = \hat{F}_n(z) = \frac{2}{r^n}C_n(z), \qquad n \geq 1,$$

(see, e.g., [16, p. 37]).

More generally, suppose that the exterior mapping function of Ω is a composition of the Joukowsky map with Moebius transformations, $\Psi(w) = (\psi_2 \circ J \circ \psi_1)(w)$, $\psi_j(w) = (a_j w + b_j)/(c_j w + d_j)$, $a_j d_j - b_j c_j \neq 0$, $j = 1, 2$. Then Ψ will have degree $(2, 1)$ and (19) holds.

A geometric interpretation of (19) therefore is: *Whenever* Ω *is Moebius-equivalent to an ellipse, its Faber polynomials can be expressed in terms of the Chebyshev polynomials of the first kind*. *In particular, when* Ω *is an ellipse, its Faber polynomials are scaled Chebyshev polynomials of the first kind*.

We next use (19) to derive an *explicit* formula for the Faber polynomials corresponding to $(2, 1)$-degree exterior mapping functions. It is well known (see, e.g., [3, p. 583]), that the Chebyshev polynomials (13) satisfy

$$C_n(z) = \sum_{j=0}^{\ulcorner n/2 \urcorner} \binom{n}{2j} z^{n-2j}(z^2 - 1)^j,$$

where $\ulcorner n/2 \urcorner$ denotes the largest integer less than or equal to $n/2$. An application of this formula to (19) yields the following corollary:

Corollary 3.1. *In the notation of Theorem* 3.1, *the shifted Faber polynomial* $\hat{F}_n(z)$ *is given by*

$$\hat{F}_n(z) = 2\sum_{j=0}^{\ulcorner n/2 \urcorner} \binom{n}{2j} W(z)^{n-2j}(W(z)^2 - V(z))^j, \qquad n \geq 1. \tag{23}$$

We point out that (23) gives explicit formulas for the Faber polynomials for a large class of sets, some of them with complicated, e.g., nonconvex or non-starlike, geometries. An important special case of (23) are the Faber polynomials for circular arcs, previously studied by He [8].

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