Combinatorial Aspects of Zonotopal Algebra

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Abstract

Zonotopal algebra is the study of a family of pairs of dual vector spaces of multivariate polynomials that can be associated with a list of vectors X. It connects objects from combinatorics, geometry, and approximation theory. The origin of zonotopal algebra is the pair $(\mathcal{D}(X), \mathcal{P}(X))$, where $\mathcal{D}(X)$ denotes the Dahmen-Micchelli space that is spanned by the local pieces of the box spline and $\mathcal{P}(X)$ is the Macaulay inverse system of a certain power ideal. Further zonotopal spaces were recently studied by Holtz-Ron and others. A common property of all these spaces is that their Hilbert series is a matroid invariant.

The present dissertation has four chapters. The first chapter contains an introduction to zonotopal algebra and some background material.

In Chapter II there are two main results. The first is the construction of a canonical basis for $\mathcal{D}(X)$ that is dual to the canonical basis for $\mathcal{P}(X)$ that is already known. The second is the construction of a new family of zonotopal spaces that is far more general than the ones that were recently studied by Ardila-Postnikov, Holtz-Ron, Holtz-Ron-Xu, Li-Ron, and others. We call the underlying combinatorial structure of those spaces forward exchange matroid. A forward exchange matroid is an ordered matroid together with a subset of its set of bases that satisfies a weak version of the basis exchange axiom.

In Chapter III we study hierarchical zonotopal power ideals and the corresponding $\mathcal{P}$-spaces. We generalise and unify results by Ardila-Postnikov on power ideals and by Holtz-Ron and Holtz-Ron-Xu on (hierarchical) zonotopal spaces.

The last chapter deals with matroid theory and its connections with zonotopal algebra. The main result is that f-vectors of matroid complexes of realisable matroids are log-concave. This was conjectured by Mason in 1972.

Zusammenfassung

Zonotopische Algebra befasst sich mit einer Familie von Paaren dualer Vektorräume, die aus multivariaten Polynomen bestehen und die anhand einer Liste von Vektoren X konstruiert werden. Sie verbindet Objekte aus der Kombinatorik, Geometrie und Approximationstheorie. Der Ursprung der zonotopischen Algebra ist das Paar $(\mathcal{D}(X), \mathcal{P}(X))$. Hierbei bezeichnet $\mathcal{D}(X)$ den Dahmen-Micchelli Raum, der von den lokalen Stücken des Box-splines aufgespannt wird und $\mathcal{P}(X)$ das Macaulaysche inverse System eines bestimmten Potenzideales. Weitere zonotopische Räume wurden kürzlich von Holtz-Ron und anderen untersucht. Eine gemeinsame Eigenschaft dieser Räume ist, dass ihre Hilbertreihen Matroidinvarianten sind.

Die vorliegende Dissertation hat vier Kapitel. Das erste Kapitel enthält eine Einführung in zonotopische Algebra und Hintergrundmaterial.

In Kapitel II gibt es zwei Hauptergebnisse. Das erste ist die Konstruktion einer kanonischen Basis für $\mathcal{D}(X)$, die zur bereits bekannten Basis für $\mathcal{P}(X)$ dual ist. Das zweite ist die Konstruktion einer neuen Familie zonotopischer Räume, die weitaus allgemeiner ist als die, die kürzlich von Ardila-Postnikov, Holtz-Ron, Holtz-Ron-Xu, Li-Ron und anderen betrachtet wurden. Wir nennen die diesen Räumen zugrundeliegende kombinatorische Struktur Vorwärtsaustauschmatroid. Ein Vorwärtsaustauschmatroid ist ein geordneter Matroid zusammen mit einer Teilmenge seiner Basen, die eine abgeschwächte Version des Basisaustauschaxiomes erfüllt.

In Kapitel III untersuchen wir hierarchische zonotopische Potenzideale und die zugehörigen $\mathcal{P}$-Räume. Wir verallgemeinern und vereinheitlichen Ergebnisse von Ardila-Postnikov über Potenzideale und von Holtz-Ron und Holtz-Ron-Xu über (hierarchische) zonotopische Räume.

Das letzte Kapitel beschäftigt sich mit Matroidtheorie und den Verbindungen zu zonotopischer Algebra. Das Hauptergbnis ist das f-Vektoren von Matroidkomplexen realisierbarer Matroide logarithmisch konkav sind. Dies wurde 1972 von Mason vermutet.

Preface

Broadly speaking, this thesis is about finite and finite-dimensional structures and about surprising connections between some of these structures that seem to be unrelated at first. This is what I like most in mathematics. Objects that we will encounter include hyperplane arrangements, polytopes, abstract simplicial complexes (matroids), graded vector spaces and ideals, graph polynomials, and piecewise polynomial functions (splines).

There are four chapters. The first chapter contains an introduction to zonotopal algebra and some background material. The remaining chapters each correspond to one of the three papers that I have written while working towards my PhD. They are largely unchanged, but some modifications were made where appropriate. For example, the introductions were shortened because some of this material is already contained in Chapter I. Chapter II is based on the preprint *Zonotopal algebra and forward exchange matroids* [**68**]. Chapter III is based on the article *Hierarchical zonotopal power ideals* [**67**] and Chapter IV relies on *The f-vector of a realizable matroid complex is log-concave* [**66**]. In Chapters II and III, we assume that the reader is familiar with the material that is presented in Chapter I while most of Chapter IV can be read independently.

Acknowledgements. I started working towards my PhD in October 2008. Since then many people have helped me in many different ways. It is now time to say thank you.

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Federico Ardila, Zhiqiang Xu, Andrew Berget, and June Huh made helpful comments on my papers. Bernd Sturmfels, Nan Li and several other people discussed research problems with me. Lars Kastner and Martin Götze were always ready to answer seemingly stupid questions. Benjamin Lorenz provided help with the mathematical software package polymake. Kaie Kubjas's comments greatly improved my thesis defence talk.

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Heather Heintzel proofread this thesis and made sure that the English is correct.

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Contents

CHAPTER I

Preliminaries

1. Introduction

A finite list of vectors X gives rise to a large number of objects in various mathematical fields. Examples include combinatorics (matroids, matroid polynomials, generalised parking functions and chip firing games if X is graphic [**42, 43, 53, 73, 78**]), discrete geometry (hyperplane arrangements, zonotopes, and tilings of zonotopes), approximation theory (box splines [**30**], least interpolation space) and algebraic geometry (Cox rings, fat point ideals [**3, 49, 86**]). Zonotopal algebra is the study of a family of pairs of dual vector spaces of multivariate polynomials that can be associated with a list of vectors. It connects all of the objects mentioned above.

In the 1980s, various authors in the approximation theory community started studying algebraic structures that capture information about splines (*e. g.* [**1, 29, 45**]). One important example is the Dahmen-Micchelli space $\mathcal{D}(X)$, that is spanned by the local pieces of the box spline and their partial derivatives. See [**54**, Section 1.2] for a historic survey and the book [**30**] for a treatment of polynomial spaces appearing in the theory of box splines. Related results were obtained independently by authors interested in hyperplane arrangements (*e. g.* [**76**]).

The space $\mathcal{P}(X)$ that is dual to $\mathcal{D}(X)$ was introduced in [**1, 45**]. It is spanned by products of linear forms and it can be written as the Macaulay inverse system (or kernel) of an ideal generated by powers of linear forms [**28**]. Ideals of this type and their inverse systems are also studied in the literature on fat point ideals (*e. g.* [**48, 49**]).

In addition to the aforementioned pair of spaces $(\mathcal{D}(X), \mathcal{P}(X))$, Olga Holtz and Amos Ron introduced two more pairs of spaces with interesting combinatorial properties [**54**]. They named the theory of those spaces *Zonotopal Algebra.* This name reflects the fact that there are various connections between zonotopal spaces and the lattice points in the zonotope defined by X. Subsequently, those results were further generalised by Olga Holtz, Amos Ron, and Zhiqiang Xu [**55**] as well as Nan Li and Amos Ron [**69**]. Federico Ardila and Alex Postnikov studied generalised $\mathcal{P}$-spaces and connections with power ideals [**3**]. Bernd Sturmfels and Zhiqiang Xu established a connection with Cox rings [**86**]. Further work on spaces of $\mathcal{P}$-type includes [**7, 14, 88**].

Zonotopal algebra is closely related to matroid theory: the Hilbert series of zonotopal spaces only depend on the matroid structure of the list X. One can show that the following statement is equivalent to the four colour theorem: for all connected planar graphs G, the evaluation of the Hilbert series $\operatorname{Hilb}(\mathcal{P}_-(X_{G^*}), q)$ at $q = -1/3$ is negative. Here, X_{G^*} denotes the reduced incidence matrix of the graph dual to G and $\mathcal{P}_-(X_{G^*})$ denotes the associated internal $\mathcal{P}$-space that we will define in Subsection 4.2.

All objects mentioned so far are part of what we call the continuous theory. If the list X lies in a lattice (*e. g.* $\mathbb{Z}^d$), an even wider spectrum of mathematical objects appears. We call this the discrete theory. Every object in the continuous theory has a discrete analogue: vector partition functions correspond to box splines and toric arrangements correspond to hyperplane arrangements. The local pieces of the vector partition function are quasi-polynomials that span the discrete Dahmen-Micchelli space $\mathrm{DM}(X)$. Both theories are nicely explained in the recent book by Corrado De Concini and Claudio Procesi [**36**] and in [**35**].

The combinatorics of the discrete case is captured by arithmetic matroids which were very recently introduced by Luca Moci and Michele D'Adderio [**22, 74**].

Vector partition functions and the related problem of counting integer points in polytopes are an active field of research (see *e. g.* [**4, 6, 13, 40**]). Vector partition functions arise for example in representation theory as Kostant partition function, when the list X is chosen to be the set of positive roots of a simple Lie algebra (*e. g.* [**21**]). There are also applications to the equivariant index theory of elliptic operators [**37, 38, 39**].

This thesis deals only with the continuous theory except for the very last section. However, the two theories overlap if the list of vectors X is totally unimodular. The author hopes that some of his results can be transferred to the discrete case in the future.

1.1. Outline of this chapter. This is an introductory chapter. We will define the objects that we study in the following chapters and give some background information.

In Section 2 we will explain our notation. In Section 3 we will introduce matroids and several objects from discrete geometry and commutative algebra. Section 4 contains a brief introduction to zonotopal algebra and least map interpolation. In Section 5 we will talk about distributions and splines. Section 6 contains some information on arithmetic matroids and in Section 7 we will discuss the level of abstraction and the ground field that should be used when studying zonotopal algebra.

2. Notation

We use the convention that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. For $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. We denote the field we are working over by $\mathbb{K}$. Sometimes, we assume that $\mathbb{K}$ has characteristic zero or even $\mathbb{K} = \mathbb{R}$. Our basic object of study is a list of vectors $X = (x_1, \ldots, x_N)$ that span an r-dimensional space $U \cong \mathbb{K}^r$. The dual space U^* is denoted by V. The subspace spanned by a set $S \subseteq U$ is denoted by $\mathrm{span}(S)$. We slightly abuse notation by using the symbol $\subseteq$ for sublists. For $Y \subseteq X$, $X \setminus Y$ denotes the deletion of a sublist, *i. e.* $(x_1, x_2) \setminus (x_1) = (x_2)$ even if $x_1 = x_2$. The list X comes with a natural ordering: we say that $x_i < x_j$ if and only if $i < j$.

Note that X can be identified with a linear map $\mathbb{K}^N \to U$ and after the choice of a basis with an $(r \times N)$-matrix with entries in $\mathbb{K}$.

We will consider families of pairs of dual spaces $(\mathcal{D}(X,\cdot),\mathcal{P}(X,\cdot))$. The space $\mathcal{D}(X,\cdot)$ is contained in $\mathrm{Sym}(V)$, the symmetric algebra over V and $\mathcal{P}(X,\cdot)$ is contained in $\mathrm{Sym}(U)$. The *symmetric algebra* is a base-free version of the ring of polynomials over a vector space. We fix a basis $(s_1,\ldots,s_r)$ for U and $(t_1,\ldots,t_r)$ denotes the dual basis for V, *i. e.* $t_i(s_j)=\delta_{ij}$, where δ_{ij} denotes the Kronecker delta. The choice of the basis determines isomorphisms $\mathrm{Sym}(U)\cong\mathbb{K}[s_1,\ldots,s_r]$ and $\mathrm{Sym}(V)\cong\mathbb{K}[t_1,\ldots,t_r]$. For $x\in U$, $x^\circ\subseteq V$ denotes the *annihilator* of x, *i. e.* $x^o:=\{f\in V : f(x)=0\}$. For more background on algebra, see **[36]** or **[46]**.

As usual, $\chi_J : J\to\{0,1\}$ denotes the *indicator function* of a set J.

3. Combinatorics and algebra

3.1. Matroids. An *ordered matroid* on N elements is a pair $\mathfrak{M}=(A,\mathbb{B})$ where A is an (ordered) list with N elements and $\mathbb{B}$ is a non-empty set of sublists of A that satisfies the following axiom:

$$\begin{aligned}&\text{let } B,B'\in\mathbb{B} \text{ and } b\in B\setminus B'.\\ &\text{Then, there exists } b'\in B'\setminus B \text{ s.t. } (B\setminus b)\cup b'\in\mathbb{B}.\end{aligned}\tag{I.1}$$

A is called the *ground set* and $\mathbb{B}$ is called the *set of bases* of the matroid $\mathfrak{M}=(A,\mathbb{B})$. One can easily show that all elements of $\mathbb{B}$ have the same cardinality. This number r is called the *rank* of the matroid $(A,\mathbb{B})$. A set $I\subseteq A$ is called *independent* if it is a subset of a basis. The *rank* of $Y\subseteq A$ is defined as the cardinality of a maximal independent set contained in Y. It is denoted $\mathrm{rk}(Y)$.

In this thesis we mainly consider matroids that are realisable over some field $\mathbb{K}$. Let $X=(x_1,\ldots,x_N)$ be a list of vectors spanning some $\mathbb{K}$-vector space W and let $\mathbb{B}(X)$ denote the set of bases of W (in the sense of linear algebra) that can be selected from X. One can easily see that $\mathfrak{M}(X):=(X,\mathbb{B}(X))$ is a matroid. The list X is called a *realisation* of this matroid and a matroid $(A,\mathbb{B})$ is called *realisable* if there is a list of vectors X and a bijection between A and X that induces a bijection between $\mathbb{B}$ and $\mathbb{B}(X)$.

The *closure* of Y is defined as $\mathrm{cl}(Y):=\{x\in A : \mathrm{rk}(Y\cup x)=\mathrm{rk}(Y)\}$. A set $C\subseteq A$ is called a *flat* if $C=\mathrm{cl}(C)$. The set of flats of a given matroid $\mathfrak{M}$ ordered by inclusion forms a lattice (*i. e.* a poset with joins and meets) called the *lattice of flats* $\mathcal{L}(\mathfrak{M})$. An *upper set* $J\subseteq\mathcal{L}(\mathfrak{M})$ is an upward closed set, *i. e.* $C\subseteq C'$, $C\in J$ implies $C'\in J$.

A *hyperplane* is a flat of rank $r-1$. Note that if we are given a realisation X of a matroid, a hyperplane in the matroid theoretic sense spans a (linear) hyperplane in the vector space spanned by X. We will use the word hyperplane to denote both, the geometric and the combinatorial object.

The set of independent sets Δ forms an abstract simplicial complex that is called the *matroid complex* of $\mathfrak{M}$.

Subsets of A that are not independent are called *dependent.* A dependent set C for which every strict subset is independent is called a *circuit.* A set $C\subseteq A$ is called a *cocircuit* if $C\cap B\neq\emptyset$ for all bases $B\in\mathbb{B}$ and C is minimal with this property. Cocircuits of cardinality one are called *coloops.* and circuits of cardinality one are called *loops.* Note that a set $\{a\}\subseteq A$ is a coloop if a is contained in every basis and a loop if a is contained in no basis.

Now fix a basis $B \in \mathbb{B}$. An element $b \in B$ is called *internally active* in B if $b = \max(A \setminus \mathrm{cl}(B \setminus b))$, *i. e.* b is the maximal element of the unique cocircuit contained in $(A \setminus B) \cup b$. The set of internally active elements in B is denoted $I(B)$. An element $x \in A \setminus B$ is called *externally active* if $x \in \mathrm{cl}\{b \in B : b \leq x\}$, *i. e.* x is the maximal element of the unique circuit contained in $B \cup x$. The set of externally active elements with respect to B is denoted $E(B)$.[1] The *Tutte polynomial*

$$T_{\mathfrak{M}}(x,y) := \sum_{B \subseteq \mathbb{B}} x^{|I(B)|} y^{|E(B)|} = \sum_{S \subseteq A} (x-1)^{r-\mathrm{rk}(S)}(y-1)^{|S|-\mathrm{rk}(S)} \qquad \text{(I.2)}$$

captures a lot of information about the matroid $\mathfrak{M}$. The equality of the two expressions for $T_{\mathfrak{M}}(x,y)$ is non-trivial but not hard to prove. In particular, it implies that the first expression is independent of the order of the elements of the ground set. This is not obvious.

A standard reference for matroid theory is Oxley's book **[77]**. Survey papers on the Tutte polynomial are **[18, 47]**.

Throughout this dissertation, we will use a running example, which we will now introduce.

Example 3.1.

$$\text{Let } X := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = (x_1, x_2, x_3). \qquad \text{(I.3)}$$

The set of bases that can be selected from X is $\mathbb{B}(X) = \{(x_1,x_2),(x_1,x_3), (x_2,x_3)\}$. The Tutte polynomial of the matroid $\mathfrak{M}(X) = (X, \mathbb{B}(X))$ is

$$T_{\mathfrak{M}(X)}(x,y) = x^2 + x + y. \qquad \text{(I.4)}$$

3.2. Discrete geometry. In this subsection we will introduce zonotopes, cones, and hyperplane arrangements. See Ziegler's book **[93]** for more details.

Definition 3.2. Let $X = (x_1, \ldots, x_N) \subseteq U \cong \mathbb{R}^r$ be a list of vectors. Then, we define the *zonotope* $Z(X)$ and the *cone* $\mathrm{cone}(X)$ by

$$Z(X) := \left\{ \sum_{i=1}^N \lambda_i x_i : 0 \leq \lambda_i \leq 1 \right\} \text{ and } \mathrm{cone}(X) := \left\{ \sum_{i=1}^N \lambda_i x_i : \lambda_i \geq 0 \right\}.$$

Let $x \in U$ and $c_x \in \mathbb{R}$. This defines a *hyperplane*

$$H_{x,c_x} := \{v \in V : v(x) = c_x\}. \qquad \text{(I.5)}$$

If we fix a vector $c \in \mathbb{R}^X$, we obtain a *hyperplane arrangement* $\mathcal{H}(X,c) = \{H_{x,c_x} : x \in X\}$.

Every basis $B \subseteq X$ determines a unique vertex $\theta_B \in V$ of the hyperplane arrangement $\mathcal{H}(X,c)$ that satisfies $\theta_B(x) = c_x$ for all $x \in B$. In matrix notation, $\theta_B = B^{-1}c_B$, where c_B denotes the restriction of c to $\mathbb{R}^B$. If the vector c is sufficiently generic, then $\theta_B \neq \theta_{B'}$ for distinct bases B and B'.

[1] Usually, combinatorialists use min instead of max in the definition of the activities. In the zonotopal algebra literature max is used. This has some notational advantages.

In this case, the hyperplane arrangement $\mathcal{H}(X,c)$ is said to be *in general position*. For more information on hyperplane arrangements, see [**75, 85**].

3.3. Commutative algebra. In this subsection we will define some commutative algebra terminology that is used in this thesis.

A *derivation* on $\mathrm{Sym}(V)$ is a $\mathbb{K}$-linear map D satisfying Leibniz's law, *i. e.* $D(fg) = D(f)g + fD(g)$ for $f,g \in \mathrm{Sym}(V)$. For $v \in V = U^*$, we define the *directional derivative* in direction v, $D_v : \mathrm{Sym}(U) \to \mathrm{Sym}(U)$ as the unique derivation which satisfies $D_v(u) = v(u)$ for all $u \in U$. For $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, this definition agrees with the analytic definition of the directional derivative. $\mathrm{Sym}(V)$ can be identified with the ring of differential operators on $\mathrm{Sym}(U)$. Namely, $v_1 \cdots v_k \in \mathrm{Sym}(V)$ acts on $\mathrm{Sym}(U)$ by mapping $f \in \mathrm{Sym}(U)$ to $D_{v_1}(\ldots(D_{v_{k-1}}(D_{v_k}f))\ldots)$.

Definition 3.3 (A pairing between symmetric algebras). We define the following pairing:

$$\begin{aligned} \langle\cdot,\cdot\rangle &: \mathbb{K}[s_1,\ldots,s_r] \times \mathbb{K}[t_1,\ldots,t_r] \to \mathbb{K} \\ \langle p,f\rangle &:= \left(p\left(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r}\right)f\right)(0), \end{aligned} \tag{I.6}$$

i. e. we let p act on f as a differential operator and take the degree zero part of the result.

Remark 3.4. One can easily show that the definition of the pairing $\langle\cdot,\cdot\rangle$ is independent of the choice of the bases for the symmetric algebras $\mathrm{Sym}(U)$ and $\mathrm{Sym}(V)$ as long as the bases are dual to each other.

Definition 3.5. Let $\mathcal{I} \subseteq \mathbb{K}[s_1,\ldots,s_r]$ be a homogeneous ideal. Its *kernel* or Macaulay inverse system [**50, 51, 70**] is defined as

$$\ker\mathcal{I} := \{f \in \mathbb{K}[t_1,\ldots,t_r] : \langle q,f\rangle = 0 \text{ for all } q \in \mathcal{I}\}. \tag{I.7}$$

Remark 3.6. $\ker\mathcal{I}$ can also be written as

$$\ker\mathcal{I} := \{f \in \mathbb{K}[t_1,\ldots,t_r] : p\left(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r}\right)f = 0\} \tag{I.8}$$

where p runs over a set of generators for the ideal $\mathcal{I}$.

Remark 3.7. For a homogeneous ideal $\mathcal{I} \subseteq \mathbb{K}[s_1,\ldots,s_r]$ of finite codimension the Hilbert series of $\ker\mathcal{I}$ and $\mathbb{K}[s_1,\ldots,s_r]/\mathcal{I}$ are equal. For instance, this follows from [**36**, Theorem 5.4].[2]

A *graded vector space* is a vector space W that decomposes into a direct sum $W = \bigoplus_{i\geq 0} W_i$. A graded linear map $f : W \to W'$ preserves the grade, *i. e.* $f(W_i)$ is contained in W'_i. For a graded vector space W, we define its *Hilbert series* as the formal power series $\mathrm{Hilb}(W,q) := \sum_{i\geq 0}\dim(W_i)q^i$. A *graded algebra* $A = \bigoplus_{i\geq 0} A_i$ has the additional property $A_iA_j \subseteq A_{i+j}$. We use the symmetric algebra $\mathrm{Sym}(U)$ with its natural grading. This grading is

[2] $\ker\mathcal{I}$ is sometimes defined slightly differently in the literature: first note that $\mathrm{Sym}(U)$ ($\approx$ polynomials) is a subspace of $\mathrm{Sym}(V)^*$ ($\approx$ formal power series). The pairing $\langle\bullet,\bullet\rangle$ is defined on $\mathrm{Sym}(V)^* \times \mathrm{Sym}(V)$ and $\ker\mathcal{I}$ is the subset of $\mathrm{Sym}(V)^*$ that is annihilated by $\mathcal{I}$. It is then proven that if $\mathcal{I}$ has finite codimension, then $\ker\mathcal{I}$ is contained in $\mathrm{Sym}(U)$, *i. e.* in this case both definitions yield the same space.

characterised by the property that the degree one elements are exactly the ones that are contained in $U \setminus \{0\}$. Note that a linear map $f : V \to W$ induces an algebra homomorphism $\mathrm{Sym}(f) : \mathrm{Sym}(V) \to \mathrm{Sym}(W)$.

For more background on algebra, see **[36]** or **[46]**.

4. Zonotopal algebra

In this Section we will give a brief introduction to zonotopal algebra. In Subsection 4.1 we will introduce the central spaces $\mathcal{D}(X)$ and $\mathcal{P}(X)$. In Subsection 4.2 we will define the internal and external zonotopal spaces that were introduced in **[54]**. More general zonotopal spaces are discussed in Chapters II and III. Quite surprisingly, these spaces can also be obtained as the least space of certain sets of lattice points. This is explained in Subsection 4.3.

4.1. Central zonotopal spaces. In this subsection we will define the Dahmen-Micchelli space $\mathcal{D}(X)$ and its dual $\mathcal{P}(X)$. The pair $(\mathcal{D}(X), \mathcal{P}(X))$, which is called the central pair of zonotopal spaces in **[54]**, is the origin of zonotopal algebra.

A vector $u \in U$ naturally defines a polynomial $p_u \in \mathbb{K}[s_1, \ldots, s_r]$ as follows: if u can be expressed in the basis $(s_1, \ldots, s_r)$ as $u = \sum_{i=1}^r \lambda_i s_i$, then we define $p_u := \sum_{i=1}^r \lambda_i s_i \in \mathbb{K}[s_1, \ldots, s_r]$. For $Y \subseteq X$, we define $p_Y := \prod_{x \in Y} p_x$. For the list X in Example 3.1 we obtain $p_X = s_1 s_2 (s_1 + s_2)$.

Definition 4.1. Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. Then, we define

$$\mathcal{J}(X) := \mathrm{ideal}\{p_T : T \subseteq X \text{ cocircuit}\} \subseteq \mathbb{K}[s_1, \ldots, s_r] \tag{I.9}$$

$$\text{and } \mathcal{D}(X) := \ker \mathcal{J}(X) \subseteq \mathbb{K}[t_1, \ldots, t_r]. \tag{I.10}$$

$\mathcal{D}(X)$ is called the *central $\mathcal{D}$-space* or *Dahmen-Micchelli* space.

It can be shown that $\mathcal{D}(X)$ is the space spanned by the local pieces of the box spline and their partial derivatives. The box spline is defined in Subsection 5.2. The space $\mathcal{D}(X)$ was introduced in **[29]** and in **[26]** it was shown that its dimension is $|\mathbb{B}(X)|$.

Definition 4.2. Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. Then, we define the *central $\mathcal{P}$-space*

$$\mathcal{P}(X) := \mathrm{span}\{p_Y : Y \subseteq X,\, X \setminus Y \text{ has full rank}\} \subseteq \mathbb{K}[s_1, \ldots, s_r]. \tag{I.11}$$

Proposition 4.3 (**[45]**). *Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. A basis for $\mathcal{P}(X)$ is given by*

$$\mathcal{B}(X) := \{Q_B : B \in \mathbb{B}(X)\}, \tag{I.12}$$

where $Q_B := p_{X \setminus (B \cup E(B))}$.

The space $\mathcal{P}(X)$ can also be written as the kernel of an ideal. The following proposition appeared in **[28]**.

PROPOSITION 4.4. *Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. Then,*

$$\mathcal{P}(X) = \ker \mathcal{I}(X) = \ker \mathcal{I}'(X), \tag{I.13}$$

where $$\mathcal{I}(X) := \text{ideal}\left\{p_\eta^{m(\eta)} : \eta \in V \setminus \{0\}\right\} \subseteq \mathbb{K}[t_1, \ldots, t_r], \tag{I.14}$$

$$\mathcal{I}'(X) := \text{ideal}\left\{p_\eta^{m(\eta)} : \eta \in V \setminus \{0\}, \operatorname{rk}(X \cap \eta^o) = r - 1\right\}, \tag{I.15}$$

and $m : V \to \mathbb{N}$ assigns to $\eta \in V$ the number of vectors in X that are not *perpendicular to η.*

Ideals like $\mathcal{I}(X)$ that are generated by products of linear forms are called *power ideals.*

EXAMPLE 4.5. Let X be the list of vectors we defined in Example 3.1. Then,

$$\mathcal{D}(X) = \ker \text{ideal}\{s_1 s_2, s_1(s_1 + s_2), s_2(s_1 + s_2)\} = \text{span}\{1, t_1, t_2\},$$
$$\mathcal{I}(X) = \text{ideal}\{t_1^2, t_2^2, (t_1 - t_2)^2\} + \mathbb{R}[t_1, t_2]_{\geq 3} = \text{ideal}\{t_1^2, t_2^2, t_1 t_2\},$$

and $\mathcal{P}(X) = \ker \mathcal{I}(X) = \text{span}\{1, s_1, s_2\}$.

PROPOSITION 4.6 ([**45, 59**]). *Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. Then, the spaces $\mathcal{P}(X)$ and $\mathcal{D}(X)$ are dual under the pairing $\langle \cdot, \cdot \rangle$, i. e.*

$$\begin{aligned} \mathcal{D}(X) &\to \mathcal{P}(X)^* \\ f &\mapsto \langle \cdot, f \rangle \end{aligned} \tag{I.16}$$

is an isomorphism.

The preceding proposition implies that the Hilbert series of $\mathcal{P}(X)$ and $\mathcal{D}(X)$ are equal. By Proposition 4.3, this Hilbert series is a matroid invariant and a specialisation of the Tutte polynomial. These facts are summarised in the following proposition.

PROPOSITION 4.7. *Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a list of N vectors that spans U. Then,*

$$\text{Hilb}(\mathcal{D}(X), q) = \text{Hilb}(\mathcal{P}(X), q) = q^{N-r} T_{\mathfrak{M}(X)}(1, \frac{1}{q}) = \sum_{B \in \mathbb{B}(X)} q^{N-r-|E(B)|}.$$

4.2. Internal and external zonotopal spaces. In this subsection we will define two more pairs of zonotopal spaces that were introduced by Holtz and Ron in [**54**]. The internal pair $(\mathcal{D}_-(X), \mathcal{P}_-(X))$ and the external pair $(\mathcal{D}_+(X), \mathcal{P}_+(X))$ have many nice properties in common with the central pair.

First, we will define internal and external bases that are used in the definition of the internal and external $\mathcal{D}$-space.

DEFINITION 4.8 (Internal and external bases). Let $X \subseteq U \cong \mathbb{K}^r$ be a list of vectors that spans U and let $B_0 = (b_1, \ldots, b_r) \subseteq \mathbb{K}^r$ be an arbitrary basis for $\mathbb{K}^r$ that is not necessarily contained in $\mathbb{B}(X)$. Let $X' = (X, B_0)$ and let

$$\text{ex} : \{I \subseteq X : I \text{ linearly independent}\} \to \mathbb{B}(X') \tag{I.17}$$

be the function that maps an independent set in X to its greedy extension in X'. This means that given an independent set $I \subseteq X$, the vectors $b_1, \dots, b_r$ are added successively to I unless the resulting set would be linearly dependent.

Then, we define the set of *external bases* $\mathbb{B}_+(X, B_0)$ and the set of *internal bases* $\mathbb{B}_-(X)$ by

$$\mathbb{B}_+(X, B_0) := \{B \in \mathbb{B}(X, B_0) : B = \operatorname{ex}(I) \text{ for some } I \subseteq X \text{ independent}\}$$
$$\text{and } \mathbb{B}_-(X) := \{B \in \mathbb{B}(X) : B \text{ contains no internally active elements}\}.$$

Definition 4.9. Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. Then, we define

$$\mathcal{J}_+(X) := \operatorname{ideal}\{p_T : T \subseteq X \ \mathbb{B}_+(X)\text{-cocircuit}\} \subseteq \mathbb{K}[s_1, \dots, s_r], \quad \text{(I.18)}$$
$$\mathcal{D}_+(X) := \ker \mathcal{J}_+(X) \subseteq \mathbb{K}[t_1, \dots, t_r], \quad \text{(I.19)}$$
$$\mathcal{J}_-(X) := \operatorname{ideal}\{p_T : T \subseteq X \ \mathbb{B}_-(X)\text{-cocircuit}\} \subseteq \mathbb{K}[s_1, \dots, s_r], \quad \text{(I.20)}$$
$$\text{and } \mathcal{D}_-(X) := \ker \mathcal{J}_-(X) \subseteq \mathbb{K}[t_1, \dots, t_r], \quad \text{(I.21)}$$

where a $\mathbb{B}_-(X)$-cocircuit (resp. a $\mathbb{B}_+(X)$-cocircuit) is a subset of X that intersects all bases in $\mathbb{B}_-(X)$ (resp. in $\mathbb{B}_+(X)$) and that is inclusion-minimal with this property. $\mathcal{D}_+(X)$ is called the *external $\mathcal{D}$-space* and $\mathcal{D}_-(X)$ is called the *internal $\mathcal{D}$-space*.

Definition 4.10. Let $\mathbb{K}$ be some field and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. Then we define

$$\mathcal{P}_+(X) := \operatorname{span}\{p_Y : Y \subseteq X\} \text{ and } \mathcal{P}_-(X) := \bigcap_{x \in X} \mathcal{P}(X \setminus x). \quad \text{(I.22)}$$

$\mathcal{P}_+(X)$ is called the *external $\mathcal{P}$-space* and $\mathcal{P}_-(X)$ is called the *internal $\mathcal{P}$-space.*

Proposition 4.11. *Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. Then*

$$\mathcal{P}_+(X) = \ker \mathcal{I}_+(X) \text{ and } \mathcal{P}_-(X) = \ker \mathcal{I}_-(X), \quad \text{(I.23)}$$
$$\text{where } \mathcal{I}_+(X) := \operatorname{ideal}\left\{p_\eta^{m(\eta)+1} : \eta \in V \setminus \{0\}\right\} \subseteq \mathbb{R}[t_1, \dots, t_r] \quad \text{(I.24)}$$
$$\text{and } \mathcal{I}_-(X) := \operatorname{ideal}\left\{p_\eta^{m(\eta)-1} : \eta \in V \setminus \{0\}\right\} \subseteq \mathbb{R}[t_1, \dots, t_r] \quad \text{(I.25)}$$

and $m : V \to \mathbb{N}$ is defined as in Proposition 4.4.

Proposition 4.12. *Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. A basis for $\mathcal{P}_+(X)$ is given by*

$$\mathcal{B}(X, B_0) := \{Q_B : B \in \mathbb{B}_+(X, B_0)\}, \quad \text{(I.26)}$$

where $Q_B := p_{X' \setminus (B \cup E(B))}$ and $E(B) \subseteq X'$.

Remark 4.13. The internal space $\mathcal{P}_-(X)$ does not have a description as a product of linear forms. See Chapter III and in particular Remark III.3.21 for more details.

PROPOSITION 4.14 ([**54**]). *Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. Then, the internal spaces $\mathcal{P}_-(X)$ and $\mathcal{D}_-(X)$ as well as the external spaces $\mathcal{P}_+(X)$ and $\mathcal{D}_+(X)$ are dual under the pairing $\langle\cdot,\cdot\rangle$.*

PROPOSITION 4.15 ([**3, 54**]). *Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a list of N vectors that spans U. Then,*

$$\begin{aligned}\operatorname{Hilb}(\mathcal{D}_+(X),q) &= \operatorname{Hilb}(\mathcal{P}_+(X),q) = q^{N-r}T_{\mathfrak{M}(X)}(1+q,\frac{1}{q}) \\ &= \sum_{B\in\mathbb{B}_+(X,B_0)} q^{N-r-|E(B)|}\end{aligned} \tag{I.27}$$

$$\begin{aligned}\textit{and } \operatorname{Hilb}(\mathcal{D}_-(X),q) &= \operatorname{Hilb}(\mathcal{P}_-(X),q) = q^{N-r}T_{\mathfrak{M}(X)}(0,\frac{1}{q}) \\ &= \sum_{B\in\mathbb{B}_-(X)} q^{N-r-|E(B)|}.\end{aligned} \tag{I.28}$$

REMARK 4.16. More zonotopal spaces that were previously studied by other authors are described in Section II.7. In Chapter II we will define far more general pairs of zonotopal spaces. That chapter focuses on $\mathcal{D}$-spaces. In Chapter III we will study various $\mathcal{P}$-spaces and the power ideals defining them.

4.3. Least map interpolation. Carl de Boor and Amos Ron introduced the so-called least map interpolation [**31, 33**]. Given a finite set $S \subseteq V$, they construct a space of polynomials $\Pi(S) \subseteq \operatorname{Sym}(V)$ of dimension $|S|$. The space $\Pi(S)$ has several nice properties related to interpolation problems.

Let $\mathbb{K}$ be a field of characteristic zero. Recall that $U \cong \mathbb{K}^r$, V denotes the dual space and a vector $v \in V$ defines a linear form $p_v \in \mathbb{K}[t_1,\ldots,t_r] \cong \operatorname{Sym}(V)$. We define the exponential function as usual by

$$e^v := \sum_{j\geq 0} \frac{p_v^j}{j!} \in \mathbb{K}[[t_1,\ldots,t_r]] \cong \operatorname{Sym}(U)^*. \tag{I.29}$$

The *least map* $\downarrow$ maps a non-zero element of the ring of formal power series $\mathbb{K}[[t_1,\ldots,t_r]]$ to its homogeneous component of lowest degree that is non-zero. The *least space* of a finite set $S \subseteq V$ is defined as

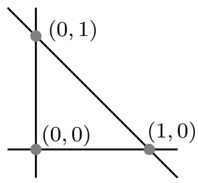
$$\Pi(S) := \operatorname{span}\{f_\downarrow : f \in \operatorname{span}\{e^v : v \in S\}\} \subseteq \mathbb{K}[t_1,\ldots,t_r]. \tag{I.30}$$

The following surprising theorem makes a connection between hyperplane arrangements and the space $\mathcal{D}(X)$. It generalises to other $\mathcal{D}$-spaces (see [**54, 55, 69**]).

THEOREM 4.17 ([**31**]). *Let $\mathbb{K}$ be a field of characteristic zero and let $X \subseteq U \cong \mathbb{K}^r$ be a finite list of vectors that spans U. Let $c \in \mathbb{K}^X$ be a vector s. t. the hyperplane arrangement $\mathcal{H}(X,c)$ is in general position and let S be the set of vertices of $\mathcal{H}(X,c)$. Then,*

$$\mathcal{D}(X) = \Pi(S). \tag{I.31}$$

Example 4.18. Let $X = (e_1, e_2, e_1 + e_2)$ be the list of vectors we introduced in Example 3.1.
Let $c_1 = c_2 = 0$ and $c_3 = 1$. The set of vertices of $\mathcal{H}(X, c)$ is $S = \{(0,0), (1,0), (0,1)\}$. Then,



$$\begin{aligned}\Pi(S) &= \operatorname{span}\{f_{\downarrow} : f \in \operatorname{span}\{1, e^{t_1}, e^{t_2}\}\} \\ &= \operatorname{span}\{1, t_1, t_2\}, \text{ since } 1 = 1_{\downarrow}, \\ &\qquad t_1 = (e^{t_1} - 1)_{\downarrow}, \text{ and } t_2 = (e^{t_2} - 1)_{\downarrow}.\end{aligned}$$

Now we will describe a connection between zonotopes and $\mathcal{P}$-spaces. Since zonotopes can only be defined in Euclidian space, we now require $\mathbb{K} = \mathbb{R}$. Let $\Lambda \subseteq U \cong \mathbb{R}$ be a lattice of covolume one, *i. e.* Λ has a lattice basis with determinant 1 or -1 (a typical example is $\Lambda = \mathbb{Z}^r$). Recall that a list of vectors $X \subseteq \Lambda$ is called *totally unimodular* if every (vector space) basis $B \subseteq X$ has determinant 1 or -1.

Theorem 4.19 ([**54**]). *Let X be a list of vectors that is contained in a lattice $\Lambda \subseteq U \cong \mathbb{R}^r$ of covolume one. Suppose that X is totally unimodular. Let $\tau \in U$ be a vector that is not contained in any strict subspace of U that is spanned by a sublist of X. Then,*

$$\Pi(Z(X) \cap \Lambda) = \mathcal{P}_+(X), \tag{I.32}$$

$$\Pi(\mathring{Z}(X) \cap \Lambda) = \mathcal{P}_-(X), \tag{I.33}$$

$$\textit{and } \Pi((Z(X) - \tau) \cap \Lambda) = \mathcal{P}(X) \tag{I.34}$$

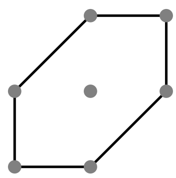
holds, where $\mathring{Z}(X)$ denotes the interior of the zonotope $Z(X)$.

Remark 4.20. Under the assumptions of Theorem 4.19,

$$|(Z(X) - \tau) \cap \Lambda)| = \sum_{B \in \mathbb{B}(X)} \det(B) = \operatorname{vol}(Z(X)) \tag{I.35}$$

holds (see *e. g.* [**36**, Proposition 2.50]).

Example 4.21. Let $X = (e_1, e_2, e_1 + e_2)$ be the list of vectors we introduced in Example 3.1.
The zonotope $Z(X)$ has volume three, seven lattice points and one interior lattice point.



$$\begin{aligned}\mathcal{P}_+(X) &= \operatorname{span}\{1,\, s_1,\, s_2,\, s_1^2,\, s_1 s_2,\, s_2^2\} \\ \mathcal{P}(X) &= \operatorname{span}\{1,\, s_1,\, s_2\} \\ \mathcal{P}_-(X) &= \operatorname{span}\{1\}\end{aligned}$$

5. Analysis

In this section we will discuss distributions and splines.

5.1. Distributions. The algebraic objects we are mainly interested in are multivariate polynomials. However, in the construction in Section II.2 more general objects appear as intermediate products. In this construction we need "generalised polynomials" whose support is contained in a subspace. Furthermore, we use convolutions and the fact that convolutions and partial derivatives commute. Distributions have all of the desired properties. In this subsection we will summarise important facts about distributions that we will need later on. For a detailed introduction to the subject, we refer the reader to Laurent Schwartz's book [**80**].

A *distribution* on a vector space $U \cong \mathbb{R}^r$ (or an open subset of U) is a continuous linear functional that maps a test function to a real number. *Test functions* are compactly supported smooth functions $U \to \mathbb{R}$. An important example is the delta distribution δ_x given by $\delta_x(\varphi) := \varphi(x)$. A locally integrable function $f : U \to \mathbb{R}$ defines a distribution T_f in the following way:

$$T_f(\varphi) := \int_U f(u)\varphi(u)\,\mathrm{d}u. \tag{I.36}$$

Recall that for two functions $f, g : U \to \mathbb{R}$, the *convolution* is defined as

$$f * g := \int_U f(u)g(\cdot - u)\,\mathrm{d}u. \tag{I.37}$$

This is well-defined only if f and g decay sufficiently rapidly at infinity in order for the integral to exist. The convolution of two distributions can also be defined under certain conditions.

A distribution T vanishes on a set $\Gamma \subseteq U$ if $T(\varphi) = 0$ for all test functions whose support is contained in Γ. The support $\operatorname{supp}(T)$ of T is the complement of the maximal open set on which T vanishes.

Let S_ξ and T_η be two distributions for which $\operatorname{supp}(S_\xi) \cap (K - \operatorname{supp}(T_\eta))$ is compact for any compact set K. Let $\varphi : U \to \mathbb{R}$ be a test function with support K. Then, we define the convolution

$$(S_\xi * T_\eta)(\varphi) := S_\xi(T_\eta(\alpha(\xi)\varphi(\xi + \eta))), \tag{I.38}$$

where α is a test function that is equal to one on a neighbourhood of $\operatorname{supp}(S_\xi) \cap (K - \operatorname{supp}(T_\eta))$. When evaluating $T_\eta(\alpha(\xi)\varphi(\xi + \eta))$, we think of $\varphi(\xi+\eta)$ as a function in η and of ξ as a fixed parameter. Then, $T_\eta(\alpha(\xi)\varphi(\xi+\eta))$ is a function in ξ with compact support that is contained in $K - \operatorname{supp}(T_\eta)$. Note that the definition of $(S_\xi * T_\eta)(\varphi)$ is independent of the choice of the function α. The multiplication by α is necessary to ensure that $T_\eta(\alpha(\xi)\varphi(\xi+\eta))$ as a function in ξ has compact support.

Note that the convolution of two distributions is a commutative operation and $T * \delta_0 = T$. Let $u \in U$. The partial derivative of a distribution T in direction u is defined by $(D_u T)(\varphi) := -T(D_u \varphi)$. Convolutions of distributions have the same nice property with respect to partial derivatives as convolutions of functions. Namely, if T_1 and T_2 are distributions on U and $u \in U$, then

$$D_u(T_1 * T_2) = (D_u T_1) * T_2 = T_1 * (D_u T_2). \tag{I.39}$$

5.2. Splines. In this subsection we will introduce multivariate splines and box splines as in [**36**, Chapter 7]. Another good reference is [**30**].

DEFINITION 5.1. Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors. The *multivariate spline* (or truncated power) T_X and the *box spline* B_X are distributions that are characterised by the formulae

$$\int_U f(u)B_X(u)\,\mathrm{d}u = \int_0^1 \cdots \int_0^1 f\left(\sum_{i=1}^N \lambda_i x_i\right) \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N \tag{I.40}$$

$$\text{and} \quad \int_U f(u)T_X(u)\,\mathrm{d}u = \int_0^\infty \cdots \int_0^\infty f\left(\sum_{i=1}^N \lambda_i x_i\right) \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N. \tag{I.41}$$

The multivariate spline is well-defined only if the convex hull of the vectors in X does not contain 0 or equivalently, if there is a functional $\varphi \in V$ s. t. $\varphi(x) > 0$ for all $x \in X$. If all vectors are non-zero, it is of course always possible to multiply certain entries of the list X by -1 s. t. this condition is satisfied. Note that in Definition 5.1, we do *not* require that X spans U in contrast to most of the rest of this thesis.

B_X and T_X can be identified with the functions

$$B_X(u) = \frac{1}{\sqrt{\det(XX^T)}} \operatorname{vol}_{N-\dim(\operatorname{span}(X))}\{z \in [0;1]^N : Xz = u\} \tag{I.42}$$

$$\text{and } T_X(u) = \frac{1}{\sqrt{\det(XX^T)}} \operatorname{vol}_{N-\dim(\operatorname{span}(X))}\{z \in \mathbb{R}^N_{\geq 0} : Xz = u\}. \tag{I.43}$$

It follows immediately from (I.42) and (I.43) that B_X is supported in the zonotope $Z(X)$ and T_X is supported in the cone cone(X). For a basis $C \subseteq U$,

$$B_C = \frac{\chi_{Z(C)}}{|\det(C)|} \text{ and } T_C = \frac{\chi_{\operatorname{cone}(C)}}{|\det(C)|}. \tag{I.44}$$

REMARK 5.2. The box spline can easily be obtained from the multivariate spline. Namely,

$$B_X(x) = \sum_{S \subseteq X} (-1)^{|S|} T_X(x - a_S), \tag{I.45}$$

where $a_S := \sum_{a \in S} a$.

The multivariate spline plays an important role in Chapter II. We introduced the box spline only because of its importance in approximation theory.

THEOREM 5.3. *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and whose convex hull does not contain* 0. *The cone* cone(X) *can be decomposed into finitely many cones C_i s. t. T_X restricted to each C_i is a homogeneous polynomial of degree $N - r$.*

THEOREM 5.4. *The space spanned by the local pieces of the multivariate spline T_X and their partial derivatives is equal to the Dahmen-Micchelli space $\mathcal{D}(X)$ that was defined in Definition* 4.1.

The multivariate spline can also be defined inductively by the convolution formula

$$T_{(X,x)} = T_X * T_{(x)} = \int_0^\infty T_X(\cdot - \lambda x)\,\mathrm{d}\lambda \tag{I.46}$$

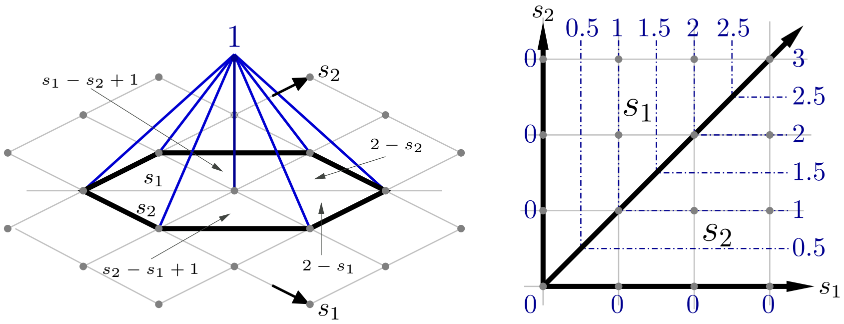


FIGURE 1. The box spline and the multivariate spline defined by the list X in Example 3.1. The multivariate spline is calculated in Example 5.5. On the right, the dashed lines are level curves.

using (I.44) as a starting point. In particular, $T_X = T_{x_1} * \cdots * T_{x_N}$. Since $D_x T_x = \delta_0$, the convolution formula implies for $Y \subseteq X$ that

$$D_Y T_X = T_{X \setminus Y} \text{ where } D_Y := \prod_{x \in Y} D_x. \tag{I.47}$$

EXAMPLE 5.5. We consider the same list X as in Example 3.1. By (I.44), $T_{(x_1,x_2)}$ is the indicator function of $\mathbb{R}^2_{\geq 0}$. Then, by (I.46), we can deduce

$$T_X(s_1, s_2) = \int_0^\infty \chi_{\mathbb{R}^2_{\geq 0}}(s_1 - \lambda, s_2 - \lambda)\, \mathrm{d}\lambda = \min(s_1, s_2). \tag{I.48}$$

See Figure 1 for a graphic description of T_X.

EXAMPLE 5.6. Let $X_i := (\underbrace{1, \ldots, 1}_{i \text{ times}})$. Then,

$$T_{X_1}(s) = \chi_{\mathbb{R}_{\geq 0}}(s) \tag{I.49}$$

$$\text{and} \quad T_{X_{i+1}}(s) = \int_0^\infty T_{X_i}(s - \lambda)\, \mathrm{d}\lambda = \int_0^s \frac{\lambda^{i-1}}{(i-1)!}\, \mathrm{d}\lambda = \frac{s^i}{i!} \text{ for } s \geq 0.$$

6. Arithmetic matroids

In this section we will introduce arithmetic matroids. They only appear again in the very last section of this thesis. Arithmetic matroids are matroids together with a so-called multiplicity function. In the realisable case, the multiplicity function records the determinants of the bases. Arithmetic matroids are the analogues of matroids in the discrete theory (cf. Section 1 and Table 1).

An arithmetic matroid is a pair $(\mathfrak{M}, m)$, where $\mathfrak{M}$ is a matroid on the ground set A and $m : 2^A \to \mathbb{Z}_{\geq 0}$ is a function that satisfies certain axioms **[22, 74]**. The function m is called a *multiplicity function*. The prototype of an arithmetic matroid is the one that is canonically associated with a finite list A of elements of a finitely generated abelian group G. Recall that such a group is isomorphic to $G = \mathbb{Z}^r \oplus G_t$ for some $r \in \mathbb{N}$ and some finite group G_t (the torsion subgroup of G). Given a sublist $S \subseteq A$, we denote by $\langle S \rangle$ the subgroup of G generated by S. We define the rank of a sublist $S \subseteq A$ as

continuous theory	discrete theory
X list of vectors in a vector space	X list of vectors in a lattice
box spline / multivariate spline	vector partition function
hyperplane arrangement	toric arrangement
continuous zonotopal spaces	discrete zonotopal spaces
matroid	arithmetic matroid

TABLE 1. Continuous and discrete zonotopal algebra

the maximal rank of a free abelian subgroup of $\langle S\rangle$. This defines a matroid structure on A. For $S \in 2^A$, let G_S be the maximal subgroup of G that contains S and in which the subgroup index $[G_S : \langle S\rangle]$ is finite. We define the multiplicity of S as $m(S) := [G_S : \langle S\rangle]$.

Recall that matroids have a nice duality theory and that they come with a Tutte polynomial $T_{\mathfrak{M}}(x,y) = \sum_{S\subseteq A}(x-1)^{\mathrm{rk}(A)-\mathrm{rk}(S)}(y-1)^{|S|-\mathrm{rk}(S)}$ which captures a lot of information about the matroid. Luca Moci and Michele D'Adderio have shown that arithmetic matroids also have a nice duality theory and that they come with an *arithmetic Tutte polynomial*:

$$M_{(\mathfrak{M},m)}(x,y) := \sum_{S\subseteq A} m(S)(x-1)^{\mathrm{rk}(A)-\mathrm{rk}(S)}(y-1)^{|S|-\mathrm{rk}(S)}. \qquad (I.50)$$

Other work on arithmetic matroids includes [**10, 23, 24**].

The *discrete Dahmen-Micchelli space* $\mathrm{DM}(X)$ is an example of a discrete zonotopal space. It is defined like $\mathcal{D}(X)$ but differential operators are replaced by difference operators. It is a space of quasi-polynomials that is spanned by the local pieces of the vector partition function. The dimension of $\mathrm{DM}(X)$ agrees with the volume of the zonotope $Z(X)$ for any list $X \subseteq \mathbb{Z}^r$ in contrast to $\mathcal{D}(X)$, where this holds only if X is totally unimodular. A discrete analogue of Proposition 4.7 [**74**, Theorem 6.3] states that

$$\dim \mathrm{DM}(X) = q^{N-r} M_{(\mathfrak{M},m)}(1, \frac{1}{q}). \qquad (I.51)$$

where $(\mathfrak{M}, m)$ denotes the arithmetic matroid defined by the list X.

7. Remarks on the notation, level of abstraction and ground fields

As zonotopal spaces have been studied by people from different fields, the notation and the level of abstraction used in the literature varies. Authors with a background in spline theory usually work over $\mathbb{R}^r$ and identify it with its dual space via the canonical inner product even though many of their results hold for other fields as well. Other authors work in a more abstract setting as we do.

So what is the "right" field to work over? $\mathcal{P}$-spaces can be defined over any field (*e. g.* [**7**]). If one studies power ideals and their kernels, it is helpful to assume that the ground field $\mathbb{K}$ has characteristic zero. Otherwise, problems might arise essentially because in characteristic p equalities like $(t_1+t_2)^p = t_1^p + t_2^p$ hold. Of course, if one is interested in connections with splines and zonotopes, one has to work in a Euclidian setting.

In Chapter II we will work over the real numbers because our construction involves splines. In Chapter III we will work over a field of characteristic zero because we study power ideals. A reader with no background in abstract algebra may safely assume $\mathbb{K} = \mathbb{R}$ and work with polynomial rings instead of symmetric algebras everywhere in this thesis. This setting captures all of the important ideas.

Some authors (*e. g.* [**3**]) work in a dual setting and consider a central hyperplane arrangement $\mathcal{A}$ instead of a finite list of vectors X. Both settings are equivalent but for us it is more natural to work with a list of vectors since we are also interested in the zonotope $Z(X)$ and the multivariate spline T_X.

CHAPTER II

Zonotopal Algebra and Forward Exchange Matroids

The first of the two main results in this chapter is the construction of a canonical basis for the Dahmen-Micchelli space $\mathcal{D}(X)$. We show that it is dual to the canonical basis for $\mathcal{P}(X)$ that is already known.

The second main result is the construction of a new family of zonotopal spaces that is far more general than the ones that were recently studied by Ardila-Postnikov, Holtz-Ron, Holtz-Ron-Xu, Li-Ron, and others. We call the underlying combinatorial structure of those spaces forward exchange matroid.

1. Introduction

Given a pair of vector spaces that are dual to each other, it is often helpful to have a pair of dual bases for the two spaces. It is known that there is a canonical way to construct bases for the spaces of $\mathcal{P}$-type (see [**3, 45, 54, 55, 69**] and Proposition I.4.3 and I.4.12).

The first of the two main results in this paper is that there is an algorithm that produces a canonical basis for the spaces of $\mathcal{D}$-type that is dual to the canonical basis for the spaces of $\mathcal{P}$-type. Here, canonical means that the basis we obtain only depends on the order of the elements in the list X and not on any further choices. The two previously known algorithms that construct a basis for spaces of $\mathcal{D}$-type depend on additional choices [**25, 32**].

Our second main result is that far more general pairs of zonotopal spaces with nice properties can be constructed than the ones that were previously known. We will define a new combinatorial structure called forward exchange matroid. A forward exchange matroid is an ordered matroid together with a subset of its set of bases that satisfies a weak version of the basis exchange axiom. This is the underlying structure of the generalised zonotopal $\mathcal{D}$-spaces and $\mathcal{P}$-spaces that we introduce.

This chapter is based on the preprint [**68**].

1.1. Outline of this chapter. This chapter is organised as follows. In Section 2 we will construct certain polynomials R^B as convolutions of differences of multivariate splines. In Section 3 we will show that the set

$$\mathcal{B}(X) := \{\det(B)R^B : B \in \mathbb{B}(X)\} \qquad \text{(II.1)}$$

is a basis for $\mathcal{D}(X)$ and we will prove that this basis is dual to the basis $\mathcal{B}(X)$ for $\mathcal{P}(X)$. In Section 4 we will discuss deletion-contraction and two short exact sequences. In Section 5 we will introduce a new combinatorial structure called forward exchange matroid. This is an ordered matroid together with a subset $\mathbb{B}'$ of its set of bases with the so-called forward exchange property. In Section 6 we will introduce the generalised $\mathcal{P}$-space $\mathcal{P}(X,\mathbb{B}') := \operatorname{span}\{Q_B : B \in \mathbb{B}'\}$ and the generalised $\mathcal{D}$-space $\mathcal{D}(X,\mathbb{B}')$. We will show that most of the results that we have described in Section I.4 and in Section 3 still hold for those spaces if $\mathbb{B}'$ has the forward exchange property. For example, the two spaces are dual and a suitable subset of $B(X)$ will turn out to be a basis for $\mathcal{D}(X,\mathbb{B}')$. Furthermore, $\mathcal{D}(X,\mathbb{B}')$ and $\mathcal{P}(X,\mathbb{B}')$ have deletion-contraction decompositions that are related to the deletion-contraction reduction of the Tutte polynomial.

In Section 7 we will review the previously known zonotopal spaces and we will show that they are special cases of our spaces $\mathcal{D}(X,\mathbb{B}')$ and $\mathcal{P}(X,\mathbb{B}')$.

2. Construction of basis elements

In this section we will construct a polynomial R_Z^B in $\mathbb{R}[s_1,\ldots,s_r]$, given a finite list $Z \subseteq U \cong \mathbb{R}^r$ and a basis $B \subseteq Z$. Later on we will show that polynomials of this type form bases for various zonotopal $\mathcal{D}$-spaces if one chooses suitable pairs (B,Z). The polynomial R_Z^B is constructed as a convolution of differences of multivariate splines.

Let $Z \subseteq U$ be a finite list and let $B = (b_1,\ldots,b_r) \subseteq Z$ be a basis. It is important that the basis is ordered and that this order is the order obtained by restricting the order on Z to B. For $i \in \{0,\ldots,r\}$, we define $S_i = S_i^B := \operatorname{span}\{b_1,\ldots,b_i\}$. Hence,

$$\{0\} = S_0^B \subsetneq S_1^B \subsetneq S_2^B \subsetneq \ldots \subsetneq S_r^B = U \cong \mathbb{R}^r \tag{II.2}$$

is a flag of subspaces. We define an orientation on each of the spaces S_i by saying that $(b_1,\ldots,b_i)$ is a positive basis for S_i. Now a basis $D = (d_1,\ldots,d_i)$ for S_i is called positive if the map that sends b_ν to d_ν for $1 \le \nu \le i$ has positive determinant.

Let $u \in S_i \setminus S_{i-1}$. If $(b_1,\ldots,b_{i-1},u)$ is a positive basis, we call u positive. Otherwise, we call u negative. We partition $Z \cap (S_i \setminus S_{i-1})$ as follows:

$$P_i^B := \{u \in Z \cap (S_i \setminus S_{i-1}) : u \text{ positive}\} \tag{II.3}$$

$$\text{and } N_i^B := \{u \in Z \cap (S_i \setminus S_{i-1}) : u \text{ negative}\}. \tag{II.4}$$

We define

$$T_i^{B+} := (-1)^{|N_i|} \cdot T_{P_i} * T_{-N_i} \text{ and } T_i^{B-} := (-1)^{|P_i|} \cdot T_{-P_i} * T_{N_i}. \tag{II.5}$$

Note that T_i^{B+} is supported in $\operatorname{cone}(P_i, -N_i)$ and that

$$T_i^{B-}(x) = (-1)^{|P_i \cup N_i|} T_i^{B+}(-x). \tag{II.6}$$

Now define

$$R_i^B := T_i^{B+} - T_i^{B-} \qquad \text{and} \qquad R_Z^B = R^B := R_1^B * \cdots * R_r^B. \tag{II.7}$$

For an example of this construction see Example 3.4 and Figure 2. In Corollary 2.4, we will see that the distribution R_Z^B can be identified with a homogeneous polynomial.

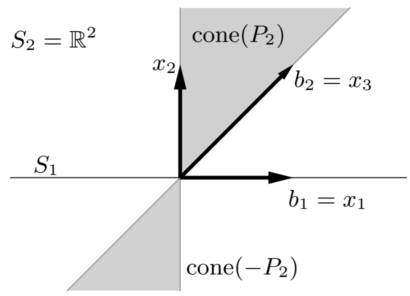


FIGURE 2. The geometry of the construction of the polynomial $R_X^{(x_1,x_3)}$ in Example 3.4. Note that $N_2 = \emptyset$.

REMARK 2.1. A similar construction of certain quasi-polynomials in the discrete case is done in [**38**, Section 3] (see also [**36**, Section 13.6]). The part of Theorem 3.2 that exhibits a basis for $\mathcal{D}(X)$ can be seen as a special case of Theorem 3.22 in [**38**].

REMARK 2.2. The construction of the polynomials R_Z^B may at first seem rather complicated in comparison with the construction of the polynomials Q_B that form bases of the $\mathcal{P}$-spaces.

Here are a few remarks to explain this construction: multivariate splines are very convenient because it is so easy to calculate their partial derivatives (cf. (I.47)). Taking differences of two splines in the definition of R_i^B ensures that R_Z^B is a polynomial and not just piecewise polynomial. In fact, $R_1^B * \ldots * R_i^B$ is a "polynomial supported in S_i" for all i.

We have to change the sign of some of the vectors before constructing the multivariate spline T_i^{B+} to ensure that all the convolutions are well-defined. For example, the convolutions in (II.7) are well-defined for the following reason: the support of $R_1^B * \cdots * R_i^B$ is contained in S_i. The support of R_{i+1}^B is $\operatorname{cone}(P_{i+1}, -N_{i+1}) \cup \operatorname{cone}(-P_{i+1}, N_{i+1})$. For every compact set K, the set

$$(S_i \cap (K - (\operatorname{cone}(P_{i+1}, -N_{i+1}) \cup \operatorname{cone}(P_{i+1}, -N_{i+1})) \qquad \text{(II.8)}$$

is compact.

PROPOSITION 2.3. *The distribution R_Z^B is a local piece of the multivariate spline $T_1^{B+} * \cdots * T_r^{B+}$.*

PROOF. Let $c \gg 0$ and let

$$\tau := b_1 + \frac{1}{c} b_2 + \ldots + \frac{1}{c^{r-2}} b_{r-1} + \frac{1}{c^{r-1}} b_r. \qquad \text{(II.9)}$$

See Figure 3 for an example of this construction. The vector τ is contained in cone(Z). By Theorem I.5.3 there exists a subcone of cone(Z) that contains τ s. t. T_Z agrees with a polynomial $p_{\tau,Z}$ on this subcone. We claim that R_Z^B is equal to $p_{\tau,Z}$. Note that

$$R_Z^B = (T_1^{B+} - T_1^{B-}) * \cdots * (T_r^{B+} - T_r^{B-}) \qquad \text{(II.10)}$$

$$= \sum_{J \subseteq [r]} (-1)^{|J| + \sum_{i \notin J} |N_i| + \sum_{i \in J} |P_i|} T_{Z_B^J}, \qquad \text{(II.11)}$$

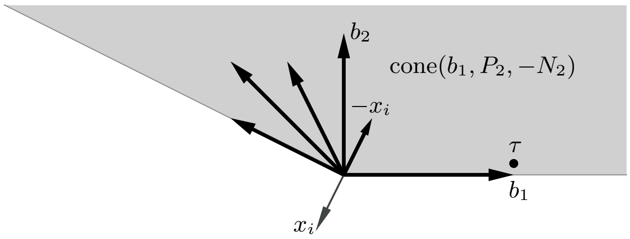


FIGURE 3. The setup in the proof of Proposition 2.3.

where

$$Z_B^J = \bigcup_{i \notin J} (P_i, -N_i) \cup \bigcup_{i \in J} (-P_i, N_i). \tag{II.12}$$

In order to prove our claim, it is sufficient to show that τ is contained in cone(Z_B^J) if and only if $J = \emptyset$. The "if" part is clear.

Let J be non-empty and let j^* be the minimal element. For $\alpha \in \mathbb{R}$ let $\phi_\alpha : U \to \mathbb{R}$ be the linear form that maps a vector x to

$$\sum_{j=j^*}^{r} (-1)^{\chi_J(j)} \alpha^j \lambda_j, \tag{II.13}$$

where λ_j denotes the coefficient of b_j when x is written in the basis $(b_1, \ldots, b_r)$. We claim that for sufficiently large α, ϕ_α is non-negative on Z_B^J and $\phi_\alpha(\tau) < 0$. By Farkas' Lemma (*e. g.* [**79**, Section 5.5]), this proves that τ is not contained in cone(Z_B^J).

If $x \in S_i \cap Z_B^J$ for $i < j^*$, then obviously $\phi_\alpha(x) = 0$. If $x \in (S_i \setminus S_{i-1}) \cap Z_B^J$ for $i \geq j^*$, then $\lambda_i \neq 0$ and $\lambda_\nu = 0$ for all $\nu \geq i+1$. In addition, $(-1)^{\chi_J(i)} \lambda_i > 0$, since all vectors in $(P_i, -N_i)$ have a positive b_i component when written in the basis $(b_1, \ldots, b_r)$. Hence, $\phi_\alpha(x) = (-1)^{\chi_J(i)} \alpha^i \lambda_i + o(\alpha^i) = \alpha^i \, |\lambda_i| + o(\alpha^i)$. This is positive for sufficiently large α. For τ, we obtain

$$\phi(\tau) = -\frac{\alpha^{j^*}}{c^{j^*-1}} \pm \frac{\alpha^{j^*+1}}{c^{j^*}} \pm \ldots = -\frac{\alpha^{j^*}}{c^{j^*-1}} + o\left(\frac{1}{c^{j^*-1}}\right). \tag{II.14}$$

This is negative for sufficiently large c. Note that we fix a large α first and then we let c grow. □

Note that the distribution R_Z^B does not change if we add or remove zero vectors from the list Z. Using Theorem I.5.3, we can deduce the following corollary.

COROLLARY 2.4. *Let $\tilde{Z}$ be the list of vectors obtained from Z by removing all copies of the zero vector. The distribution $R_Z^B = R_{\tilde{Z}}^B$ can be identified with a homogeneous polynomial of degree $|\tilde{Z}| - r$.*

REMARK 2.5. The local pieces of the multivariate spline are uniquely determined by a certain equation (cf. [**36**, Theorems 9.5 and 9.7]). Taking into account Proposition 2.3, this gives us a different method to calculate the polynomials R_Z^B.

The following theorem that is due to Zhiqiang Xu will yield another formula for the polynomials R_Z^B. It is a variant of Brion's formula [**12**].

THEOREM 2.6 ([**91**, Theorem 3.1.]). *Let $X \subseteq U \cong \mathbb{R}^r$ be a list of N vectors that spans U. Let $c \in \mathbb{R}^X$ be a vector s. t. the hyperplane arrangement $\mathcal{H}(X,c)$ is in general position. For a basis $B \in \mathbb{B}(X)$, let $\theta_B \in V$ denote the vertex of $\mathcal{H}(X,c)$ corresponding to B (cf. Subsection* I.3.2*). Then*

$$T_X(u) = \frac{1}{(N-r)!} \sum_{B \in \mathbb{B}(X)} \frac{(-\theta_B u)^{N-r}}{|\det(B)| \prod_{x \in X \setminus B} (\theta_B x - c_x)} \chi_{\mathrm{cone}(B)}(u). \quad \text{(II.15)}$$

Note that the numerator (II.15) is non-zero because c is chosen s. t. $\mathcal{H}(X,c)$ is in general position. Using Proposition 2.3, one can deduce the following corollary.

COROLLARY 2.7. *Let $Z \subseteq U \cong \mathbb{R}^r$ be a list of n vectors that spans U. Let c and θ_B as in Theorem* 2.6*. Then, the polynomial $R_Z^B(u)$ is given by*

$$R_Z^B(u) = \frac{1}{(n-r)!} \sum_{\substack{B' \in \mathbb{B}(Z^{B+}) \\ \tau \in \mathrm{cone}(B')}} \frac{(-\theta_{B'} u)^{n-r}}{|\det(B')| \prod_{x \in Z \setminus B'} (\theta_{B'} x - c_x)}, \quad \text{(II.16)}$$

where τ denotes the vector defined in (II.9) *and Z^{B+} denotes the reorientation of the list Z s. t. all vectors are positive with respect to B, i. e. $Z^{B+} = \bigcup_{i=1}^r (P_i, -N_i)$.*

3. A basis for the Dahmen-Micchelli space $\mathcal{D}(X)$

In this section we will define a set $Б(X)$ and we will show that this set is a basis for the central $\mathcal{D}$-space $\mathcal{D}(X)$. Furthermore, we will show that this basis is dual to the basis $\mathcal{B}(X)$ of the central $\mathcal{P}$-space $\mathcal{P}(X)$. Note that $Б$ is the equivalent of the letter B in the Cyrillic alphabet.

DEFINITION 3.1 (Basis for $\mathcal{D}(X)$). Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U. Recall that $\mathbb{B}(X)$ denotes the set of bases that can be selected from X and that $E(B)$ denotes the set of externally active elements with respect to a basis B. We define

$$Б(X) := \{\det(B) R^B_{X \setminus E(B)} : B \in \mathbb{B}(X)\}. \quad \text{(II.17)}$$

THEOREM 3.2. *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U. Then, $Б(X)$ is a basis for the central Dahmen-Micchelli space $\mathcal{D}(X)$ and this basis is dual to the basis $\mathcal{B}(X)$ for the central $\mathcal{P}$-space $\mathcal{P}(X)$.*

REMARK 3.3. $\mathcal{D}(X)$ and $\mathcal{P}(X)$ are independent of the order of the elements of X. The bases $\mathcal{B}(X)$ and $Б(X)$ both depend on that order. In Theorem 3.2, we assume that both bases are constructed using the same order.

EXAMPLE 3.4. This is a continuation of Example I.3.1. See also Figure 2 on page 27. The elements of $\mathcal{B}(X)$ are

$$R^{(x_1,x_2)}_{(x_1,x_2)} = 1, \tag{II.18}$$

$$R^{(x_1,x_3)}_{X} = (T_{x_1} - T_{-x_1}) * (T_{(x_2,x_3)} - T_{(-x_2,-x_3)}) = s_2, \tag{II.19}$$

$$\text{and } R^{(x_2,x_3)}_{X} = (T_{x_2} - T_{-x_2}) * (T_{(x_1,x_3)} - T_{(-x_1,-x_3)}) = s_1. \tag{II.20}$$

The elements of $\mathcal{B}(X)$ are

$$Q_{(x_1,x_2)} = p_\emptyset = 1, \tag{II.21}$$

$$Q_{(x_1,x_3)} = p_{x_2} = t_2, \tag{II.22}$$

$$\text{and } Q_{(x_2,x_3)} = p_{x_1} = t_1. \tag{II.23}$$

$\mathcal{B}(X)$ and $\mathcal{B}(X)$ are obviously dual bases.

The proof of Theorem 3.2 is split into four lemmas. Recall that for a basis $B = (b_1, \ldots, b_r)$ we defined a flag of subspaces $\{0\} = S_0^B \subsetneq S_1^B \subsetneq \ldots \subsetneq S_r^B = U \cong \mathbb{R}^r$, where $S_i^B := \operatorname{span}(b_1, \ldots, b_i)$.

LEMMA 3.5 (Annihilation criterion). *Let $Z \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors and let $B \subseteq Z$ be a basis. Let R_Z^B be the polynomial that is defined in* (II.7). *Let $C \subseteq Z$. Suppose there exists $i \in [r]$ s. t. $Z \cap (S_i^B \setminus S_{i-1}^B) \subseteq C$. Then, $D_C R_Z^B = 0$.*

PROOF. Note that $D_a T_{-a} = -\delta_0$. Using (I.47), we obtain

$$\begin{aligned} D_C R_i^B &= D_C((-1)^{|N_i|} T_{P_i} * T_{-N_i} - (-1)^{|P_i|} T_{-P_i} * T_{N_i})) \\ &= D_{C \setminus (S_i \setminus S_{i-1})}(\delta_0 - \delta_0) = 0. \end{aligned}$$

This implies

$$D_C R_Z^B = D_{C \setminus (S_i \setminus S_{i-1})} R_1^B * \cdots * R_{i-1}^B * 0 * R_{i+1}^B * \cdots * R_r^B = 0. \quad \square$$

LEMMA 3.6 (Inclusion). *The polynomial $R^B_{X \setminus E(B)}$ is contained in $\mathcal{D}(X \setminus E(B))$ for all $B \in \mathbb{B}(X)$. Since $\mathcal{D}(X \setminus E(B)) \subseteq \mathcal{D}(X)$, this implies*

$$\mathcal{B}(X) \subseteq \mathcal{D}(X). \tag{II.24}$$

PROOF. Let $\det(B) R^B_{X \setminus E(B)} \in \mathcal{B}(X)$ and let $C \subseteq X \setminus E(B)$ be a cocircuit, *i. e.* C intersects all bases that can be selected from $X \setminus E(B)$. We need to show that $D_C R^B_{X \setminus E(B)} = 0$. C can be written as $C = X \setminus (H \cup E(B))$ for some hyperplane $H \subseteq U$.

Let i be minimal s. t. $S_i \not\subseteq H$. Such an i must exist since $S_r = U$. Even $(S_i \setminus S_{i-1}) \cap H = \emptyset$ holds. This implies

$$(X \setminus E(B)) \cap (S_i \setminus S_{i-1}) \subseteq X \setminus (H \cup E(B)) = C. \tag{II.25}$$

By Lemma 3.5, this implies $D_C R^B_{X \setminus E(B)} = 0$. $\square$

The following lemma will be used only in the proof of Lemma 3.8.

LEMMA 3.7. *Let $B, D \in \mathbb{B}(X)$. Suppose that both bases are distinct but have the same number of externally active elements.*

Then, there exists $i \in [r]$ s. t.

$$(X \setminus E(D)) \cap (S_i^D \setminus S_{i-1}^D) \subseteq X \setminus (B \cup E(B)). \qquad \text{(II.26)}$$

PROOF. Let $B = (b_1, \ldots, b_r)$ and $D = (d_1, \ldots, d_r)$. Suppose that the lemma is false. Then, there exist vectors $z_1, \ldots, z_r$ s. t.

$$z_i \in (X \setminus E(D)) \cap (S_i^D \setminus S_{i-1}^D) \cap (B \cup E(B)). \qquad \text{(II.27)}$$

These vectors form a basis because $z_i \in S_i^D \setminus S_{i-1}^D$. Since z_i is not contained in $E(D)$, $z_i \le d_i$ must hold. This implies $E(D) \subseteq E(z_1, \ldots, z_r)$. On the other hand, $E(z_1, \ldots, z_r) \subseteq E(B)$ since all z_i are contained in $B \cup E(B)$.

We have shown that $E(D) \subseteq E(B)$. This is a contradiction since no finite set can be contained in a distinct set of the same cardinality. □

LEMMA 3.8 (Duality). *Let $B, D \in \mathbb{B}(X)$. Let $Q_B = p_{X \setminus (B \cup E(B))} \in \mathcal{B}(X)$ and let $R^D_{X \setminus E(D)}$ be the polynomial that is defined in* (II.7). *Then,*

$$\langle Q_B, R^D_{X \setminus E(D)} \rangle = \frac{\delta_{B,D}}{\det(D)}. \qquad \text{(II.28)}$$

$\delta_{B,D}$ denotes the Kronecker delta and we consider B and D to be equal if there exist $1 \le i_1 < \ldots < i_r \le N$ s. t. $B = (x_{i_1}, \ldots, x_{i_r}) = D$.

PROOF. By Corollary 2.4, $R^D_{X \setminus E(D)}$ is a homogeneous polynomial of degree $N - r - |E(D)|$. Thus, if $|E(B)| \neq |E(D)|$, then Q_B and $R^D_{X \setminus E(D)}$ are homogeneous polynomials of different degrees and $\langle Q_B, R^D_{X \setminus E(D)} \rangle = 0$.

Now suppose that $B \neq D$ and both bases have the same number of externally active elements. In this case, the statement follows from Lemma 3.5 and Lemma 3.7.

The only case that remains is $B = D$. Recall that $R^B_{X \setminus E(B)} = R^B_1 * \ldots * R^B_r$. Consider the ith factor R^B_i. The elements of $(X \setminus E(B)) \cap (S_i \setminus S_{i-1})$ are used for the construction of R^B_i. Exactly one basis element is contained in this set: b_i. Recall that in Section 2 we defined a partition $P_i \cup N_i = (X \setminus E(B)) \cap (S_i \setminus S_{i-1})$. By construction, b_i is positive, *i. e.* $b_i \in P_i$. Now we apply the differential operator $D_{(P_i \setminus b_i) \cup N_i}$ to R^B_i:

$$D_{(P_i \cup N_i) \setminus b_i}((-1)^{|N_i|} \cdot T_{P_i} * T_{-N_i} - (-1)^{|P_i|} \cdot T_{-P_i} * T_{N_i}) = (T_{b_i} + T_{-b_i}).$$

Now, we can put things together. Note that $X \setminus (B \cup E(B)) = \bigcup_{i=1}^r ((P_i \setminus b_i) \cup N_i)$. Hence,

$$D_{X \setminus (B \cup E(B))} R_B = (T_{b_1} + T_{-b_1}) * \cdots * (T_{b_r} + T_{-b_r}) = \frac{1}{\det(B)}. \qquad \text{(II.29)}$$

This finishes the proof. □

PROOF OF THEOREM 3.2. We know that $\mathcal{P}(X)$ and $\mathcal{D}(X)$ are dual via the pairing $\langle \cdot, \cdot \rangle$ and that $\mathcal{B}(X)$ is a basis for $\mathcal{P}(X)$. By Lemma 3.8, $B(X)$ and $\mathcal{B}(X)$ are dual to each other and by Lemma 3.6, $B(X)$ is contained in $\mathcal{D}(X)$. Hence, $B(X)$ is a basis for $\mathcal{D}(X)$. □

3.1. Previously known methods for constructing bases for $\mathcal{D}$-spaces. Two other methods are known to construct a basis for $\mathcal{D}(X)$. However, our algorithm has several advantages over the other two: it is canonical, *i. e.* it only depends on the order of the list X and it yields a basis that is dual to the known basis $\mathcal{B}(X)$ for the $\mathcal{P}$-space.

In Wolfgang Dahmen's construction [**25**], polynomials are chosen as basis elements that are local pieces of certain multivariate splines. For certain choices of the parameters in his construction, it might yield the same basis as ours.

The second construction uses least map interpolation that was introduced in Subsection I.4.3. Recall that $\mathcal{D}(X)$ equals the least space $\Pi(S)$, where S is the set of vertices of a certain hyperplane arrangement. De Boor and Ron give a method to select a basis from $\Pi(S)$ in [**32**] (see also [**30**, Chapter II] for a summary). Their construction depends on the choice of the vector c, an ordering of the bases and an ordering of $\mathbb{N}^r$ while our construction only depends on the order on X.

4. Deletion-contraction and exact sequences

By Proposition I.4.7, the Hilbert series of $\mathcal{D}(X)$ and $\mathcal{P}(X)$ are equal and an evaluation of the Tutte polynomial. In particular, they satisfy a deletion-contraction identity that extends in a natural way to our algebraic setting. This is reflected by two dual short exact sequences.

In this section we will define deletion and contraction and we explain these two exact sequences. While the two sequences were known before, their duality has not yet been stated explicitly in the literature.

Two important matroid operations are deletion and contraction. For realisations of matroids, they are defined as follows. Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors and let $x \in X$. The *deletion* of x is the list $X \setminus x$. The *contraction* of x is the list X/x, which is defined to be the image of $X \setminus x$ under the projection $\pi_x : U \to U/x$.

The space $\mathcal{P}(X/x)$ is contained in the symmetric algebra $\mathrm{Sym}(U/x)$. If $x = s_r$, then there is a natural isomorphism $\mathrm{Sym}(U/x) \cong \mathbb{R}[s_1, \ldots, s_{r-1}]$ that maps $\bar{s}_i$ to s_i. This isomorphism depends on the choice of the basis $(s_1, \ldots, s_r)$ for U. Under this identification, $\mathrm{Sym}(\pi_x)$ is the map from $\mathbb{R}[s_1, \ldots, s_r]$ to $\mathbb{R}[s_1, \ldots, s_{r-1}]$ that sends s_r to zero and $s_1, \ldots, s_{r-1}$ to themselves.

For $\mathcal{D}(X/x)$, the situation is simpler: this space is contained in the symmetric algebra $\mathrm{Sym}((U/x)^*) \cong \mathrm{Sym}(x^o)$. This is a subspace of $\mathrm{Sym}(V)$. We denote the inclusion map by j_x. If $x = s_r$, then $\mathrm{Sym}((U/x)^*)$ is isomorphic to $\mathbb{R}[t_1, \ldots, t_{r-1}]$. This is a canonical isomorphism that is independent of the choice of the basis elements $s_1, \ldots, s_{r-1}$.

For a graded vector space S, we write $S[1]$ for the vector space with the degree shifted up by one.

Proposition 4.1 ([**3**]). *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $x \in X$ be neither a loop nor a coloop. Then, the following sequence of graded vector spaces is exact:*

$$0 \to \mathcal{P}(X \setminus x)[1] \overset{\cdot p_x}{\to} \mathcal{P}(X) \overset{\mathrm{Sym}(\pi_x)}{\longrightarrow} \mathcal{P}(X/x) \to 0. \tag{II.30}$$

Proposition 4.2 ([**34**]). *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $x \in X$ be neither a loop nor a coloop. Then, the following sequence of graded vector spaces is exact:*

$$0 \to \mathcal{D}(X/x) \overset{j_x}{\to} \mathcal{D}(X) \overset{D_x}{\to} \mathcal{D}(X \setminus x)[1] \to 0. \tag{II.31}$$

Note that (II.31) is a special case of (1.12) in [**34**] and it is exact by the results in that paper.

Remark 4.3. Proposition 4.1 and Proposition 4.2 are equivalent because of the duality of $\mathcal{P}(X)$ and $\mathcal{D}(X)$.

Proof of Remark 4.3. We only show that Proposition 4.2 implies Proposition 4.1. The other implication is similar.

Since dualisation of finite dimensional vector spaces is a contravariant exact functor, the following sequence is exact by Proposition 4.2:

$$0 \to \mathcal{D}(X \setminus x)^* \overset{(D_x)^*}{\longrightarrow} \mathcal{D}(X)^* \overset{(j_x)^*}{\longrightarrow} \mathcal{D}(X/x)^* \to 0. \tag{II.32}$$

By Proposition I.4.6, $\mathcal{P}(X)$ is isomorphic to $\mathcal{D}(X)^*$ via $q \mapsto \langle q, \cdot \rangle$. Hence, it is sufficient to show that the following two diagrams commute:

$$\begin{array}{ccc} \mathcal{P}(X \setminus x) & \xrightarrow{q \mapsto \langle q, \cdot \rangle} & \mathcal{D}(X \setminus x)^* \\ \downarrow \cdot p_x & & \downarrow (D_x)^* \\ \mathcal{P}(X) & \xrightarrow{q \mapsto \langle q, \cdot \rangle} & \mathcal{D}(X)^* \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{q \mapsto \langle q, \cdot \rangle} & \mathcal{D}(X)^* \\ \downarrow \mathrm{Sym}(\pi_x) & & \downarrow (j_x)^* \\ \mathcal{P}(X/x) & \xrightarrow{q \mapsto \langle q, \cdot \rangle} & \mathcal{D}(X/x)^* \end{array} . \tag{II.33}$$

For the diagram on the left, we have to show that $\langle p_x q, \cdot \rangle = \langle q, D_x \cdot \rangle$ for all $q \in \mathcal{P}(X \setminus x)$. This is easy.

For the diagram on the right, we have to show that $\langle \mathrm{Sym}(\pi_x) q, \cdot \rangle = \langle q, j_x(\cdot) \rangle$ for all $q \in \mathcal{P}(X)$. If we choose a basis with $s_r = x$ this follows from the fact that $\frac{\partial}{\partial t_r} f = 0$ for all $f \in \mathbb{R}[t_1, \ldots, t_{r-1}]$. □

5. Forward exchange matroids

In this section we will introduce forward exchange matroids. A forward exchange matroid is an ordered matroid together with a subset of its set of bases that satisfies a weak version of the basis exchange axiom (I.1).

The motivation for this definition is the following: we noticed that most of the results in Section I.4 and Section 3 hold in a far more general context. An important ingredient of the definitions of the spaces $\mathcal{P}(X)$ and $\mathcal{D}(X)$ and their bases is the set of bases $\mathbb{B}(X)$ of the list X. These two spaces still have nice properties if we modify their definitions and use only a suitable subset $\mathbb{B}'$ of $\mathbb{B}(X)$. It turned out that forward exchange matroids are the right axiomatisation of "suitable subset".

Let $(A, \mathbb{B})$ be an ordered matroid of rank r and let $B = (b_1, \ldots, b_r) \in \mathbb{B}$ be an ordered basis. The flag (II.2) can be defined in combinatorial terms: for $i \in \{0, \ldots, r\}$, we define $S_i = S_i^B := \mathrm{cl}\{b_1, \ldots, b_i\} \subseteq A$. Hence, we obtain a flag of flats

$$\{x \in A : x \text{ loop}\} = S_0^B \subsetneq S_1^B \subsetneq S_2^B \subsetneq \ldots \subsetneq S_r^B = A. \tag{II.34}$$

One can easily show that for a basis $B \in \mathbb{B}$ and $i \in [r]$, the following statement holds:

$$\text{let } x \in S_i^B \setminus S_{i-1}^B. \text{ Then } B' = (B \setminus b_i) \cup x \text{ is also in } \mathbb{B}. \tag{II.35}$$

Note that $x \in S_i^B \setminus S_{i-1}^B$ satisfies $x > b_i$ if and only if x is externally active with respect to B. This motivates the name of the following definition.

Definition 5.1 (Forward exchange property). Let $(A, \mathbb{B})$ be an ordered matroid and let $\mathbb{B}' \subseteq \mathbb{B}$. We say that the set of bases $\mathbb{B}'$ has the *forward exchange property* if the following holds for all bases $B \in \mathbb{B}'$ and all $i \in [r]$:

$$\text{let } x \in S_i^B \setminus (S_{i-1}^B \cup E(B)). \text{ Then } B' = (B \setminus b_i) \cup x \text{ is also in } \mathbb{B}'. \tag{II.36}$$

Remark 5.2. Note that $S_j^B = S_j^{B'}$ holds for all $j \geq i$. If x is the ith vector in B', this equality holds for all j, *i. e.* B and B' define the same flag.

Definition 5.3 (Forward exchange matroid). A triple $(A, \mathbb{B}, \mathbb{B}')$ is called a *forward exchange matroid* if $(A, \mathbb{B})$ is an ordered matroid and $\mathbb{B}'$ is a subset of the set of bases $\mathbb{B}$ with the forward exchange property.

Remark 5.4. In this thesis, we mainly consider realisations of forward exchange matroids, *i. e.* pairs $(X, \mathbb{B}')$ where X is a list of vectors and $\mathbb{B}' \subseteq \mathbb{B}(X)$ is a set of bases with the forward exchange property.

Definition 5.5 (Tutte polynomial for forward exchange matroids). Let $(A, \mathbb{B}, \mathbb{B}')$ be a forward exchange matroid.

Then, we define its *Tutte polynomial* to be

$$T_{(A,\mathbb{B},\mathbb{B}')}(x, y) := \sum_{B \in \mathbb{B}'} x^{|I(B)|} y^{|E(B)|}, \tag{II.37}$$

where $I(B)$ and $E(B)$ denote the sets of internally and externally active elements with respect to B in the ordered matroid $(A, \mathbb{B})$.

Remark 5.6. It would be interesting to clarify the relationship between forward-exchange matroids and other set systems studied in combinatorics such as greedoids [**9, 62**].

6. Generalised $\mathcal{D}$-spaces and $\mathcal{P}$-spaces

Earlier, we considered the spaces $\mathcal{D}(X)$ and $\mathcal{P}(X)$ for a given list of vectors X. The construction of these spaces relied mainly on the matroidal properties of the list X, namely on the sets of bases and cocircuits.

Motivated by questions in approximation theory, various authors generalised these constructions. Given a list X and a subset $\mathbb{B}'$ of its set of bases $\mathbb{B}(X)$, one can define a set $\mathcal{D}(X, \mathbb{B}')$ as the kernel of the ideal generated by the $\mathbb{B}'$-cocircuits (*i. e.* sets that intersect all bases in $\mathbb{B}'$). Under certain conditions, $\dim \mathcal{D}(X, \mathbb{B}') = |\mathbb{B}'|$ still holds. In this section we will show that if the set $\mathbb{B}'$ has the forward exchange property, this equality holds and there is a canonical dual space $\mathcal{P}(X, \mathbb{B}')$. Both, the generalised $\mathcal{D}$-spaces and the generalised $\mathcal{P}$-spaces satisfy deletion-contraction identities as in Section 4 and there are canonical bases for both spaces that are dual.

6.1. Definitions and Main Result.

DEFINITION 6.1 (generalised $\mathcal{D}$-spaces). Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}'$ be an arbitrary subset of its set of bases $\mathbb{B}(X)$. A set $C \subseteq X$ is called a $\mathbb{B}'$-*cocircuit* if C intersects every basis in $\mathbb{B}'$ and C is inclusion-minimal with this property.

The *generalised $\mathcal{D}$-space* defined by X and $\mathbb{B}'$ is

$$\mathcal{D}(X,\mathbb{B}') := \{f : D_C f = 0 \text{ for all } \mathbb{B}'\text{-cocircuits } C\} = \ker \mathcal{J}(X,\mathbb{B}'),$$

where $\mathcal{J}(X,\mathbb{B}') := \operatorname{ideal}\{p_C : C \subseteq X \text{ is a } \mathbb{B}'\text{-cocircuit}\}$.

PROPOSITION 6.2 ([**31**, Theorem 6.6]). *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}'$ be an arbitrary subset of its set of bases $\mathbb{B}(X)$. Then*

$$\dim \mathcal{D}(X,\mathbb{B}') \geq \left|\mathbb{B}'\right|. \tag{II.38}$$

DEFINITION 6.3 (generalised $\mathcal{P}$-spaces). Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}'$ be an arbitrary subset of its set of bases $\mathbb{B}(X)$. Then, we define

$$\mathcal{B}(X,\mathbb{B}') := \{Q_B : B \in \mathbb{B}'\} = \{p_{X\setminus(B\cup E(B))} : B \in \mathbb{B}'\} \tag{II.39}$$

$$\text{and } \mathcal{P}(X,\mathbb{B}') := \operatorname{span} \mathcal{B}(X,\mathbb{B}'). \tag{II.40}$$

We call $\mathcal{P}(X,\mathbb{B}')$ the *generalised $\mathcal{P}$-space* defined by X and $\mathbb{B}'$.

REMARK 6.4. The set $\mathcal{B}(X,\mathbb{B}')$ is a basis for $\mathcal{P}(X,\mathbb{B}')$. By definition, it is spanning and it is linearly independent because it is a subset of $\mathcal{B}(X)$.

If the set $\mathbb{B}'$ has the forward exchange property, the spaces $\mathcal{D}(X,\mathbb{B}')$ and $\mathcal{P}(X,\mathbb{B}')$ have many nice properties. Here is the Main Theorem of this section.

THEOREM 6.5. *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}' \subseteq \mathbb{B}(X)$ be a set of bases with the forward exchange property.*

Then, the generalised $\mathcal{D}$-space $\mathcal{D}(X,\mathbb{B}')$ and the generalised $\mathcal{P}$-space $\mathcal{P}(X,\mathbb{B}')$ are dual via the pairing $\langle\cdot,\cdot\rangle$. In addition, the set

$$\bar{\mathcal{B}}(X,\mathbb{B}') := \{\det(B) R^B_{X\setminus E(B)} : B \in \mathbb{B}'\} \tag{II.41}$$

forms a basis for $\mathcal{D}(X,\mathbb{B}')$ and this basis is dual to the basis $\mathcal{B}(X,\mathbb{B}')$ for $\mathcal{P}(X,\mathbb{B}')$.

COROLLARY 6.6. *Let $X \subseteq U \cong \mathbb{R}^r$ be a list of N vectors that spans U and let $\mathbb{B}' \subseteq \mathbb{B}(X)$ be a set of bases with the forward exchange property. Then*

$$\begin{aligned}\operatorname{Hilb}(\mathcal{P}(X,\mathbb{B}'),q) = \operatorname{Hilb}(\mathcal{D}(X,\mathbb{B}'),q) &= \sum_{B\in\mathbb{B}'} q^{N-r-|E(B)|} \\ &= q^{N-r} T_{(X,\mathbb{B}(X),\mathbb{B}')}(1,\frac{1}{q}).\end{aligned}$$

Here are two examples that help to understand generalised $\mathcal{D}$-spaces, generalised $\mathcal{P}$-spaces, and Theorem 6.5.

EXAMPLE 6.7. Let $X = (e_1, e_2, e_3, a, b) \subseteq \mathbb{R}^3$ where e_1, e_2, e_3 denote the unit vectors and $a = (\alpha, \beta, \gamma)$ and b are generic. In particular, $\alpha, \beta, \gamma \neq 0$. Let

$$\mathbb{B}' := \{(e_1e_2e_3),\ (e_1e_2a), (e_1e_2b),\ (e_1e_3a), (e_1e_3b),\ (e_2e_3a), (e_2e_3b)\} \subseteq \mathbb{B}(X).$$

The reader is invited to check that $\mathbb{B}'$ has the forward exchange property. The Tutte polynomial is $T_{(X,\mathbb{B}(X),\mathbb{B}')}(x, y) = 3x+3y+y^2$ and the $\mathbb{B}'$-cocircuits are $\{e_1e_2,\ e_1e_3,\ e_2e_3,\ e_1ab,\ e_2ab,\ e_3ab\}$. Hence,

$$\begin{aligned}
\mathcal{D}(X, \mathbb{B}') &= \ker \operatorname{ideal}\{s_1s_2,\ s_1s_3,\ s_2s_3,\ s_1p_{ab},\ s_2p_{ab},\ s_3p_{ab}\} \\
&= \operatorname{span}\{1,\ t_1,\ t_2,\ t_3,\ t_1^2,\ t_2^2,\ t_3^2\}, \\
\mathcal{B}(X, \mathbb{B}') &= \{1,\ p_{e_3},\ p_{e_3a},\ p_{e_2},\ p_{e_2a},\ p_{e_1},\ p_{e_1a}\}, \\
\mathcal{P}(X, \mathbb{B}') &= \operatorname{span}\{1,\ s_1,\ s_2,\ s_3,\ s_1(\alpha s_1 + \beta s_2 + \gamma s_3), \\
&\qquad\qquad s_2(\alpha s_1 + \beta s_2 + \gamma s_3),\ s_3(\alpha s_1 + \beta s_2 + \gamma s_3)\}, \text{ and} \\
\mathcal{B}\!\!\!/(X, \mathbb{B}') &= \left\{1,\ t_3,\ \frac{t_3^2}{2\gamma},\ t_2,\ \frac{t_2^2}{2\beta},\ t_1,\ \frac{t_1^2}{2\alpha}\right\}.
\end{aligned}$$

EXAMPLE 6.8. Let $N \geq 3$ be an integer. Let $X_N = (x_1, \ldots, x_N)$ be a list of vectors in general position in $\mathbb{R}^2$ with $x_1 = e_1$, $x_2 = e_2$, and $x_3 = e_1 + e_2$. In addition, we suppose that the second coordinate of all vectors x_i ($i \geq 3$) is one. Let $\mathbb{B}' := \{(x_1, x_i) : i \in \{2, \ldots, N\}\} \cup \{(x_2, x_3)\}$. Note that $\mathbb{B}'$ is totally unimodular, *i. e.* all elements have determinant 1 or -1 and $\mathbb{B}'$ has the forward exchange property. Then,

$$\mathcal{D}(X_N, \mathbb{B}') = \ker \operatorname{ideal}\{p_{x_1x_2}, p_{x_1x_3}, p_{x_2\cdots x_N}\} = \operatorname{span}\{1, t_1, t_2, t_2^2, \ldots, t_2^{N-2}\},$$

$$\mathcal{B}\!\!\!/(X_N, \mathbb{B}') = \left\{1, t_1, t_2, \frac{t_2^2}{2}, \ldots, \frac{t_2^{N-2}}{(N-2)!}\right\}, \text{ and} \tag{II.42}$$

$$\mathcal{B}(X_N, \mathbb{B}') = \{1, p_{x_1}\} \cup \{p_{x_2\cdots x_i} : i \in \{2, \ldots, N-1\}\}. \tag{II.43}$$

Now we will embark on the proof of Theorem 6.5. We will start with the following simple lemma.

LEMMA 6.9 (Inclusion). *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}' \subseteq \mathbb{B}(X)$ be a set of bases with the forward exchange property. Then*

$$\mathcal{B}\!\!\!/(X, \mathbb{B}') \subseteq \mathcal{D}(X, \mathbb{B}'). \tag{II.44}$$

PROOF. Let $B \in \mathbb{B}'$ and let $\det(B)R^B_{X\setminus E(B)} \in \mathcal{B}\!\!\!/(X, \mathbb{B}')$ be the corresponding basis element. Let $C \subseteq X$ be a $\mathbb{B}'$-cocircuit, *i. e.* an inclusion-minimal subset of X that intersects every basis in $\mathbb{B}'$.

Let $B = (b_1, \ldots, b_r)$. If there exists an i s. t. $(S_i^B \setminus S_{i-1}^B) \cap (X \setminus E(B)) \subseteq C$, we are done by Lemma 3.5. Now suppose that this is not the case, *i. e.* for every $i \in [r]$, there is a $z_i \in (S_i^B \setminus S_{i-1}^B) \cap (X \setminus (E(B) \cup C))$. Then we define a sequence of bases $B_0, \ldots, B_r$ by

$$B_0 := B \quad \text{and} \quad B_i := (B_{i-1} \setminus b_i) \cup z_i \text{ for } i \in [r]. \tag{II.45}$$

The lists B_i are indeed bases and even though in general, they might define different flags, they satisfy

$$(S_i^{B_{i-1}} \setminus S_{i-1}^{B_{i-1}}) \cap (X \setminus E(B_{i-1})) = (S_i^B \setminus S_{i-1}^B) \cap (X \setminus E(B)), \qquad \text{(II.46)}$$

because $\operatorname{span}(b_1, \ldots, b_i) = \operatorname{span}(z_1, \ldots, z_i)$ for all $i \in [r]$. Hence, $B_i \in \mathbb{B}'$ implies $B_{i+1} \in \mathbb{B}'$ because $\mathbb{B}'$ has the forward exchange property. In particular, $B_r = (z_1, \ldots, z_r) \in \mathbb{B}'$. By construction, $B_r \cap C = \emptyset$. This is a contradiction. □

Definition 6.10. Let $(A, \mathbb{B})$ be a matroid and let $\mathbb{B}' \subseteq \mathbb{B}$. Let $x \in A$. $\mathbb{B}'$ can be partitioned as $\mathbb{B}' = \mathbb{B}_{\setminus x} \cup \mathbb{B}_{|x}$, where

$$\mathbb{B}'_{\setminus x} := \{B \in \mathbb{B}' : x \notin B\} \text{ denotes the } \textit{deletion} \text{ of } x \text{ and} \qquad \text{(II.47)}$$

$$\mathbb{B}'_{|x} := \{B \in \mathbb{B}' : x \in B\} \text{ the } \textit{restriction} \text{ to } x. \qquad \text{(II.48)}$$

If we are given a list of vectors $X \subseteq U$ and a set of bases $\mathbb{B}' \subseteq \mathbb{B}(X)$, we can also define the *contraction* $\mathbb{B}'_{/x}$. Recall that $\pi_x : U \to U/x$ denotes the canonical projection. Then, we define

$$\mathbb{B}'_{/x} := \{\pi_x(B \setminus x) : x \in B \in \mathbb{B}'_{|x}\}. \qquad \text{(II.49)}$$

Remark 6.11. For technical reasons, it is helpful to distinguish the contraction $\mathbb{B}'_{/x}$ and the restriction $\mathbb{B}'_{|x}$ although there is a canonical bijection between both sets.

We will now introduce the concept of placibility. This is a condition on a set of bases $\mathbb{B}'$ which implies equality in (II.38).

Definition 6.12 ([**34**], see also [**69**]). Let $(A, \mathbb{B})$ be a matroid and let $\mathbb{B}' \subseteq \mathbb{B}$ be a non-empty set of bases.

(i) We call an element $x \in A$ *placeable* in $\mathbb{B}'$ if for each $B \in \mathbb{B}'$, there exists an element $b \in B$ such that $(B \setminus b) \cup x \in \mathbb{B}'$.
(ii) We say that $\mathbb{B}'$ is *placible* if one of the following two conditions holds:
  (a) $\mathbb{B}'$ is a singleton or
  (b) there exists $x \in A$ s. t. x is placeable in $\mathbb{B}'$ and both, $\mathbb{B}'_{|x}$ and $\mathbb{B}'_{\setminus x}$ are non-empty and placible.

Proposition 6.13 ([**34**]). *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}'$ be an arbitrary subset of its set of bases $\mathbb{B}(X)$.*
If $\mathbb{B}'$ is placible, then $\dim \mathcal{D}(X, \mathbb{B}') = |\mathbb{B}'|$.

Lemma 6.14. *Let $(A, \mathbb{B}, \mathbb{B}')$ be a forward exchange matroid. Then $\mathbb{B}'$ is placible.*

Proof. If $|\mathbb{B}'| = 1$, then $\mathbb{B}'$ is placible by definition. Now let $|\mathbb{B}'| \geq 2$. Let x be the minimal element in A s. t. both, $\mathbb{B}'_{|x}$ and $\mathbb{B}'_{\setminus x}$ are non-empty. Such an element must exist if $|\mathbb{B}'| \geq 2$.

We will now show that x is placeable in $\mathbb{B}'$. Let $B = (b_1, \ldots, b_r)$ be a basis in $\mathbb{B}'$ and let $i \in [r]$ s. t. $x \in S_i^B \setminus S_{i-1}^B$. We claim that $x \leq b_i$. Suppose it is not. Because of the minimality of x, this implies that $b_1, \ldots, b_i$ are contained in all bases in $\mathbb{B}'$. Since $x \in \operatorname{span}(b_1, \ldots, b_i)$, this implies that x is not contained in any basis. This is a contradiction because we assumed that $\mathbb{B}'_{|x}$ is non-empty.

Now we have established that $x \leq b_i$. This implies that x is not externally active. Hence, because of the forward exchange property, $(B \backslash b_i) \cup x \in \mathbb{B}'$, *i. e.* x is placeable in $B \in \mathbb{B}'$.

It remains to be shown that $\mathbb{B}'_{\backslash x}$ and $\mathbb{B}'_{/x}$ are both placible. By induction, it is sufficient to show that both sets have the forward exchange property. For $\mathbb{B}'_{\backslash x}$, this is clear. For $\mathbb{B}'_{|x}$, this follows from the following fact: by the choice of x, all $a \in A$ that satisfy $a < x$ are either contained in all bases or in no basis in $\mathbb{B}'_{|x}$. □

Proof of Theorem 6.5. By Lemma 6.9, $B(X, \mathbb{B}') \subseteq \mathcal{D}(X, \mathbb{B}')$. By Proposition 6.13 and by Lemma 6.14, $\dim \mathcal{D}(X, \mathbb{B}') = |\mathbb{B}'| = |B(X, \mathbb{B}')|$. Linear independence of $B(X, \mathbb{B}')$ is clear because it is a subset of $B(X)$. For the same reason, the duality with $\mathcal{B}(X, \mathbb{B}') \subseteq \mathcal{B}(X)$ follows from Lemma 3.8. □

Remark 6.15. The correspondence between $\mathcal{D}(X)$ and the set of vertices S of a hyperplane arrangement $\mathcal{H}(X, c)$ in general position that is stated in Theorem I.4.17 generalises in a straightforward way to a correspondence between $\mathcal{D}(X, \mathbb{B}')$ and the subset of S that is defined by $\mathbb{B}'$.

6.2. Deletion-contraction and exact sequences. In this subsection we will show that the results in Section 4 about deletion-contraction and exact sequences naturally extend to generalised $\mathcal{D}$-spaces and $\mathcal{P}$-spaces. We will use the same terminology as in that section.

Recall that for a graded vector space S, we write $S[1]$ for the vector space with the degree shifted up by one.

Proposition 6.16. *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}' \subseteq \mathbb{B}(X)$ be a set of bases with the forward exchange property. Let x be the minimal element of X that is neither a loop nor a coloop. Then, the following sequence of graded vector spaces is exact:*

$$0 \to \mathcal{D}(X/x, \mathbb{B}'_{/x}) \overset{j_x}{\to} \mathcal{D}(X, \mathbb{B}') \overset{D_x}{\to} \mathcal{D}(X \setminus x, \mathbb{B}'_{\backslash x})[1] \to 0. \qquad \text{(II.50)}$$

Proof. Let $B \in \mathbb{B}'$ be a basis that does not contain x. Because of the minimality, x is not externally active with respect to B. This implies $D_x R^B_{X \setminus E(B)} = R^B_{X \setminus (E(B) \cup x)}$. Hence, $D_x : \{\det(B) R^B \in B(X, \mathbb{B}') : x \notin B\} \to B(X \setminus x, \mathbb{B}'_{\backslash x})$ is a bijection and consequently, D_x maps $\mathcal{D}(X, \mathbb{B}')$ surjectively to $\mathcal{D}(X \setminus x, \mathbb{B}'_{\backslash x})$. For the rest of the proof, we refer the reader to [**34**], in particular to the explanations following (1.12) and to Theorem 2.16. □

Proposition 6.17. *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}' \subseteq \mathbb{B}(X)$ be a set of bases with the forward exchange property. Let x be the minimal element of X that is neither a loop nor a coloop. Then, the following sequence of graded vector spaces is exact:*

$$0 \to \mathcal{P}(X \setminus x, \mathbb{B}'_{\backslash x})[1] \overset{\cdot p_x}{\to} \mathcal{P}(X, \mathbb{B}') \overset{\mathrm{Sym}(\pi_x)}{\longrightarrow} \mathcal{P}(X/x, \mathbb{B}'_{/x}) \to 0. \qquad \text{(II.51)}$$

Proof. One can easily check this for the bases of the $\mathcal{P}$-spaces. Alternatively, it can be deduced from Proposition 6.16 using a duality argument as in the proof of Remark 4.3. □

REMARK 6.18. The exact sequences in this section require x to be minimal in contrast to the ones Section 4, where x can be any element that is neither a loop nor a coloop. This reflects the fact that matroids have an (unordered) ground set, while forward exchange matroids have an (ordered) ground list.

REMARK 6.19. One could replace $\mathcal{D}(X/x, \mathbb{B}'_{/x})$ by $\mathcal{D}(X, \mathbb{B}'_{|x})$ in Proposition 6.16. The analogous replacement in Proposition 6.17 would be problematic. The reasons for that are explained in Section 4.

6.3. $\mathcal{P}(X, \mathbb{B}')$ as the kernel of a power ideal. By now we have seen that most of the results regarding $\mathcal{D}(X)$ and $\mathcal{P}(X)$ that we stated earlier also hold for the generalised $\mathcal{D}$-spaces and $\mathcal{P}$-spaces. The only thing that is missing is a power ideal $\mathcal{I}(X, \mathbb{B}')$ s. t. $\mathcal{P}(X, \mathbb{B}') = \ker \mathcal{I}(X, \mathbb{B}')$. Unfortunately, such an ideal does not always exist.

In this section we will describe the natural candidate for this power ideal and we will give an example where its kernel is equal to $\mathcal{P}(X, \mathbb{B}')$ and one where it is not.

DEFINITION 6.20. Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}'$ be an arbitrary subset of its set of bases $\mathbb{B}(X)$. Recall that $V := U^*$. We define a function $\kappa : V \to \mathbb{N}$ by

$$\kappa(\eta) := \max_{B \in \mathbb{B}'} |X \setminus (B \cup E(B) \cup \eta^o)| \tag{II.52}$$

$$\text{and } \mathcal{I}(X, \mathbb{B}') := \operatorname{ideal}\{p_\eta^{\kappa(\eta)+1} : \eta \in V \setminus \{0\}\}. \tag{II.53}$$

LEMMA 6.21. *Let $X \subseteq U \cong \mathbb{R}^r$ be a finite list of vectors that spans U and let $\mathbb{B}'$ be an arbitrary subset of its set of bases $\mathbb{B}(X)$. Then*

$$\mathcal{P}(X, \mathbb{B}') \subseteq \ker \mathcal{I}(X, \mathbb{B}'). \tag{II.54}$$

PROOF. It is sufficient to show that all elements of the basis $\mathcal{B}(X, \mathbb{B}')$ are contained in $\ker \mathcal{I}(X, \mathbb{B}')$. Let $B \in \mathbb{B}'$ and let $\eta \in V \setminus \{0\}$. Then,

$$D_\eta^{\kappa(\eta)} p_{X \setminus (B \cup E(B))} = p_{(X \cap \eta^o) \setminus (B \cup E(B))} D_\eta^{\kappa(\eta)+1} p_{X \setminus (B \cup E(B) \cup \eta^o)} = 0 \tag{II.55}$$

The first equality follows from Leibniz's law. The second follows from the fact that by definition, $\kappa(\eta) \geq |X \setminus (B \cup E(B) \cup \eta^o)|$. □

REMARK 6.22. If one examines the proof of Lemma 6.21, one immediately sees that $\mathcal{I}(X, \mathbb{B})$ is the only power ideal for which $\mathcal{P}(X, \mathbb{B}') = \ker \mathcal{I}(X, \mathbb{B}')$ can possibly hold.

REMARK 6.23. In some cases, $\mathcal{P}(X, \mathbb{B}')$ and $\ker \mathcal{I}(X, \mathbb{B}')$ are equal (see Example 6.25). In other cases however, $\mathcal{P}(X, \mathbb{B}')$ is not even closed under differentiation (see Example 6.8).

Remark 6.23 naturally leads to the following question.

QUESTION 6.24. Is there a simple criterion to decide whether $\mathcal{P}(X, \mathbb{B}')$ is closed under differentiation or if $\mathcal{P}(X, \mathbb{B}') = \ker \mathcal{I}(X, \mathbb{B}')$ holds?

EXAMPLE 6.25. This is a continuation of Example 6.7. Recall that we considered the list $X = (e_1, e_2, e_3, a, b) \subseteq \mathbb{R}^3$ where a and b are generic vectors and $a = (\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \neq 0$. The set of bases is

$$\mathbb{B}' = \{(e_1e_2e_3), (e_1e_2a), (e_1e_2b), (e_1e_3a), (e_1e_3b), (e_2e_3a), (e_2e_3b)\} \subseteq \mathbb{B}(X).$$

In order to calculate the function κ, we first determine the inclusion-maximal lists in $\{X \setminus (B \cup E(B)) : B \in \mathbb{B}\}$. Those are $(e_1 a)$, $(e_2 a)$, and $(e_3 a)$. We can deduce that $\kappa(\eta)$ is one if $\eta \in a^o$ and two otherwise. We obtain

$$\begin{aligned}\mathcal{I}(X, \mathbb{B}') &= \operatorname{ideal}\{p^2_{(\alpha,-\beta,0)}, p^2_{(0,\beta,-\gamma)}, p^2_{(\alpha,0,-\gamma)}\} + \mathbb{R}[s_1, s_2, s_3]_{\geq 3} \text{ and} \\ \mathcal{P}(X, \mathbb{B}') &= \ker \mathcal{I}(X, \mathbb{B}') = \operatorname{span}\{1, s_1, s_2, s_3,\ s_1(\alpha s_1 + \beta s_2 + \gamma s_3), \\ &\qquad s_2(\alpha s_1 + \beta s_2 + \gamma s_3), s_3(\alpha s_1 + \beta s_2 + \gamma s_3)\}.\end{aligned}$$

The degree two component of $\mathbb{R}[s_1, s_2, s_3]/\mathcal{I}(X, \mathbb{B}')$ is three-dimensional. This implies that $\mathcal{P}(X, \mathbb{B}') = \ker \mathcal{I}(X, \mathbb{B}')$.

7. Comparison with previously known zonotopal spaces

In this section we will review the definitions of various zonotopal spaces that have been studied previously by other authors. It turns out that they are all special cases of the generalised $\mathcal{D}$-spaces and $\mathcal{P}$-spaces that we introduced in Section 6. The most prominent examples are of course the central spaces $\mathcal{D}(X)$ and $\mathcal{P}(X)$ that we obtain if we choose $\mathbb{B}' = \mathbb{B}(X)$.

Let $A = (a_1, \ldots, a_n) \subseteq \mathbb{R}^r$ be a list of vectors that spans $\mathbb{R}^r$ and let $X = (x_1, \ldots, x_N) \subseteq \mathbb{R}^r$, where $N \geq n$ and $a_i = x_i$ for $i \in [n]$. In [**54, 55, 69**], the spaces $\mathcal{D}(X, \mathbb{B}')$ and $\mathcal{P}(X, \mathbb{B}')$ are studied for certain sets of bases $\mathbb{B}' \subseteq \mathbb{B}(X)$.

We have already seen internal and external bases in Definition I.4.8. They are special cases of semi-internal and semi-external.

Recall that the lattice of flats $\mathcal{L}(\mathfrak{M})$ of the matroid $\mathfrak{M} = (A, \mathbb{B}(A))$ is the set $\{C \subseteq A : \operatorname{cl}(C) = C\}$ ordered by inclusion. An *upper set* $J \subseteq \mathcal{L}(\mathfrak{M})$ is an upward closed set, *i. e.* $C_1 \subseteq C_2 \in J$ implies $C_1 \in J$.

DEFINITION 7.1 (Semi-internal and semi-external bases [**55**]). Let $A \subseteq \mathbb{R}^r$ be a list of vectors that spans $\mathbb{R}^r$ and let $B_0 = (b_1, \ldots, b_r) \subseteq \mathbb{R}^r$ be an arbitrary basis for $\mathbb{R}^r$ that is not necessarily contained in $\mathbb{B}(A)$. Let $X = (A, B_0)$ and let

$$\operatorname{ex} : \{I \subseteq A : I \text{ linearly independent}\} \to \mathbb{B}(X) \qquad \text{(II.56)}$$

be the function that maps an independent set in A to its greedy extension. This means that given an independent set $I \subseteq A$, the vectors $b_1, \ldots, b_r$ are added successively to I unless the resulting set would be linearly dependent.

In addition, we fix an upper set J in the lattice of flats $\mathcal{L}(\mathfrak{M})$ of the matroid $\mathfrak{M} = (A, \mathbb{B}(A))$. For the semi-internal space, we fix an independent set $I_0 \subseteq A$ whose elements are maximal in A.

Then we define the set of *semi-external bases* $\mathbb{B}_+(A, B_0, J)$ and the set of *semi-internal bases* $\mathbb{B}_-(A, I_0)$ by

$$\begin{aligned}\mathbb{B}_+(A, B_0, J) &:= \{B \in \mathbb{B}(X) : B = \operatorname{ex}(I) \text{ for some } I \subseteq A \text{ independent} \\ &\qquad \text{and } \operatorname{cl}(I) \in J\} \text{ and} \\ \mathbb{B}_-(A, I_0) &:= \{B \in \mathbb{B}(A) : B \cap I_0 \text{ contains no internally active elements}\}.\end{aligned}$$

DEFINITION 7.2 (Generalised external bases [**69**]). Let $A \subseteq \mathbb{R}^r$ be a list of vectors. Let $\kappa : \mathcal{L}(A) \to \{0, 1, 2, \ldots\}$ be an increasing function, *i. e.* $C_1 \subseteq C_2$ implies $\kappa(C_1) \leq \kappa(C_2)$.

Let $X = (A, Y)$, where $Y = (y_1, y_2, \ldots, y_{\kappa(A)+r})$ is a list of generic vectors, *i. e.* if y_i is in the span of $Z \subseteq X \setminus y_i$, then $\operatorname{span}(Z) = \operatorname{span}(X)$.

Then we define

$$\mathbb{B}_\kappa(A,Y) := \{B \in \mathbb{B}(X) : B \cap Y \subseteq (y_1,\ldots,y_{\kappa(\mathrm{cl}(A\cap B))+|B\cap Y|})\}. \tag{II.57}$$

REMARK 7.3. The spaces $\mathcal{P}(X,\mathbb{B}')$ and $\mathcal{D}(X,\mathbb{B}')$ are equal to

- the external spaces $\mathcal{P}_+(X)$ and $\mathcal{D}_+(X)$ in [**54**] resp. Definition I.4.8 if $\mathbb{B}'$ is the set of external bases;
- the semi-external spaces $\mathcal{P}_+(X,J)$ and $\mathcal{D}_+(X,J)$ in [**55**] resp. Definition 7.1 if $\mathbb{B}'$ is the set of semi-external bases;
- the generalised external spaces $\mathcal{P}_\kappa(X)$ and $\mathcal{D}_\kappa(X)$ in [**69**] resp. Defition 7.2 if $\mathbb{B}'$ is the set of generalised external bases. For $\mathcal{P}_\kappa(X)$, we need to assume in addition that κ is incremental, *i. e.* for two flats $C_1 \subseteq C_2$, $\kappa(C_2) - \kappa(C_1) \leq \dim(C_2) - \dim(C_1)$.

Furthermore, the space $\mathcal{D}(X,\mathbb{B}')$ is equal to the (semi-)internal space $\mathcal{D}_-(X)$ resp. $\mathcal{D}_-(X,I_0)$ in [**54, 55**] if $\mathbb{B}'$ is the set of (semi-)internal bases.

REMARK 7.4. The (semi-)internal spaces $\mathcal{P}(X,\mathbb{B}_-(X))$ and $\mathcal{P}(X,\mathbb{B}_-(X,I_0))$ are in general different from the spaces $\mathcal{P}_-(X)$ and $\mathcal{P}_-(X,I_0)$ in [**54, 55**], but they have the same Hilbert series.

REMARK 7.5. The theorems about duality of certain $\mathcal{P}$-spaces and $\mathcal{D}$-spaces in [**54, 55, 69**] are all special cases of Theorem 6.5. This is a consequence of Lemma 7.6 below.

LEMMA 7.6. *The sets of bases defined in Definitions* I.4.8*,* 7.1*, and* 7.2 *all have the forward exchange property.*

PROOF. We use the following notation throughout the proof: $B = (b_1,\ldots,b_r)$ is a basis and $x \in (S_i^B \setminus S_{i-1}^B) \cap (X \setminus E(B))$ for some $i \in [r]$. In addition, $B' := (B \cup x) \setminus b_i$. Since x is not externally active, $x \leq b_i$ holds. We may even assume $x < b_i$ because if equality occurs, nothing needs to be shown.

Internal and external bases are special cases of semi-internal and semi-external bases so we do not consider them separately.

Let us start with the semi-external bases. Let $B \in \mathbb{B}_+(A,B_0,J)$, *i. e.* B is the greedy extension of an independent set $I \subseteq A$. Recall that $x < b_i$. Hence, $x \in A$ because if x was in B_0, the greedy extension of I would contain x instead of b_i. Now one can easily check that $B' \cap A$ is independent and that $\mathrm{ex}(B' \cap A) = B'$. This is equivalent to $B' \in \mathbb{B}_+(A,B_0,J)$.

Now we will consider the semi-internal bases. Let $B \in \mathbb{B}_-(A,I_0)$. By construction, the fundamental cocircuits of B and b_i resp. x are equal. As $x < b_i$, inactivity of b_i implies inactivity of x. Hence, $B' \in \mathbb{B}_-(A,I_0)$.

Last, let us consider the generalised external bases. Let $B \in \mathbb{B}_\kappa(A,Y)$. If $b_i \in A$, then $\kappa(\mathrm{cl}(A\cap B)) + |B \cap Y| = \kappa(\mathrm{cl}(A \cap B')) + |B' \cap Y|$. This implies $B' \in \mathbb{B}_\kappa(A,Y)$. If $b_j \in Y$, then $j = r$ must hold because the vectors in Y are generic and we are supposing $b_j \neq x$. Replacing b_r by x reduces the index of the maximal element in Y that is permitted in B' by at most one since κ is non-decreasing on $\mathcal{L}(A)$. Since we remove the maximal element of the basis B, this causes no problems. □

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CHAPTER III

Hierarchical Zonotopal Power Ideals

Given a finite list of vectors X, an integer $k \geq -1$ and an upper set in the lattice of flats of the matroid defined by X, we will define and study the associated *hierarchical zonotopal power ideal*. This ideal is generated by powers of linear forms. We will show that its kernel is generated by products of linear forms and we will give a method to select a basis for it.

This work unifies and generalises results by Ardila-Postnikov on power ideals and by Holtz-Ron and Holtz-Ron-Xu on (hierarchical) zonotopal algebra. We will also generalise a result on zonotopal Cox modules that were introduced by Sturmfels-Xu.

The zonotopal spaces studied in this chapter can be seen as special cases of the generalised $\mathcal{P}$-spaces in Chapter II.

1. Introduction

Let $X = (x_1, \ldots, x_N) \subseteq \mathbb{R}^r$ be a list of vectors that span $\mathbb{R}^r$. For a vector η, let $m(\eta)$ denote the number of vectors in X that are *not* perpendicular to η. A vector $v \in \mathbb{R}^r$ defines a linear polynomial $p_v := \sum_i v_i t_i \in \mathbb{R}[t_1, \ldots, t_r]$. For $Y \subseteq X$, let $p_Y := \prod_{x \in Y} p_x$. Then define

$$\mathcal{P}(X) := \operatorname{span}\{p_Y : X \setminus Y \text{ spans } \mathbb{R}^r\} \tag{III.1}$$

$$\text{and } \mathcal{I}(X) := \operatorname{ideal}\{p_\eta^{m(\eta)} : \eta \neq 0\}. \tag{III.2}$$

The following theorem and several generalisations are well-known (cf. Proposition I.4.4 and [**3, 28, 54**]).

THEOREM 1.1.

$$\mathcal{P}(X) = \ker \mathcal{I}(X) := \operatorname{span}\{q \in \mathbb{R}[t_1, \ldots, t_r] : f(D)q = 0 \textit{ for all } f \in \mathcal{I}(X)\},$$

where $f(D) := f\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}\right)$. *In addition,* $\mathcal{I}(X)$ *is equal to the ideal*

$$\mathcal{I}'(X) := \{p_\eta^{m(\eta)} : \textit{the vectors in } X \textit{ perpendicular to } \eta \textit{ span a hyperplane}\}.$$

In this chapter we will show that a statement as in Theorem 1.1 holds in a far more general setting: we will study the kernel of the *hierarchical zonotopal power ideal*

$$\mathcal{I}(X, k, J) := \operatorname{ideal}\{p_\eta^{m(\eta)+k+\chi_J(\eta)} : \eta \neq 0\} \tag{III.3}$$

where $k \geq -1$ is an integer and χ_J is the indicator function of an upper set J in the lattice of flats of the matroid defined by X. We will examine these spaces in a slightly more abstract setting, *e. g.* $\mathcal{P}(X, k, J)$ is contained in the symmetric algebra over an r-dimensional $\mathbb{K}$-vector space, where $\mathbb{K}$ is a field of characteristic zero.

Central $\mathcal{P}$-spaces (in our terminology the kernel of $\mathcal{I}'(X)$) were introduced in the literature on approximation theory around 1990 [**1, 28, 45**]. The dual space $\mathcal{D}(X)$ appeared almost 30 years ago [**29**]. Recently, Olga Holtz and Amos Ron introduced internal ($k = -1$) and external ($k = +1$) $\mathcal{P}$-spaces and $\mathcal{D}$-spaces and coined the term *zonotopal algebra* [**54**]

Federico Ardila and Alexander Postnikov [**3**] constructed $\mathcal{P}$-spaces for arbitrary integers $k \geq -1$. Olga Holtz, Amos Ron, and Zhiqiang Xu [**55**] introduced hierarchical zonotopal spaces, *i. e.* structures that depend on the choice of an upper set J in addition to X and k. They studied semi-internal and semi-external spaces (*i. e.* $k = -1$ and $k = 0$ and some special upper sets J). A result related to semi-external spaces (a decomposition of the external space in terms of the lattice of flats of X) appeared in the work of Peter Orlik and Hiroaki Terao [**76**] on hyperplane arrangements.

This chapter is based on [**67**]. An extended abstract of that paper has appeared in the proceedings of FPSAC 2011 [**65**].

1.1. Comparison with the results in Chapter II. The general idea in Chapter II was to define nice $\mathcal{D}$-spaces together with some dual $\mathcal{P}$-space that does not even have to be closed under differentiation.

In this chapter, it is the other way around. We will define nice $\mathcal{P}$-spaces and not mention $\mathcal{D}$-spaces at all. However, if we assume $k \geq 0$, dual $\mathcal{D}$-spaces exist since in this case the $\mathcal{P}$-spaces in this chapter are special cases of the generalised $\mathcal{P}$-spaces in Chapter II. This is not obvious and can be seen as follows: Li and Ron [**69**] have shown that the $\mathcal{P}$-spaces for $k \geq 0$ in this chapter fit into their framework and in Section II.7 we have seen that the spaces in [**69**] are special cases of our generalised $\mathcal{P}$-spaces and $\mathcal{D}$-spaces.

The construction of the $\mathcal{P}$-space in Chapter II is quite different from the one in this chapter. In the former construction, it was eventually necessary to add additional elements to the list X in order to obtain arbitrarily large $\mathcal{P}$-spaces. In this chapter it is sufficient to let the integer parameter k grow while keeping the list X fixed. The construction of the bases for the $\mathcal{P}$-spaces in this chapter takes into account the internal activity of the bases in $\mathbb{B}(X)$. Every element of $\mathbb{B}(X)$ with internally active elements may define multiple elements of the basis for the $\mathcal{P}$-space.

1.2. Outline of this chapter. In Section 2 we will briefly review the notation. In Section 3 we will describe the kernels of the ideals $\mathcal{I}(X, k, J)$ and define a subideal $\mathcal{I}'(X, k, J)$ with finitely many generators. We will show that the two ideals are equal for $k \leq 0$.

In Section 4 we will construct bases for the vector spaces $\mathcal{P}(X, k, J)$ for $k \geq 0$. We will deduce formulae for the Hilbert series of the spaces $\mathcal{P}(X, k, J)$ in Section 5.

In Section 6 we will apply our results to prove a statement about zonotopal Cox modules that were defined by Bernd Sturmfels and Zhiqiang Xu [**86**].

Finally, in Section 7 we will give plenty of examples.

2. Notation

In this chapter we work over a fixed field $\mathbb{K}$ of characteristic zero. As usual, $U \cong \mathbb{K}^r$ denotes a finite-dimensional $\mathbb{K}$-vector space of dimension $r \geq 1$ and $V := U^*$ its dual. Our main object of study is a finite list $X = (x_1, \ldots, x_N) \subseteq U$ whose elements span U. The order of the elements in X is irrelevant for us except in a few cases, where this is explicitly mentioned.

Recall that the set of flats of the matroid $\mathfrak{M}(X)$ defined by X forms a lattice (*i. e.* a poset with joins and meets) that is called the *lattice of flats* $\mathcal{L}(X) = \mathcal{L}(\mathfrak{M}(X))$. An *upper set* $J \subseteq \mathcal{L}(X)$ is an upward closed set, *i. e.* $C \subseteq C'$, $C \in J$ implies $C' \in J$. We call $C \in \mathcal{L}(X)$ a *maximal missing flat* if $C \notin J$ and C is maximal with this property. In this chapter, *hyperplane* always refers to the matroid-theoretic object, *i. e.* a flat $H \subseteq X$ of rank $r-1$.

Given an upper set $J \subseteq \mathcal{L}(X)$, $\chi_J : \mathcal{L}(X) \to \{0, 1\}$ denotes its indicator function. The index is omitted if it is clear which upper set is meant. χ can be extended to the power set of X by $\chi(A) := \chi(\mathrm{cl}(A))$ for $A \subseteq X$.

A vector $\eta \in V$ defines a flat $C = \{x \in X : \eta(x) = 0\} \subseteq X$. Recall that $m_X(C) = m_X(\eta) := |X \setminus C|$. Sometimes, we write $m(C)$ instead of $m_X(C)$. If η defines the flat C, we call η a *defining normal* for C. Note that for hyperplanes, there is a unique defining normal (up to scaling).

In this chapter we study $\mathcal{P}$-spaces that are contained in $\mathrm{Sym}(U)$ (the symmetric algebra over U) and ideals that are contained in $\mathrm{Sym}(V)$. For a given $x \in U$, we denote the image of X under the canonical injection $U \hookrightarrow \mathrm{Sym}(U)$ by p_x. For $Y \subseteq X$, we define $p_Y := \prod_{x \in Y} p_x$. For $\eta \in V$, we write D_η for the image of η under the canonical injection $V \hookrightarrow \mathrm{Sym}(V)$ in order to stress the fact that we mostly think of $\mathrm{Sym}(V)$ as the algebra generated by the directional derivatives on $\mathrm{Sym}(U)$.

For more explanations and background information, see Chapter I.

3. Hierarchical zonotopal power ideals and their kernels

In this section we will define hierarchical zonotopal power ideals and show that their kernels have a nice description as $\mathcal{P}$-spaces, *i. e.* they are spanned by products of linear forms.

The first subsection contains the definitions and the statement of the Main Theorem. In the second subsection we will prove some simple facts and give explicit formulae for the $\mathcal{P}$-spaces in two simple cases. In the third subsection we will define deletion and contraction for pairs consisting of a matroid and an upper set in its lattice of flats. This will then be used to give an inductive proof of the Main Theorem.

3.1. Definitions and the Main Theorem.

Definition 3.1 (Hierarchical zonotopal power ideals and $\mathcal{P}$-spaces). Let $\mathbb{K}$ be a field of characteristic zero, V be a finite-dimensional $\mathbb{K}$-vector space of dimension $r \geq 1$ and $U = V^*$. Let $X = (x_1, \ldots, x_N) \subseteq U$ be a finite list of vectors whose elements span U. Let $k \geq -1$ be an integer and let $J \subseteq \mathcal{L}(X)$ be a non-empty upper set, where $\mathcal{L}(X)$ denotes the lattice of flats of the matroid defined by X.

Let $\chi_J : \mathcal{L}(X) \to \{0, 1\}$ denote the indicator function of J. Let $E : \mathcal{L}(X) \to V$ be a *normal selector function*, *i. e.* a function that assigns a defining normal to every flat. Now define

$$\mathcal{I}'(X,k,J,E) := \operatorname{ideal}\Big\{ D_{E(C)}^{m(C)+k+\chi(C)} : C \text{ hyperplane or max. missing flat} \Big\}$$

$$\mathcal{I}(X,k,J) := \operatorname{ideal}\Big\{ D_{\eta}^{m(\eta)+k+\chi(\eta)} : \eta \in V \setminus \{0\} \Big\} \subseteq \operatorname{Sym}(V) \tag{III.4}$$

$$\mathcal{P}(X,k,J) := \operatorname{span} S(X,k,J) \subseteq \operatorname{Sym}(U) \tag{III.5}$$

where

$$S(X,k,J) := \{ f p_Y : Y \subseteq X, 0 \leq \deg f \leq \chi(X \setminus Y) + k - 1 \} \quad \text{for } k \geq 1 \tag{III.6}$$

$$S(X,0,J) := \{ p_Y : Y \subseteq X, \operatorname{cl}(X \setminus Y) \in J \}$$

$$S(X,-1,J) := \{ p_Y : |Y \setminus C| < m(C) - 1 + \chi(C) \text{ for all } C \in \mathcal{L}(X) \setminus \{X\} \}$$

Note that the definition of $S(X,0,J)$ can be seen as a special case of the definition of $S(X,k,J)$ for $k \geq 1$. Therefore, we distinguish only the two cases $k \geq 0$ and $k = -1$ in the proofs.

The condition $X \in J$ is only relevant in the case $k = 0$. Then it ensures $1 \in S(X,0,J)$. One can easily see that in the definition of $S(X,-1,J)$, it is sufficient to check only the inequalities associated to hyperplanes and to maximal missing flats. If x is a coloop and $X \setminus x \notin J$, then $S(X,-1,J) = \emptyset$.

For examples, see Section 7, Remark 3.11, and Proposition 3.12.

THEOREM 3.2 (Main Theorem). *We are using the same terminology as in Definition* 3.1. *For $k = -1$, we assume in addition that J contains all hyperplanes in X. Then*

$$\mathcal{P}(X,k,J) = \ker \mathcal{I}(X,k,J) \subseteq \ker \mathcal{I}'(X,k,J,E). \tag{III.7}$$

Furthermore, for $k \in \{-1,0\}$, $\mathcal{I}'(X,k,J,E)$ is independent of the choice of the normal selector function E and

$$\mathcal{P}(X,k,J) = \ker \mathcal{I}(X,k,J) = \ker \mathcal{I}'(X,k,J,E). \tag{III.8}$$

REMARK 3.3. Example 7.3 explains why there is an additional condition for $k = -1$ (see also Remark 3.21). Holtz, Ron, and Xu [**55**] define a different semi-internal structure. For a fixed $C_0 \in \mathcal{L}(X)$ and $J_{C_0} := \{C \in \mathcal{L}(X) : C \supseteq C_0\}$, they show $\ker \mathcal{I}'(X,-1,J_{C_0}) = \bigcap_{x \in C_0} \mathcal{P}(X \setminus x, 0, \{X\})$. However, they do not have a canonical generating set for this space. See Subsection 5.3 for more details. In the same paper, Holtz, Ron, and Xu define semi-external spaces that are the same as ours. However, they only identify them with the kernel of a power ideal in the special case where all maximal missing flats are hyperplanes.

From the Main Theorem and the results in Section 5, one can easily deduce the following two corollaries:

COROLLARY 3.4. *In the setting of the Main Theorem,*

$$\mathcal{P}(X,k,\mathcal{L}(X)) = \mathcal{P}(X,k+1,\{X\}) \tag{III.9}$$

COROLLARY 3.5. *The Hilbert series of $\mathcal{P}(X,k,J)$ depends only on the matroid $\mathfrak{M}(X)$ and k and J, but not on the realisation X.*

Remark 3.6. One might wonder if similar theorems can be proven for $k \le -2$. One would of course need to impose extra conditions on X to ensure that the exponents appearing in the definition of the ideals are non-negative. It is easy to see that $\mathcal{I}$ and $\mathcal{I}'$ are equal in this case (Lemma 3.13). However, we do not know how to construct a generating set for their kernel. A different approach would be required: in general, their kernel is not spanned by a set of polynomials of type p_Y for some $Y \subseteq X$ [**3**].

Remark 3.7. The internal, central, and external $\mathcal{P}$-space that were defined in Section I.4 are special cases of the hierarchical zonotopal $\mathcal{P}$-spaces that where defined in this section. Namely,

$$\mathcal{P}_-(X) = \mathcal{P}_-(X, -1, \{X\}), \tag{III.10}$$
$$\mathcal{P}(X) = \mathcal{P}(X, 0, \{X\}), \tag{III.11}$$
$$\text{and } \mathcal{P}_+(X) = \mathcal{P}(X, 1, \{X\}). \tag{III.12}$$

3.2. Basic results. In this subsection we will prove three lemmas that will be needed later on and we will prove the Main Theorem in two special cases that will be the base cases for the inductive proof in the next subsection.

Lemma 3.8. *Let $Y \subseteq X$ and let $\eta \in V$ be a defining normal for $C \subseteq X$. Then*

$$D_\eta p_Y = p_{Y \cap C} D_\eta p_{Y \setminus C}. \tag{III.13}$$

Proof. This is a direct consequence of Leibniz's law. □

The following lemma is related to Waring decompositions of monomials (cf. [**20**]).

Lemma 3.9. *Let $u_1, \ldots, u_s \in U$ and let $k \in \mathbb{N}$. Then,*

$$\operatorname{span}\{(\alpha_1 u_1 + \ldots + \alpha_s u_s)^k : \alpha_i \in \mathbb{K} \setminus \{0\}\} = \operatorname{span}\left\{ \boldsymbol{u}^{\boldsymbol{a}} : \sum_{i=1}^s a_i = k \right\},$$

where $\boldsymbol{u}^{\boldsymbol{a}} := \prod_{i=1}^s u_i^{a_i}$ and $a_i \in \mathbb{N}$.

Proof. "$\subseteq$" is clear. Let L denote the number of monomials of the form $\prod_{i=1}^s u_i^{a_i}$ ($\sum_i a_i = k$, $a_i \in \mathbb{N}$). Order those monomials lexicographically. By induction, we can see that there are polynomials $p_1, \ldots, p_L$ contained in the set on the left s. t. the leading term of p_i is the ith monomial. This implies that all those monomials are contained in the set on the left side.

Alternatively, the statement follows from the following beautiful formula that is mentioned in [**5**]:

$$\boldsymbol{u}^{\boldsymbol{a}} = \frac{1}{k!} \sum_{0 \le \lambda_i \le a_i} (-1)^{k-(\lambda_1+\ldots+\lambda_s)} \binom{a_1}{\lambda_1} \cdots \binom{a_s}{\lambda_s} (\lambda_1 u_1 + \ldots + \lambda_s u_s)^k. \quad \square$$

Lemma 3.10. *$\mathcal{P}(X, k, J) \subseteq \ker \mathcal{I}(X, k, J) \subseteq \ker \mathcal{I}'(X, k, J)$ holds for all $k \ge -1$ and all $J \subseteq \mathcal{L}(X)$.*

Proof. The second inclusion is clear. For the first, we generalise the proof of [**54**, Theorem 3.5]: it suffices to prove that every generator of $\mathcal{I}(X, k, J)$ annihilates every element of $S(X, k, J)$. For $k = -1$, this is obvious. Now consider the case $k \ge 0$. Let C be a flat, η a defining normal for C, $Y \subseteq X$, $\deg f \le k + \chi(X \setminus Y) - 1$. Set $e(C) := m(C) + k + \chi(C)$. By Lemma 3.8,

$$D_\eta^{e(C)} f p_Y = p_{Y\cap C} \sum_{i=0}^{e(C)} \binom{e(C)}{i} D_\eta^i f D_\eta^{e(C)-i} p_{Y\setminus C} \tag{III.14}$$

$$\overset{(*)}{=} p_{Y\cap C} \sum_{i=k+\chi(C)}^{k+\chi(X\setminus Y)-1} \binom{e(C)}{i} D_\eta^i f D_\eta^{e(C)-i} p_{Y\setminus C}. \tag{III.15}$$

$(*)$ holds because f does not survive $k+\chi(X\setminus Y)$ differentiations and $p_{Y\setminus C}$ is annihilated by $m(C)+1$ differentiations. Suppose the term in (III.15) is not zero. Then $\chi(X \setminus Y) = 1$ and $\chi(C) = 0$. Furthermore, $m(C)$ differentiations in direction η do not annihilate $p_{Y\setminus C}$. This is only possible if $Y \setminus C = X \setminus C$. This implies $X\setminus Y \subseteq C$. Then $\chi(X\setminus Y) \le \chi(C)$. This is a contradiction. □

Now we will give explicit formulae for $\mathcal{P}(X, k, J)$ and $\mathcal{I}(X, k, J)$ in two particularly simple cases.

Remark 3.11. Suppose that $\dim U = 1$ and that X contains N' non-zero entries. Let $x \in U$ and $y \in V$ be non-zero vectors. Note that $\mathrm{cl}(\emptyset)$ is the only hyperplane in X. Hence, $\mathcal{I}'(X, k, J) = \mathcal{I}(X, k, J) = \mathrm{ideal}\{D_y^{N'+k+\chi(\emptyset)}\}$ and $\mathcal{P}(X, k, J) = \mathrm{span}\{p_x^i : i \in \{0, 1, \ldots, N' - 1 + k + \chi(\emptyset)\}\}$.

Proposition 3.12. *We are using the same terminology as in Definition 3.1. Let $X = (x_1, \ldots, x_r)$ be a basis for U. Let $(y_1, \ldots, y_r)$ denote the dual basis of V. Then, $\mathcal{P}(X, k, J) = \ker \mathcal{I}(X, k, J) \subseteq \ker \mathcal{I}'(X, k, J, E)$.*

Furthermore, for $k \in \{-1, 0\}$, $\ker \mathcal{I}(X, k, J) = \ker \mathcal{I}'(X, k, J, E)$ for all normal selector functions E. More precisely, writing $p_i := p_{x_i}$ and $D_i := D_{y_i}$ as shorthand notation, we get

$$\mathcal{I}(X, k, J) = \mathrm{ideal}\left\{ \prod_{i\in I} D_i^{a_i+1} : \sum_{i\in I} a_i = k + \chi(X \setminus \{x_i : i \in I\}) \right\} \text{ and}$$
$$\mathcal{P}(X, k, J) = \mathrm{span}\left\{ \prod_{i\in I} p_i^{a_i+1} : \sum_{i\in I} a_i \le k + \chi(X \setminus \{x_i : i \in I\}) - 1 \right\}, \tag{III.16}$$

where $I \subseteq [r]$ and $a_i \in \mathbb{N}$.
For $k = 0$, this specialises to $\mathcal{P}(X, 0, J) = \mathrm{span}\,\{p_Y : X \setminus Y \in J\}$.
For $k = -1$, $\mathcal{I}(X, -1, J) = \mathrm{ideal}\{D_1, \ldots, D_r\}$ if J contains all hyperplanes in X and $\mathcal{I}(X, -1, J) = \mathrm{ideal}\{1\}$ otherwise.

For a two-dimensional example of this construction, see Example 7.1 and Figure 4 on page 64.

Proof. This proof generalises the proof of Proposition 4.3 in [**3**]. The statements about $k = -1$ are trivial.

Every flat of X can be written as $C = X \setminus \{x_i : i \in I\}$ for some $I \subseteq [r]$. The set of defining normals for C is given by $\left\{\sum_{i\in I} \alpha_i y_i : \alpha_i \in \mathbb{K} \setminus \{0\}\right\}$.

First, we show that for $k = 0$, $\mathcal{I}(X,k,J) = \mathcal{I}'(X,k,J,E)$. y_i is the defining normal for the hyperplane $X \setminus x_i$. Hence,

$$D_i^{1+\chi(X\setminus x_i)} \in \mathcal{I}'(X,k,J,E) \qquad \text{for } i = 1,\ldots,r. \tag{III.17}$$

Fix a flat C. Let $D_{\eta_C}^{m(C)+\chi(C)}$ be a generator of $\mathcal{I}(X,k,J)$. We will prove now that $D_{\eta_C}^{m(C)+\chi(C)}$ is contained in $\mathcal{I}'(X,k,J,E)$.

Case 1: $C \in J$. Hence, $D_{\eta_C}^{m(C)+\chi(C)} = (\sum_{i\in I}\alpha_i D_i)^{|I|+1}$ for some $I \subseteq [r]$ and $\alpha_i \in \mathbb{K}\setminus\{0\}$. In the monomial expansion of this term, every monomial contains a square. By (III.17), all squares are contained in $\mathcal{I}'(X,k,J,E)$.

Case 2: $C \notin J$. Let C' be a maximal missing flat that contains C. Then,

$$D_{E(C')}^{m(C')+\chi(C')} = \big(\sum_{x_i\notin C'} \lambda_i D_i\big)^{m(C')} \in \mathcal{I}'(X,k,J,E). \tag{III.18}$$

In the monomial expansion of this polynomial, there is only one monomial that does not contain a square: $q := \prod_{x_i\notin C'} D_i$. It follows from the definition of $\mathcal{I}'(X,k,J,E)$ and (III.17) that $q \in \mathcal{I}'(X,k,J,E)$. In the monomial expansion of $D_{\eta_C}^{m(C)+\chi(C)}$, there are only monomials containing squares and a monomial that is a multiple of q. Hence, $D_{\eta_C}^{m(C)+\chi(C)} \in \mathcal{I}'(X,k,J,E)$.

One can easily see that (III.16) describes the $\mathcal{P}$-space by comparing (III.16) with (III.6) and taking into account that X is a basis for U.

Using Lemma 3.9, we can calculate $\mathcal{I}(X,k,J)$:

$$\begin{aligned}\mathcal{I}(X,k,J) &= \operatorname{ideal}\left\{\left(\sum_{i\in I}\alpha_i D_i\right)^{|I|+k+\chi(X\setminus\{x_i:i\in I\})} : I\subseteq[r], \alpha_i\in\mathbb{K}\setminus\{0\}\right\}\\ &= \operatorname{ideal}\left\{\prod_{i\in I} D_i^{a_i+1} : I\subseteq[r], \sum_{i\in I} a_i = k+\chi(X\setminus\{x_i : i\in I\})\right\}.\end{aligned}$$

It is now clear that $\ker\mathcal{I}(X,k,J) = \mathcal{P}(X,k,J)$. $\square$

The following Lemma implies $\mathcal{I}(X,k,J) = \mathcal{I}'(X,k,J,E)$ for $k \le 0$, using the Main Theorem for $k = 0$ as base case (cf. Remark 3.6).

Lemma 3.13. *Let $J \subseteq \mathcal{L}(X)$ be an arbitrary upper set and k be an arbitrary integer. If $\mathcal{I}(X,k,J)$ is contained in* $\mathrm{Sym}(V)$ *(i. e. $m(C) \ge k$ for all flats C) and $\mathcal{I}'(X,k,J,E) = \mathcal{I}(X,k,J)$ for all normal selector functions E, then $\mathcal{I}'(X,l,J,E) = \mathcal{I}(X,l,J)$ for all $l \le k$ which satisfy $\mathcal{I}(X,l,J) \subseteq$* $\mathrm{Sym}(V)$ *and all normal selector functions E.*

Proof. Let $D_\eta^{m(\eta)+l+\chi(\eta)}$ be a generator of $\mathcal{I}(X,l,J)$. We show that this generator is contained in $\mathcal{I}'(X,l,J,E)$. By induction, we may suppose that $D_\eta^{m(\eta)+l+1+\chi(\eta)} \in \mathcal{I}'(X,l+1,J)$, *i. e.* there exist $q_i \in \mathrm{Sym}(V)$ and $D_{\eta_i}^{m(\eta_i)+l+1+\chi(\eta_i)}$ generators of $\mathcal{I}'(X,l+1,J)$ s. t.

$$D_\eta^{m(\eta)+l+1+\chi(\eta)} = \sum_i q_i D_{\eta_i}^{m(\eta_i)+l+1+\chi(\eta_i)} \tag{III.19}$$

Let $u \in U$ be a vector s. t. $\eta(u) = 1$. We consider p_u as a differential operator on $\mathrm{Sym}(V)$. By applying p_u to (III.19), we see that $D_\eta^{m(\eta)+l+\chi(\eta)}$ is contained in $\mathcal{I}'(X,l,J,E)$. $\square$

3.3. Deletion and contraction. In the third paragraph of this subsection we will prove the Main theorem. The proof is inductive using deletion and contraction. In the first paragraph we will define those two operations for realisations of matroids, *i. e.* for lists of vectors as in Section II.4. In the second paragraph we will define them for upper sets.

3.3.1. *Matroids under deletion and contraction.* Two important matroid operations are deletion and contraction of an element. For realisations of matroids, they are defined as follows. Let X be a finite list of vectors and let $x \in X$. The *deletion* of x is (the matroid defined by) the list $X \setminus x$.

For the rest of this paragraph, fix an element $x \in X$ that is not a loop. Let $\pi_x : U \to U/x$ denote the projection to the quotient space. The *contraction* of x is (the matroid defined by) the list X/x which contains the images of the elements of $X \setminus x$ under π_x.

We want to be able to see $\mathrm{Sym}(U/x)$ as a subspace of $\mathrm{Sym}(U)$. For that, pick a basis $B = \{b_1, \ldots, b_r\} \subseteq U$ with $b_r = x$. Let $W := \mathrm{span}\{b_1, \ldots, b_{r-1}\}$. Then we have an isomorphism $U/x \cong W$ which extends to an isomorphism $\mathrm{Sym}(U/x) \cong \mathrm{Sym}(W) \subseteq \mathrm{Sym}(U)$. Under this identification, $\mathrm{Sym}(\pi_x)$ becomes the map that sends p_x to zero and maps all other basis vectors to themselves. Then $\mathrm{Sym}(U) \cong \mathrm{Sym}(W) \oplus p_x \mathrm{Sym}(U)$.

Let $Y \subseteq X \setminus x$. We write $\bar{Y}$ to denote the sublist of X/x with the same index set as Y and vice versa. Let $\bar{C} \subseteq X/x \subseteq W$ be a flat and $\eta \in W^*$ be a defining normal for $\bar{C}$. Since $W^* \cong x^o := \{v \in V : v(x) = 0\}$, η is also a defining normal for the flat $C \cup \{x\} \subseteq X$.

3.3.2. *The lattice of flats under deletion and contraction.* In this paragraph we will discuss how the lattice of flats of a matroid behaves under deletion and contraction and for a given upper set J we define upper sets $J \setminus x \subseteq \mathcal{L}(X \setminus x)$ and $J/x \subseteq \mathcal{L}(X/x)$.

For the whole paragraph we fix an element $x \in X$ that is not a loop. First, we will exhibit some relations between the lattices of flats of X, $X \setminus x$ and X/x. There are two bijective maps

$$L_x : \mathcal{L}(X \setminus x) \to \{C \in \mathcal{L}(X) : C = \mathrm{cl}(C \setminus x)\} \tag{III.20}$$

$$\text{and } L^x : \mathcal{L}(X/x) \to \{C \in \mathcal{L}(X) : x \in C\}. \tag{III.21}$$

The maps are given by $L_x(C) := \mathrm{cl}_X(C)$, $L_x^{-1}(C) := C \setminus x$, $L^x(\bar{C}) := C \cup x$ and $(L^x)^{-1}(C) := \overline{C \setminus x}$.

Definition 3.14. Let $J \subseteq \mathcal{L}(X)$ be an upper set. Then define

$$J \setminus x := \{C \setminus x : C \in J \text{ and } C = \mathrm{cl}(C \setminus x)\} = L_x^{-1}(J \cap L_x(\mathcal{L}(X \setminus x)))$$

$$\text{and } J/x := \{(\overline{C \setminus x}) : x \in C \in J\} = (L^x)^{-1}(J \cap L^x(\mathcal{L}(X/x))) \subseteq \mathcal{L}(X/x).$$

It is easy to check that these two sets are upper sets. The following statement on the indicator functions is also easy to prove.

Lemma 3.15. *Let $x \notin Y \subseteq X$. Then $\chi_{J \setminus x}(Y) = \chi_J(Y)$ and $\chi_{J/x}(\bar{Y}) = \chi_J(Y \cup x)$.*

From this, we can deduce the following lemma.

Lemma 3.16. *(1) If $C \subseteq X \setminus x$ is a maximal missing flat for $J \setminus x$ then C or $C \cup x$ is a maximal missing flat for J.*

(2) If $\bar{C} \subseteq X/x$ is a maximal missing flat for J/x then $C \cup x$ is a maximal missing flat for J.

We will also need the following two facts.

REMARK 3.17. Let $x \in X$ be neither a loop nor a coloop. Suppose that J contains all hyperplanes in X. Then,

(1) $J \setminus x$ contains all hyperplanes in $X \setminus x$. This follows from the fact that $\mathcal{L}(X \setminus x)$ contains exactly the flats C that satisfy $\mathrm{rk}(C) = \mathrm{rk}(C \setminus x)$.
(2) J/x contains all hyperplanes in X/x. This follows from the fact that $\mathcal{L}(X/x)$ contains exactly the flats containing x and the fact that contraction reduces the rank of a flat containing x by one.

3.3.3. *Proof of the Main Theorem.* In this paragraph we will prove the Main Theorem. The following proposition is a side product of the deletion-contraction proof.

PROPOSITION 3.18. *We are using the same terminology as in Definition 3.1. Suppose that $x \in X$ is neither a loop nor a coloop. For $k = -1$, we assume in addition that J contains all hyperplanes in X or $J = \{X\}$.*

Then the following is an exact sequence of graded vector spaces:

$$\begin{aligned} 0 \quad \longrightarrow \quad & \ker \mathcal{I}(X \setminus x, k, J \setminus x)[-1] \xrightarrow{\cdot p_x} \ker \mathcal{I}(X, k, J) \\ \xrightarrow{\mathrm{Sym}(\pi_x)} & \ker \mathcal{I}(X/x, k, J/x) \to 0. \end{aligned} \tag{III.22}$$

If $x \in X$ is a loop, then $\ker \mathcal{I}(X \setminus x, k, J \setminus x) = \ker \mathcal{I}(X, k, J)$. For $k \in \{-1, 0\}$, both statements also hold if we replace $\mathcal{I}$ by $\mathcal{I}'$.

Here, $(\cdot)[-1]$ denotes the graded vector space $(\cdot)$ with the degree shifted up by one and $\mathrm{Sym}(\pi_x)$ denotes the algebra homomorphism that maps p_v to $p_{\pi_x(v)}$.

REMARK 3.19. Proposition 3.18 will turn out to be a special case of Proposition II.6.17 for $k \geq 0$ once we have proven Theorem 3.2 (cf. Section 1).

The proof of Proposition 3.18 is inductive. It uses the following lemma.

LEMMA 3.20. *Suppose that we are in the same setting as in Proposition 3.18. Let $x \in X$ be neither a loop nor a coloop. Suppose that $\mathcal{P}(X \setminus x, k, J \setminus x) = \ker \mathcal{I}(X \setminus x, k, J \setminus x)$ and $\mathcal{P}(X/x, k, J/x) = \ker \mathcal{I}(X/x, k, J/x)$.*

(1) Then, the following sequence is exact:

$$\begin{aligned} 0 \quad \longrightarrow \quad & \ker \mathcal{I}(X \setminus x, k, J \setminus x)[-1] \xrightarrow{\cdot p_x} \ker \mathcal{I}(X, k, J) \\ \xrightarrow{\mathrm{Sym}(\pi_x)} & \ker \mathcal{I}(X/x, k, J/x) \to 0. \end{aligned} \tag{III.23}$$

(2) If we suppose in addition that $\mathcal{I}'(X \setminus x, k, J \setminus x, E') = \mathcal{I}(X \setminus x, k, J \setminus x)$ and $\mathcal{I}'(X/x, k, J/x, E'') = \mathcal{I}(X/x, k, J/x)$, the following sequence is exact:

$$\begin{aligned} 0 \quad \longrightarrow \quad & \ker \mathcal{I}'(X \setminus x, k, J \setminus x, E')[-1] \xrightarrow{\cdot p_x} \ker \mathcal{I}'(X, k, J, E) \\ \xrightarrow{\mathrm{Sym}(\pi_x)} & \ker \mathcal{I}'(X/x, k, J/x, E'') \longrightarrow 0. \end{aligned} \tag{III.24}$$

Here, E' and E'' denote the restrictions of E to $\mathcal{L}(X \setminus x)$ and $\mathcal{L}(X/x)$, i. e. $E'(C) := E(\mathrm{cl}_X(C))$ and $E''(\bar{C}) := E(C \cup x)$.

Proof of Lemma 3.20. We only prove part (2). The reader will notice that the same proof with some obvious modifications can be used to prove part (1), unless $k = -1$ and $J = \{X\}$. In that case, both parts are equivalent by Lemma 3.13.

Before starting with the proof, we will introduce some additional notation, which is only used here. For a flat C defined by η, we write $e_X(\eta) = e_X(C) := m_X(C) + k + \chi(C)$. As described above, we fix a subspace $W \subseteq U$ that is complementary to the space $\operatorname{span}(x)$ and identify U/x with W. Hence, $\ker \mathcal{I}'(X/x, k, J/x, E'') \subseteq \operatorname{Sym}(W)$.

Let $f \in \operatorname{Sym}(U)$ and $v, \eta \in V$. Let t be a formal symbol. Then $f(v + t\eta) \in \mathbb{K}[t]$ and the following Taylor expansion formula holds: $f(v + t\eta) = \sum_{k\geq 0} \frac{D_\eta^k}{k!} f(v) t^k$. Now define $\rho_f : V \to \mathbb{N}$, the *directional degree function* of f [**3**], as the function which assigns to η the degree of the univariate polynomial $f(v + t\eta) \in \mathbb{K}[t]$ for generic v. We obtain $\rho_{fg} = \rho_f + \rho_g$ by comparing the Taylor expansion of $f \cdot g$ with the product of the Taylor expansions of f and g. The number $\rho_f(\eta)$ tells us how many derivations f survives in direction η. Hence, ρ can be used to describe $\ker \mathcal{I}(X, k, J)$ and $\ker \mathcal{I}'(X, k, J, E)$. Namely,

$$\ker \mathcal{I}(X, k, J) = \{f \in \operatorname{Sym}(U) : \rho_f(\eta) < e(\eta),\, \eta \in V \setminus \{0\}\}. \qquad \text{(III.25)}$$

Now we come to the main part of the proof. It is split into five parts.

(i) $\cdot p_x$ is well-defined, i. e. really maps to $\ker \mathcal{I}'(X, k, J, E)$*:* due to Lemma 3.10, it suffices to prove $S(X \setminus x, k, J \setminus x) \overset{\cdot p_x}{\hookrightarrow} S(X, k, J)$. For $k \geq 0$, this follows directly from Lemma 3.15. For $k = -1$, consider $p_Y \in S(X \setminus x, -1, J \setminus x)$ and $C \in \mathcal{L}(X)$. Then $C \setminus x \in \mathcal{L}(X \setminus x)$ and $\chi_{J\setminus x}(C \setminus x) \leq \chi_J(C)$. One can easily deduce $|(Y \cup x) \setminus C| < m(C) - 1 + \chi_J(C)$ from the corresponding inequality for $C \setminus x$.

(ii) $\operatorname{Sym}(\pi_x)$ *is well-defined:* let $g \in \ker \mathcal{I}'(X, k, J, E)$ and let $h := (\operatorname{Sym}(\pi_x))(g)$. Let $\bar{C} \in \mathcal{L}(X/x)$ be a maximal missing flat or a hyperplane, respectively. By Lemma 3.16, $C \cup x \in \mathcal{L}(X)$ is a maximal missing flat or a hyperplane, respectively. Let $\eta := E(C \cup x)$. This implies $E''(\bar{C}) = \eta$. We need to prove $\rho_h(\eta) < e_{X/x}(\bar{C})$.

Note that $m_{X/x}(\bar{C}) = m_X(C \cup x)$ and by Lemma 3.15, $\chi_{J/x}(\bar{C}) = \chi_J(C \cup x)$. Hence, $e_{X/x}(\bar{C}) = e_X(C \cup x)$. The polynomial g can be uniquely written as $g = h + p_x g_1$ for some $g_1 \in \operatorname{Sym}(U)$. For all $k \in \mathbb{N}$, $D_\eta^k g = D_\eta^k h + p_x D_\eta^k g_1$. As p_x does not divide h, this implies $\rho_h(\eta) \leq \rho_g(\eta)$. In summary, we get

$$e_{X/x}(\bar{C}) = e_X(C \cup x) > \rho_g(\eta) \geq \rho_h(\eta). \qquad \text{(III.26)}$$

(iii) Injectivity of $\cdot p_x$: clear.

(iv) Exactness in the middle: let $g \in \ker \mathcal{I}'(X, k, J, E)$ and $\operatorname{Sym}(\pi_x)(g) = 0$. This implies that g can be written as $g = p_x h$ for some $h \in \operatorname{Sym}(U)$. We need to show that $h \in \ker \mathcal{I}'(X \setminus x, k, J \setminus x, E') = \ker \mathcal{I}(X \setminus x, k, J \setminus x)$.

Let C be a maximal missing flat (resp. hyperplane) in $X \setminus x$. By Lemma 3.16, $C' = C$ or $C' = C \cup x$ is a maximal missing flat (resp. hyperplane) in X. Let $\eta := E(C')$. The vector η is also a defining normal for $C \subseteq X \setminus x$. By definition of E', $\eta = E'(C)$. We will now show that $\rho_h(\eta) < e_{X\setminus x}(C)$.

If $x \in C'$, then $\rho_{p_x} = 0$, $m_{X\setminus x}(C) = m_X(C')$, and $\chi_{J\setminus x}(C) = \chi_J(C')$. If $x \notin C'$, then $\rho_{p_x}(\eta) = 1$, $m_{X\setminus x}(C) + 1 = m_X(C')$, and $\chi_{J\setminus x}(C) = \chi_J(C')$. So, in both cases, $e_{X\setminus x}(\eta) + \rho_{p_x}(\eta) = e_X(\eta)$. This implies

$$e_{X\setminus x}(\eta) = e_X(\eta) - \rho_{p_x}(\eta) > \rho_{p_x h}(\eta) - \rho_{p_x}(\eta) = \rho_h(\eta). \tag{III.27}$$

(v) Surjectivity of $\mathrm{Sym}(\pi_x)$*:* we consider the case $k \geq 0$ first. Let $fp_{\bar{Y}} \in S(X/x, k, J/x)$. It suffices to prove that $fp_Y \in S(X, k, J)$. Since $x \notin Y$, by Lemma 3.15, $\chi_{J/x}((X/x) \setminus \bar{Y}) = \chi_J(X \setminus Y)$. This implies $fp_Y \in S(X, k, J)$.

Now consider the case $k = -1$. This requires a little more work. There are two subcases.

(a) *J contains all hyperplanes:* let $p_{\bar{Y}} \in S(X/x, -1, J/x)$. We will now show that $p_Y \in S(X, -1, J)$. Let $C \in \mathcal{L}(X) \setminus \{X\}$. Suppose first that $x \in C$ or $\operatorname{codim} C \geq 2$. Then $D := \operatorname{cl}(C \cup x) \neq X$. By assumption, $\left|\bar{Y} \setminus (\overline{D \setminus x})\right| < m_{X/x}(\overline{D \setminus x}) - 1 + \chi_{J\setminus x}(\overline{D \setminus x})$. By Lemma 3.15, this implies

$$|Y \setminus D| < m_X(D) - 1 + \chi_J(D). \tag{III.28}$$

Since $x \in D \setminus C$ and $x \notin Y$, $|Y \setminus C| - |Y \setminus D| \leq m_X(C) - m_X(D) - 1$. Adding this inequality to (III.28), we obtain the desired inequality:

$$|Y \setminus C| < m_X(C) - 1 + \chi_J(C). \tag{III.29}$$

Now suppose that C is a hyperplane and $x \notin C$. By assumption, $\chi_J(C) = 1$. Since $x \notin Y \cup C$, we can deduce that (III.29) holds for this C.

(b) $J = \{X\}$*:* this can be shown by a dimension argument using the fact that the dimension of $\mathcal{P}(X - 1, \{X\})$ equals the cardinality of the set of internal bases $\mathbb{B}_-(X)$ (see [**54**, Theorem 5.9] or Proposition I.4.15). If $x \in X$ is the minimal element, the following deletion-contraction equality holds:

$$|\mathbb{B}_-(X)| = |\mathbb{B}_-(X \setminus x)| + |\mathbb{B}_-(X/x)|. \quad \square$$

Proof of Proposition 3.18 and of the Main Theorem. We generalise the proof of [**3**, Propositions 4.4 and 4.5].

Loops can safely be ignored: they are contained in every flat C, thus $m_X(C) = m_{X\setminus x}(C)$ and $\mathcal{L}(X) \cong \mathcal{L}(X \setminus x)$ if x is a loop. From now on, we suppose that X does not contain loops.

We prove both statements by induction on the number of elements of X that are not coloops. The reader should check that our reasoning below also works for $k = -1$, although in that case, $\mathcal{P}$-spaces might be equal to $\{0\}$. Remark 3.17 ensures that an upper set that contains all hyperplanes preserves this structure under deletion and contraction.

If X contains only coloops the Main Theorem follows from Proposition 3.12. Now suppose that $x \in X$ is not a coloop and that the Main Theorem holds for X/x and $X \setminus x$. In addition, we assume $\dim U \geq 2$. If $\dim U = 1$, the statement follows from Remark 3.11.

By Lemma 3.20, the following sequence is exact:

$$\begin{aligned} 0 \;\longrightarrow\; & \ker \mathcal{I}(X \setminus x, k, J \setminus x)[-1] \xrightarrow{\cdot p_x} \ker \mathcal{I}(X, k, J) \\ \xrightarrow{\mathrm{Sym}(\pi_x)}\; & \ker \mathcal{I}(X/x, k, J/x) \to 0. \end{aligned} \tag{III.30}$$

Every short exact sequence of vector spaces splits. Hence, $\ker \mathcal{I}(X,k,J) \cong p_x \cdot \ker \mathcal{I}(X \setminus x, k, J \setminus x) \oplus \ker \mathcal{I}(X/x, k, J/x)$. For $k \in \{-1, 0\}$, the same argumentation also works for $\mathcal{I}'(X,k,J,E)$. To conclude, we recall the following two statements that were shown in the proof of Lemma 3.20: (i) $p_x \cdot S(X \setminus x, k, J \setminus x) \subseteq S(X,k,J)$ and (ii) $\mathrm{Sym}(\pi_x) : S(X,k,J) \to S(X/x,k,J/x)$ is surjective, if $(k,J) \neq (-1,\{X\})$. □

Remark 3.21. In general, the map

$$\mathrm{Sym}(\pi_x) : S(X,-1,\{X\}) \to S(X/x,-1,\{X/x\}) \tag{III.31}$$

is not surjective (cf. Example 7.4). Proposition 3.18 is false for arbitrary upper sets J that do not contain all hyperplanes (cf. Example 7.3). The difficulty of the case $k = -1$ was already observed by Holtz and Ron. They conjectured that the Main Theorem holds in the internal case, *i. e.* for $k = -1$ and $J = \{X\}$ [**54**, Conjecture 6.1]. An incorrect "proof" of this conjecture appeared in [**3**], which assumed that $\mathrm{Sym}(\pi_x) : S(X,-1,\{X\}) \to S(X/x,-1,\{X/x\})$ is always surjective. The authors of [**3**] later found a counterexample to Holtz's and Ron's conjecture [**2**].

4. Bases for $\mathcal{P}$-spaces

In this section we will show how a basis for $\mathcal{P}(X,k,J)$ can be selected from $S(X,k,J)$ for $k \geq 0$. Our construction depends on the order on X. This order is used to define internal and external activity. Our result is a generalization of [**3**, Proposition 4.21] to hierarchical spaces. At the end of this section, there is a remark on the case $k = -1$.

Recall that $\mathbb{B}(X)$ denotes the set of all bases $B \subseteq X$. In Subsection I.3.1 we defined the set $I(B)$ of internally active elements and the set $E(B)$ of externally active elements with respect to a basis $B \in \mathbb{B}(X)$.

Definition 4.1. We are using the same terminology as in Definition 3.1. In addition, let $k \geq 0$. Then define

$$\begin{aligned}\Gamma(X,k,J) := \big\{ &(B, I, \boldsymbol{a}_I) : B \in \mathbb{B}(X),\ I \subseteq I(B),\ \boldsymbol{a}_I \in \mathbb{N}^I, \\ &\sum_{x \in I} a_x \leq k + \chi((B \cup E(B)) \setminus I) - 1 \big\} \text{ and}\end{aligned} \tag{III.32}$$

$$\mathcal{B}(X,k,J) := \Big\{ p_{X \setminus (B \cup E(B))} \prod_{x \in I} p_x^{a_x+1} : (B, I, \boldsymbol{a}_I) \in \Gamma(X,k,J) \Big\} \subseteq \mathrm{Sym}(U).$$

For $k = 0$, this specialises to

$$\begin{aligned}\mathcal{B}(X,0,J) = \{ p_{(X \setminus (B \cup E(B))) \cup I} : &\ B \in \mathbb{B}(X),\ I \subseteq I(B), \\ &\mathrm{cl}((B \cup E(B)) \setminus I) \in J \}.\end{aligned} \tag{III.33}$$

Note that *a priori*, it is unclear whether the set $\Gamma(X,k,J)$ has the same cardinality as the set $\mathcal{B}(X,k,J)$ since we do not know if distinct elements of $\Gamma(X,k,J)$ correspond to distinct polynomials in $\mathcal{B}(X,k,J)$. This desired property only becomes clear in the proof of the following theorem.

Theorem 4.2 (Basis Theorem). *We are using the same terminology as in Definition* 3.1. *In addition, let* $k \geq 0$. *Then* $\mathcal{B}(X,k,J)$ *is a basis for* $\mathcal{P}(X,k,J)$.

PROOF. As in the proof of the Main Theorem, we may suppose that X does not contain any loops: if x is a loop, it is not contained in any basis in $\mathbb{B}(X)$, but x is contained in every flat and always externally active. Hence, the removal of a loop changes neither $\mathcal{P}(X,k,J)$ nor $\Gamma(X,k,J)$.
The remainder of this proof is split into four parts.

(i) Let $x \in X$ be the minimal element. Let $B, B' \in \mathbb{B}(X)$ with $x \notin B$ and $x \in B'$. x is externally active with respect to B if and only if x is a loop and x is internally active in B' if and only if x is a coloop.

(ii) $|\Gamma(X,k,J)| = \dim \mathcal{P}(X,k,J)$*:* we prove this by induction over the number of elements that are not coloops. Suppose that X contains only coloops. In this case there is only one basis and all its elements are internally active. The spanning set given in (III.16) is a basis and it coincides with $\mathcal{B}(X,k,J)$.

Now suppose that there is at least one element in x which is not a coloop. In addition, we may assume $\dim U \geq 1$. If $\dim U = 1$, the statement follows from Remark 3.11. As $\dim \mathcal{P}(X,k,J)$ and by induction also $|\Gamma(X/x,k,J/x)|$ and $|\Gamma(X \setminus x, k, J \setminus x)|$ are independent of the order on X, we may assume that x is the minimal element.

$\mathbb{B}(X)$ can be partitioned as $\mathbb{B}(X) = \mathbb{B}(X \setminus x) \,\dot\cup\, \iota(\mathbb{B}(X/x))$, where ι denotes the map that sends a basis $\bar{B} \in \mathbb{B}(X/x)$ to $B \cup x$. It follows from (i) and Lemma 3.15 that $\Gamma(X,k,J)$ can also be written as a disjoint union of two sets: $\Gamma(X,k,J) = \Gamma(X \setminus x, k, J \setminus x) \,\dot\cup\, \iota_1(\Gamma(X/x,k,J/x))$, where ι_1 denotes the map that sends $(\bar{B}, \bar{I}, \boldsymbol{a}_{\bar{I}})$ to $(B \cup x, I, \boldsymbol{a}_I)$. Comparing this with Proposition 3.18, we see that $|\Gamma(X,k,J)| = \dim \mathcal{P}(X,k,J)$.

(iii) $\mathcal{B}(X,k,J) \subseteq S(X,k,J) \subseteq \mathcal{P}(X,k,J)$*:* if $Y = (X \setminus (B \cup E(B))) \cup I$, then $X \setminus Y = (B \cup E(B)) \setminus I$. Hence, by comparison of (III.32) and (III.6), the statement follows.

(iv) $\mathcal{B}(X,k,J)$ *is linearly independent:* by [**3**, Proposition 4.21], the set $\mathcal{B}(X,k,\mathcal{L}(X)) = \mathcal{B}(X,k+1,\{X\})$ is linearly independent. As $\mathcal{B}(X,k,J)$ is contained in this set, it is also linearly independent. ☐

REMARK 4.3. We do not know if there is a simple method to construct a basis for $\mathcal{P}(X,-1,J)$. This difficulty was already observed for the internal case by Holtz and Ron [**54**]. In Section 5.3 we will define a set of semi-internal bases $\mathbb{B}_-(X,J) \subseteq \mathbb{B}(X)$ whose cardinality is in some cases equal to the dimension of $\mathcal{P}(X,-1,J)$. A natural candidate for $\mathcal{B}(X,-1,J)$ would be the set $\tilde{\mathcal{B}}(X,-1,J) := \{p_{X \setminus (B \cup E(B))} : B \in \mathbb{B}_-(X,J)\}$. In some cases, this is indeed a basis, but in general it has the wrong cardinality or it fails to be contained in $\mathcal{P}(X,-1,J)$ (see Example 7.2).

REMARK 4.4. The external space has a vector space decomposition

$$\mathcal{P}(X,1,\{X\}) = \bigoplus_{C \in \mathcal{L}(X)} \mathcal{P}(X)_C \tag{III.34}$$

where $\mathcal{P}(X)_C := \operatorname{span}\{p_Y : \operatorname{cl}(X \setminus Y) = C\} = p_{X \setminus C}\mathcal{P}(C,0,\{C\})$ [**7, 76**]. This decomposition can be used to deduce Theorem 4.2 for $k = 0$ from the well-known fact that $\{p_{X \setminus (B \cup E(B)) \cup I} : B \in \mathbb{B}(X),\, I \subseteq B\}$ is a basis for $\mathcal{P}(X,1,\{X\})$ [**3, 7, 54**].

A related theorem due to Andrew Berget states that the Tutte polynomial is equal to the Hilbert series of the external space $\mathcal{P}_+(X)$ equipped with a certain bigrading.

THEOREM 4.5 ([**7**]). *Let X be a list of vectors in some vector space over an arbitrary field. Then*

$$T_{\mathfrak{M}(X)}(x,y) = \sum_{j,k\geq 0} (x-1)^{r-j} y^{k-j} \dim \mathcal{P}(X)_{j,k}, \tag{III.35}$$

where $\mathcal{P}(X)_{j,k} := \operatorname{span}\{p_Y : \operatorname{rk}(X\setminus Y) = j \text{ and } |X\setminus Y| = k\}$.

REMARK 4.6. Corrado De Concini, Claudio Procesi, and Michèle Vergne defined a space $G(X)$ that also has a decomposition in terms of the lattice of flats $\mathcal{L}(X)$ [**37**, Theorem 4.5]. $\mathcal{P}(X)_C$ and the summand of $G(X)$ that corresponds to the flat C have the same dimension. Furthermore, the two spaces are connected via the duality between $\mathcal{P}$ and $\mathcal{D}$-spaces.

5. Hilbert series

In this section we will give several formulae for the Hilbert series of the spaces $\mathcal{P}(X,k,J)$. The formulae in the first subsection are recursive. In the second subsection we will give combinatorial formulae for the case $k\geq 0$. The last subsection is devoted to the case $k=-1$. All formulae only depend on the matroid $\mathfrak{M}(X)$, the integer k, and the upper set J, but not on the realisation X.

5.1. Recursive formulae.

In this subsection we will give recursive formulae for the calculation of $\operatorname{Hilb}(\mathcal{P}(X,k,J),t)$. The following statement is a direct consequence of Proposition 3.18 and of the Main Theorem:

COROLLARY 5.1. *We are using the same terminology as in Definition* 3.1. *Let $x\in X$ be an element that is not a coloop. For $k=-1$, we assume in addition that J contains all hyperplanes in X or $J=\{X\}$. Then,*

$$\operatorname{Hilb}(\mathcal{P}(X,k,J),t) = \begin{cases} \operatorname{Hilb}(\mathcal{P}(X\setminus x,k,J\setminus x),t) & \textit{if } x \textit{ is a loop} \\ t\operatorname{Hilb}(\mathcal{P}(X\setminus x,k,J\setminus x),t) & \\ \qquad +\operatorname{Hilb}(\mathcal{P}(X/x,k,J/x),t) & \textit{otherwise} \end{cases}$$

For coloops, the situation is more complicated and requires an additional definition. Fix a coloop $x\in X$. Then, $X\setminus x$ is a hyperplane and the following is an upper set:

$$\widehat{J/x} := \{\bar{C} : x\notin C\in J\} \cup \{\overline{X\setminus x}\} \subseteq \mathcal{L}(X/x). \tag{III.36}$$

$\widehat{J/x}$ forgets about the flats containing x, whereas J/x forgets about the flats not containing x. While the latter is always an upper set in $\mathcal{L}(X/x)$, some elements of $\widehat{J/x}$ are not closed unless $X\setminus x$ is a hyperplane.

PROPOSITION 5.2. *We are using the same terminology as in Definition* 3.1. *Let $x \in X$ be a coloop and $k \geq 0$. Then*

$$\operatorname{Hilb}(\mathcal{P}(X,k,J),t) = \begin{cases} \sum_{j=0}^{k} t^{j+1} \operatorname{Hilb}(\mathcal{P}(X/x, k-j, \widehat{J/x}), t) \\ \qquad + \operatorname{Hilb}(\mathcal{P}(X/x,k,J/x),t) & \text{if } X \setminus x \in J \\ \sum_{j=0}^{k-1} t^{j+1} \operatorname{Hilb}(\mathcal{P}(X/x, k-j, \widehat{J/x}), t) \\ \qquad + \operatorname{Hilb}(\mathcal{P}(X/x,k,J/x),t) & \text{if } X \setminus x \notin J \end{cases}.$$

For $k = -1$, we have

$$\operatorname{Hilb}(\ker \mathcal{I}(X,-1,J),t) = \begin{cases} \operatorname{Hilb}(\ker \mathcal{I}(X/x,k,J/x),t) & \text{if } X \setminus x \in J \\ 0 & \text{if } X \setminus x \notin J \end{cases}.$$

This formula holds for arbitrary non-empty upper sets $J \subseteq \mathcal{L}(X)$.

For an example, see Example 7.1. Actually, we will prove a more general statement, namely decomposition formulae for the $\mathcal{P}$-spaces of type

$$\mathcal{P}(X,k,J) \cong \mathcal{P}(X/x,k,J/x) \oplus \bigoplus_{j} p_x^{j+1} \mathcal{P}(X/x, k-j, \widehat{J/x}) \qquad \text{(III.37)}$$

PROOF OF PROPOSITION 5.2. We will first prove the equation for $k \geq 0$ using Theorem 4.2 by showing that there exists a bijection between the bases of the $\mathcal{P}$-spaces appearing on each side.

Fix a basis $B \in \mathbb{B}(X)$. Let $\Gamma_B(X,k,J) := \{(B,I,\boldsymbol{a}_I) \in \Gamma(X,k,J)\}$. Since x is a coloop, x is contained in every basis and always internally active. Hence, $\overline{I(B)} = I(B/x) \cup \bar{x}$. A similar relationship between the sets of externally active elements with respect to B and B/x does not exist. However, we do not need this since $\operatorname{cl}((B \cup E(B)) \setminus I) = \operatorname{cl}(B \setminus I)$ for all $I \subseteq I(B)$ [**8**, (7.13)].

Consider the following map:

$$\Phi_B : \Gamma_B(X,k,J) \to \Gamma_B(X/x,k,J/x) \,\dot\cup \dot{\bigcup_{j=0}^{k-\varepsilon}} \Gamma_B(X/x, k-j, \widehat{J/x}) \qquad \text{(III.38)}$$

$$(B,I,\boldsymbol{a}_I) \mapsto \begin{cases} (\overline{B \setminus x}, \bar{I}, \boldsymbol{a}_{\bar{I}}) \in \Gamma_B(X/x,k,\widehat{J/x}) & \text{if } x \notin I \\ (\overline{B \setminus x}, \overline{I \setminus x}, \boldsymbol{a}_{\bar{I}}) \in \Gamma_B(X/x, k-a_x, \widehat{J/x}) & \text{if } x \in I \end{cases},$$

where $\varepsilon = 1$ if $X \setminus x \notin J$ and 0 otherwise and $\boldsymbol{a}_{\bar{I}}$ denotes the restriction of $\boldsymbol{a}_I$ to $\mathbb{N}^{I \setminus x}$.

From the following three facts, we can deduce that Φ_B is a bijection:

(i) If $x \notin I$, then by Lemma 3.15, $\chi_J(B \setminus I) = \chi_{J/x}(B \setminus (I \cup x))$.

(ii) If $x \in I$ and $X \setminus x \in J$ (*i. e.* we are in the first case of (III.37)), then $\chi_J(B \setminus I) = \chi_{\widehat{J/x}}(\overline{B \setminus I})$.

(iii) If $x \in I$ and $X \setminus x \notin J$ (*i. e.* we are in the second case of (III.37)), then $\chi_J(B \setminus I) = \chi_{\widehat{J/x}}(\overline{B \setminus I}) = 0$.

We have to distinguish the cases $X \setminus x \in J$ and $X \setminus x \notin J$ for the following reason: if $I = \{x\}$ and $\chi_J(B \setminus I) = 0$, then $\Gamma_B(X,k,J)$ contains no elements with $a_x = k$. However, $\Gamma_B(X, k-k, \widehat{J/x})$ is non-empty, since by definition, $\chi_{\widehat{J/x}}(\overline{B}) = 1$.

Furthermore, note that the degrees of the polynomials corresponding to $(B, I, \boldsymbol{a}_I)$ and $(\overline{B \setminus x}, \overline{I \setminus x}, \boldsymbol{a}_{\bar{I}})$ differ by $a_x + 1$ if $x \in I$. If $x \notin I$, the polynomials corresponding to $(B, I, \boldsymbol{a}_I)$ and $(\overline{B \setminus x}, \bar{I}, \boldsymbol{a}_{\bar{I}})$ have the same degree. This completes the proof for $k \geq 0$.

Now we consider the case $k = -1$. If $X \setminus x \notin J$, then $\mathcal{I}(X, -1, J) = \operatorname{ideal}\{1\}$. Suppose that $X \setminus x \in J$. Let η be a defining normal for $X \setminus x$. Then $D_\eta \in \mathcal{I}(X, -1, J)$ and it is easy to check that $\ker \mathcal{I}(X, -1, J) \cong \ker \mathcal{I}(X/x, -1, J/x)$. □

5.2. Combinatorial formulae for $k \geq 0$. In this subsection we will prove several combinatorial formulae for $\operatorname{Hilb}(\mathcal{P}(X, k, J), t)$. As in the case of the Tutte polynomial, there is a formula that depends on the internal and external activity of the bases of X. For $k = 0$, there is also a subset expansion formula and a particularly simple formula for the dimension.

Theorem 4.2 provides a method to compute the Hilbert series of a $\mathcal{P}$-space combinatorially.

COROLLARY 5.3. *We are using the same terminology as in Definition* 3.1. *Let $k \geq 0$. Then*

$$\operatorname{Hilb}(\mathcal{P}(X, k, J), t) = \sum_{B \in \mathbb{B}(X)} t^{N - r - |E(B)|} \left(1 + \sum_{\emptyset \neq I \subseteq I(B)} \sum_{j=0}^{\chi((B \cup E(B)) \setminus I) + k - 1} t^{|I| + j} \binom{j + |I| - 1}{|I| - 1} \right),$$

where $E(B)$ and $I(B)$ denote the sets of externally resp. internally active elements. For $k = 0$, this specialises to

$$\operatorname{Hilb}(\mathcal{P}(X, 0, J), t) = \sum_{B \in \mathbb{B}(X):} t^{N - r - |E(B)|} \left(1 + \sum_{\substack{\emptyset \neq I \subseteq I(B) \\ \chi((B \cup E(B)) \setminus I) = 1}} t^{|I|} \right).$$

Corollary 5.3 gives a formula in terms of the internal and external activity of the bases of X. For $k = 0$, there is also a subset expansion formula similar to the one for the Tutte polynomial. Recall that in the internal, central and external case, the Hilbert series of the $\mathcal{P}$-spaces are evaluations of the Tutte polynomial ([**3**] resp. Section I.4). In particular,

$$\operatorname{Hilb}(\mathcal{P}(X, 0, \{X\}), t) = t^{N-r} \sum_{\substack{A \subseteq X \\ \operatorname{rk}(A) = r}} \left(\frac{1}{t} - 1 \right)^{|A| - \operatorname{rk}(A)}, \text{ and} \tag{III.39}$$

$$\operatorname{Hilb}(\mathcal{P}(X, 1, \{X\}), t) = t^{N-r} \sum_{A \subseteq X} t^{r - \operatorname{rk}(A)} \left(\frac{1}{t} - 1 \right)^{|A| - \operatorname{rk}(A)}. \tag{III.40}$$

When looking at these two formulae, one might wonder if it is possible to find an "interpolating" formula for the semi-external case. Indeed, the natural guess works: if $\chi(A) = 1$, we take the corresponding summand from (III.40) and if $\chi(A) = 0$, we take the corresponding summand from (III.39).

Note that the latter term is always 0. In the semi-internal case however, the analogous statement is false.

PROPOSITION 5.4. *We are using the same terminology as in Definition* 3.1. *Then*

$$\operatorname{Hilb}(\mathcal{P}(X,0,J),t) = t^{N-r} \sum_{\substack{A\subseteq X \\ \chi(A)=1}} t^{r-\operatorname{rk}(A)} \left(\frac{1}{t}-1\right)^{|A|-\operatorname{rk}(A)}. \qquad \text{(III.41)}$$

PROOF. We prove this statement by deletion-contraction. In this proof we denote the polynomial on the right side of (III.41) by $T_{(X,J)}(t)$.

Let $x \in X$ be a loop and let $A \subseteq X \setminus x$. If $\chi_J(A) = 0$, A contributes neither to $T_{(X\setminus x, J\setminus x)}(t)$ nor to $T_{(X\setminus x, J\setminus x)}(t)$. If $\chi_J(A) = 1$, A contributes to $T_{(X\setminus x, J\setminus x)}(t)$ the term $t^{N-r-1}t^{r-\operatorname{rk}(A)}(1/t-1)^{|A|-\operatorname{rk}(A)} =: f_A$. To $T_{(X,J)}(t)$, A contributes the term tf_A and $A\cup x$ contributes $t(1/t-1)f_A$. This implies $T_{(X\setminus x, J\setminus x)}(t) = T_{(X,J)}(t)$.

From now on, we suppose that X does not contain any loops. Suppose that X contains only coloops. Then by Proposition 3.12,

$$\mathcal{P}(X,0,J) = \operatorname{span}\{p_Y : X\setminus Y \in J\} \qquad \text{(III.42)}$$

and it is easy to see that $T_{(X,J)}(t)$ is the Hilbert series of (III.42):

$$T_{(X,J)}(t) = t^{N-r} \sum_{\substack{A\subseteq X \\ A\in J}} t^{r-|A|} = \sum_{\substack{A\subseteq X \\ X\setminus A\in J}} t^{|A|}. \qquad \text{(III.43)}$$

Now suppose that $x \in X$ is neither a loop nor a coloop. By induction, we may suppose that (III.41) holds for X/x and $X \setminus x$. Using Corollary 5.1 and Lemma 3.15, we obtain

$$\begin{aligned}
\operatorname{Hilb}(\mathcal{P}(X,0,J),t) &= t\operatorname{Hilb}(\mathcal{P}(X\setminus x,k,J\setminus x),t) + \operatorname{Hilb}(\mathcal{P}(X/x,k,J/x),t) \\
&= t^{N-r} \sum_{\substack{A\subseteq X\setminus x \\ \chi_{J\setminus x}(A)=1}} t^{r-\operatorname{rk}(A)} \left(\frac{1}{t}-1\right)^{|A|-\operatorname{rk}(A)} \\
&\quad + t^{N-r} \sum_{\substack{\bar{A}\in X/x \\ \chi_{J/x}(\bar{A})=1}} t^{(r-1)-\operatorname{rk}(\bar{A})} \left(\frac{1}{t}-1\right)^{|\bar{A}|-\operatorname{rk}(\bar{A})} \\
&= t^{N-r} \sum_{\substack{x\notin A \\ \chi_J(A)=1}} t^{r-\operatorname{rk}(A)} \left(\frac{1}{t}-1\right)^{|A|-\operatorname{rk}(A)} \\
&\quad + t^{N-r} \sum_{\substack{x\in A \\ \chi_J(A)=1}} t^{r-\operatorname{rk}(A)} \left(\frac{1}{t}-1\right)^{|A|-\operatorname{rk}(A)} \\
&= t^{N-r} \sum_{\substack{A\subseteq X \\ \chi(A)=1}} t^{r-\operatorname{rk}(A)} \left(\frac{1}{t}-1\right)^{|A|-\operatorname{rk}(A)} = T_{(X,J)}(t) \qquad \square
\end{aligned}$$

If we set $t = 1$ in Proposition 5.4, we immediately obtain a result which relates the dimension of $\mathcal{P}(X, 0, J)$ and the number of independent sets satisfying a certain property. This has already been proven with a different method by Holtz, Ron, and Xu [**55**].

Corollary 5.5. *We are using the same terminology as in Definition* 3.1. *Then*

$$\dim \mathcal{P}(X, 0, J) = |\{Y \subseteq X : Y \textit{ independent}, \operatorname{cl}(Y) \in J\}|. \tag{III.44}$$

Remark 5.6. It is possible to deduce Proposition 5.4 directly from (III.39). This can be done using the decomposition (III.34) of the external space.

5.3. The case $k = -1$**.** For $k = -1$, we do not know if there is such a nice formula as in Corollary 5.3 or Proposition 5.4. The set $\mathbb{B}_-(X, J) := \{B \in \mathbb{B}(X) : \chi(B \setminus I(B)) = 1\}$ can in some cases be used to calculate the Hilbert series of $\mathcal{P}(X, -1, J)$, but in general the cardinality of $\mathbb{B}_-(X, J)$ depends on the order imposed on X (cf. Remark 4.3). Consider for example a sequence X of three vectors a, b, c in general position in a two-dimensional vector space and the ideal $J = \{X, \{a\}\}$. Depending on the order, $\mathbb{B}_-(X, J)$ may have cardinality 1 or 2.

Fix $C_0 \in \mathcal{L}(X)$ and set $J_{C_0} := \{C \in \mathcal{L}(X) : C \supseteq C_0\}$. All maximal missing flats in J_{C_0} are hyperplanes. They have unique defining normals (up to scaling). Then $\ker \mathcal{I}(X, -1, J_{C_0}) = \ker \mathcal{I}'(X, -1, J_{C_0}) = \bigcap_{x \in C_0} \mathcal{P}(X \setminus x, 0, \{X \setminus x\})$. This was shown by Holtz, Ron, and Xu [**55**]. They also show that for a specific order on X (see below), the dimension of $\ker \mathcal{I}(X, -1, J_{C_0})$ is equal to $|\mathbb{B}_-(X, J_{C_0})|$ and that $\mathbb{B}_-(X, J_{C_0})$ can be used to calculate the Hilbert series.

Theorem 5.7 ([**55**, p. 20]). *We are using the same terminology as in Definition* 3.1. *In addition, let* $C_0 \in \mathcal{L}(X)$. *Then*

$$\operatorname{Hilb}(\ker \mathcal{I}(X, -1, J_{C_0}), t) = \sum_{B \in \mathbb{B}_-(X, J_{C_0})} t^{N - r - |E(B)|}. \tag{III.45}$$

The proof in [**55**] relies on the following construction: an independent spanning subset $I \subseteq C_0$ is fixed and the order is chosen s. t. the elements of I are maximal. This makes it difficult or impossible to adjust this proof to a more general setting.

6. Zonotopal Cox rings

In this section we will briefly describe the zonotopal Cox rings defined by Sturmfels and Xu [**86**] and we show that our Main Theorem can be used to generalise a result on zonotopal Cox modules due to Ardila and Postnikov [**3**].

Fix m vectors $D_1, \ldots, D_m \in V$ and $\boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$. Sturmfels and Xu [**86**] introduced the Cox-Nagata ring $R^G \subseteq \mathbb{K}[s_1, \ldots, s_m, t_1, \ldots, t_m]$. This is the ring of polynomials that are invariant under the action of a certain group G which depends on the vectors $D_1, \ldots, D_m$. It is multigraded with a $\mathbb{Z}^{m+1}$-grading. A ring R is $\mathbb{Z}^{m+1}$-*multigraded* if it decomposes into a direct sum $R = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^{m+1}} R_{\boldsymbol{a}}$ and $R_{\boldsymbol{a}} R_{\boldsymbol{b}} \subseteq R_{\boldsymbol{a}+\boldsymbol{b}}$. For $r \geq 3$, R^G is equal to the Cox ring of the variety X_G which is obtained from $\mathbb{P}^{r-1}$ by blowing up the points $D_1, \ldots, D_m$. Cox rings have received a considerable amount of attention in the recent literature in algebraic geometry. See [**64**] for a survey.

Cox-Nagata rings are closely related to power ideals: we consider the ideal $\mathcal{I}_{\boldsymbol{u}} := \operatorname{ideal}\{D_1^{u_1+1}, \ldots, D_m^{u_m+1}\}$. Let $\mathcal{I}_{d,\boldsymbol{u}}^{-1}$ denote the homogeneous component of grade d of $\ker \mathcal{I}_{\boldsymbol{u}}$. Then, $R^G_{(d,\boldsymbol{u})}$, the homogeneous component of R^G of grade $(d, \boldsymbol{u})$, is naturally isomorphic to $\mathcal{I}_{d,\boldsymbol{u}}$ ([**86**, Proposition 2.1]).

Cox-Nagata rings are an object of great interest but in general, it is quite difficult to understand their structure. However, for some choices of the vectors $D_1, \ldots, D_m$, we understand a natural subring of the Cox-Nagata ring very well.

Let $\mathcal{H} = \{H_1, \ldots, H_m\}$ be the set of hyperplanes in $\mathcal{L}(X)$. $\mathfrak{H} \in \{0,1\}^{m \times N}$ denotes the non-containment hyperplane-vector matrix, *i. e.* the 0-1 matrix whose (i, j) entry is 1 if and only if H_i does not contain x_j.

Sturmfels and Xu defined the following structures: the *zonotopal Cox ring*

$$\mathcal{Z}(X) := \bigoplus_{(d,\boldsymbol{a}) \in \mathbb{N}^{N+1}} R^G_{(d,\mathfrak{H}\boldsymbol{a})} \tag{III.46}$$

and for $\boldsymbol{w} \in \mathbb{Z}^n$ the *zonotopal Cox module* of shift $\boldsymbol{w}$

$$\mathcal{Z}(X, \boldsymbol{w}) := \bigoplus_{(d,\boldsymbol{a}) \in \mathbb{N}^{N+1}} R^G_{(d,\mathfrak{H}\boldsymbol{a}+\boldsymbol{w})}. \tag{III.47}$$

Fix a vector $\boldsymbol{a} \in \mathbb{N}^N$. Let $X(\boldsymbol{a})$ denote the sequence of $\sum_i a_i$ vectors in U that is obtained from X by replacing each x_i by a_i copies of itself and let $\boldsymbol{e} := (1, \ldots, 1) \in \mathbb{N}^m$. Ardila and Postnikov show the following isomorphisms [**3**, Proposition 6.3]:

$$R^G_{(d,\mathfrak{H}\boldsymbol{a})} \cong \mathcal{P}(X(\boldsymbol{a}), 1, \{X\})_d, \tag{III.48}$$

$$R^G_{(d,\mathfrak{H}\boldsymbol{a}-\boldsymbol{e})} \cong \mathcal{P}(X(\boldsymbol{a}), 0, \{X\})_d, \tag{III.49}$$

$$\text{and } R^G_{(d,\mathfrak{H}\boldsymbol{a}-2\boldsymbol{e})} \cong \mathcal{P}(X(\boldsymbol{a}), -1, \{X\})_d. \tag{III.50}$$

They prove these isomorphisms by showing a statement similar to the following lemma.

LEMMA 6.1. *We are using the same terminology as in Definition* 3.1. *Let* $\boldsymbol{b} \in \{0,1\}^{\mathcal{H}}$ *and let* $J_{\boldsymbol{b}} := \{C \in \mathcal{L}(X) : b_H = 1$ *for all* $H \supseteq C\}$, *i. e. the maximal missing flats in* $J_{\boldsymbol{b}}$ *are exactly the hyperplanes that satisfy* $b_H = 0$.

Suppose that $\mathcal{I}(X(\boldsymbol{a}), k, J_{\boldsymbol{b}}) = \mathcal{I}'(X(\boldsymbol{a}), k, J_{\boldsymbol{b}})$ *for all* $\boldsymbol{a} \in \mathbb{N}^N$. *Then,*

$$R^G_{(d,\mathfrak{H}\boldsymbol{a}+(k-1)\boldsymbol{e}+\boldsymbol{b})} \cong (\ker \mathcal{I}(X, k, J_{\boldsymbol{b}}))_d \text{ for all } d. \tag{III.51}$$

Using the Main Theorem of this chapter, we can deduce the following results about *hierarchical zonotopal Cox modules.*

PROPOSITION 6.2. *We are using the same terminology as in Lemma* 6.1. *For the graded components of the* semi-external zonotopal Cox module $\mathcal{Z}(X, \mathfrak{H}\boldsymbol{a} - \boldsymbol{e} + \boldsymbol{b})$, *the following holds:*

$$R^G_{(d,\mathfrak{H}\boldsymbol{a}-\boldsymbol{e}+\boldsymbol{b})} \cong \mathcal{P}(X(\boldsymbol{a}), 0, J_{\boldsymbol{b}})_d \text{ for all } d. \tag{III.52}$$

PROPOSITION 6.3. *We use the same terminology as in Lemma* 6.1. *Let* $C_0 \in \mathcal{L}(X)$ *be a fixed flat and* $J_{C_0} := \{C \in \mathcal{L}(X) : C \supseteq C_0\}$ *(cf. Subsection* 5.3*). If* $\boldsymbol{b} \in \{0,1\}^{\mathcal{H}}$ *satisfies* $b_H = 1$ *if and only if* $H \supseteq C_0$, *then for the graded components of the* semi-internal zonotopal Cox module $\mathcal{Z}(X, \mathfrak{H}\boldsymbol{a} - 2\boldsymbol{e} + \boldsymbol{b})$, *the following holds:*

$$R^G_{(d,\mathfrak{H}\boldsymbol{a}-2\boldsymbol{e}+\boldsymbol{b})} \cong \ker \mathcal{I}(X(\boldsymbol{a}), -1, J_{C_0})_d \text{ for all } d. \qquad \text{(III.53)}$$

Using Theorems 5.4 and 5.7, we can calculate the multigraded Hilbert series of the semi-external and the semi-internal zonotopal Cox modules.

COROLLARY 6.4. *In the setting of Proposition* 6.2, *the dimension of* $R^G_{(d,\mathfrak{H}\boldsymbol{a}-\boldsymbol{e}+\boldsymbol{b})}$ *equals the coefficient of* t^d *in*

$$\operatorname{Hilb}(\mathcal{P}(X(\boldsymbol{a}),0,J_{\boldsymbol{b}}),t) = t^{|\boldsymbol{a}|-r} \sum_{\substack{A\subseteq X\\ \chi(A)=1}} t^{r-\operatorname{rk}(A)} \sum_{\substack{1\le s_i\le a_i\\ \boldsymbol{s}\in\mathbb{N}^A,\, x_i\in A}} \left(\prod_i \binom{a_i}{s_i}\right) \left(\frac{1}{t}-1\right)^{|\boldsymbol{s}|-\operatorname{rk}(A)},$$

where $|\boldsymbol{a}| := \sum_i a_i$.

PROOF. Apply Proposition 5.4 to $X(\boldsymbol{a})$. Take into account that for every $S \subseteq X(\boldsymbol{a})$, there is a unique pair $(A, \boldsymbol{s})$ with $A \subseteq X$ and $\boldsymbol{s} \in \mathbb{N}^A$ s. t. S is obtained from A by replacing each $x_i \in A$ by s_i copies of itself. Furthermore, $\operatorname{rk}(S) = \operatorname{rk}(A)$. For a fixed pair $(A, \boldsymbol{s})$, there are $\binom{a_i}{s_i}$ options to choose the corresponding vectors in $X(\boldsymbol{a})$ for every i. □

COROLLARY 6.5. *In the setting of Proposition* 6.3, *the dimension of* $R^G_{(d,\mathfrak{H}\boldsymbol{a}-2\boldsymbol{e}+\boldsymbol{b})}$ *equals the coefficient of* t^d *in*

$$\operatorname{Hilb}(\ker \mathcal{I}(X(\boldsymbol{a}),-1,J_{C_0}),t) = \sum_{B\in\mathbb{B}_-(X,J_{C_0})} \sum_{\substack{0\le s_i\le a_i-1\\ \boldsymbol{s}\in\mathbb{N}^B}} t^{e(B,\boldsymbol{s})} \qquad \text{(III.54)}$$

where $e(B, \boldsymbol{s}) := \sum_{i : x_i \notin E(B)} a_i - r - \sum_{x_i \in B} s_i$.

PROOF. Apply Theorem 5.7. Choose an order on $X(\boldsymbol{a})$ that is compatible with the order on X, *i. e.* if $x', y' \in X(\boldsymbol{a})$ are copies of $x, y \in X$ $(x \neq y)$ then $x' < y'$ if and only if $x < y$. Fix a basis $B \subseteq X$. Now we examine the copies of B that are contained in $X(\boldsymbol{a})$. All copies of the elements that are externally active with respect to B in X are externally active with respect to every copy of B in $X(\boldsymbol{a})$. Let $x \in B \subseteq X$. If the ith copy (the maximal one being the first) of x in $X(\boldsymbol{a})$ is chosen, $i-1$ copies of x are externally active in $X(\boldsymbol{a})$. Hence, for the copy of B in $X(\boldsymbol{a})$ that corresponds to $\boldsymbol{s}$, the exponent of t equals $e(B, \boldsymbol{s})$. □

7. Examples

This section contains a large number of examples. In the first subsection we will give explicit examples for the various structures appearing in this chapter (X, J, S, $\mathcal{P}$, $\mathcal{I}$, Γ, $\mathcal{B}$, $\mathbb{B}$, $\mathbb{B}_-$). In the second subsection we will give an example for deletion and contraction as defined in Section 3.3. In the third subsection we will exemplify the decomposition of $\mathcal{P}$-spaces that appears in the proof of Proposition 5.2. In the last subsection we will explain several problems that occur in the semi-internal case.

In this section we work over the polynomial rings $\mathbb{K}[x,y]$ and $\mathbb{K}[x,y,z]$ instead of the symmetric algebras $\mathrm{Sym}(U)$ and $\mathrm{Sym}(V)$.

7.1. Structures.

$$\text{Let } X_1 := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = (x_1, x_2, x_3). \tag{III.55}$$

Define two ideals $J_1 := \{X_1\}$ and $J_2 := \{X_1, (x_1), (x_3)\}$. The set of bases is $\mathbb{B}(X_1) = \{(x_1x_2), (x_1x_3), (x_2x_3)\}$. The sets of semi-internal bases are $\mathbb{B}_-(X_1, J_1) = \{(x_1x_2)\}$ and $\mathbb{B}_-(X_1, J_2) = \{(x_1x_2), (x_1x_3)\}$.

$$\begin{aligned}
&S(X_1,-1,J_1) = \{1\} && S(X_1,0,J_1) = \{1,\, p_{x_1},\, p_{x_2},\, p_{x_3}\} \\
&\mathcal{P}(X_1,-1,J_1) = \mathrm{span}\{1\} && \mathcal{P}(X_1,0,J_1) = \mathrm{span}\{1,x,y\} \\
&\mathcal{I}(X_1,-1,J_1) = \mathrm{ideal}\{x,y\} && \mathcal{I}(X_1,0,J_1) = \mathrm{ideal}\{x^2,xy,y^2\} \\
& && \Gamma(X_1,0,J_1) = \{((x_1x_2),\emptyset,0),((x_1x_3),\emptyset,0), \\
& && \qquad ((x_2x_3),\emptyset,0)\} \\
& && \mathcal{B}(X_1,0,J_1) = \{p_\emptyset, p_{x_2}, p_{x_1}\} \\
&S(X_1,-1,J_2) = \{1,p_{x_2}\} && S(X_1,0,J_2) = \{1,\, p_{x_1},\, p_{x_2},\, p_{x_3},\, p_{x_1x_2},\, p_{x_2x_3}\} \\
&\mathcal{P}(X_1,-1,J_2) = \mathrm{span}\{1,y\} && \mathcal{P}(X,0,J_2) = \mathrm{span}\{1,x,y,xy,y^2\} \\
&\mathcal{I}(X_1,-1,J_2) = \mathrm{ideal}\{x,y^2\} && \mathcal{I}(X_1,0,J_2) = \mathrm{ideal}\{x^2,xy^2,y^3\} \\
& && \Gamma(X_1,0,J_2) = \{((x_1x_2),\emptyset,0),((x_1x_3),\emptyset,0), \\
& && \qquad ((x_1x_3),(x_3),0),((x_2x_3),\emptyset,0), \\
& && \qquad ((x_2x_3),(x_2),\emptyset,0)\} \\
& && \mathcal{B}(X_1,0,J_2) = \{p_\emptyset,\, p_{x_2},\, p_{x_2x_3},\, p_{x_1},\, p_{x_1x_2}\}
\end{aligned}$$

7.2. Deletion and contraction. In this subsection we will give examples explaining deletion and contraction for pairs (X, J).

$$X_1 \setminus x_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (x_2, x_3) \quad X_1/x_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} = (\bar{x}_2, \bar{x}_3) \tag{III.56}$$

$$J_1 \setminus x_1 = \{(x_2,x_3)\} \qquad J_1/x_1 = \{(\bar{x}_2,\bar{x}_3)\} \tag{III.57}$$

$$J_2 \setminus x_1 = \{(x_2,x_3),(x_3)\} \qquad J_2/x_1 = \{(\bar{x}_2,\bar{x}_3),\bar{\emptyset}\} = \mathcal{L}(X_1/x_1) \tag{III.58}$$

Recall that we identify $\mathbb{K}[x,y]/x$ and $\mathbb{K}[y]$. Then

$$\begin{aligned}
\mathcal{I}(X_1 \setminus x_1, 0, J_1 \setminus x_1) &= \mathrm{ideal}\{x,y\}, & \mathcal{I}(X_1 \setminus x_1, 0, J_2 \setminus x_1) &= \mathrm{ideal}\{x,y^2\}, \\
\mathcal{P}(X_1 \setminus x_1, 0, J_1 \setminus x_1) &= \mathrm{span}\{1\}, & \mathcal{P}(X_1 \setminus x_1, 0, J_2 \setminus x_1) &= \mathrm{span}\{1,y\}, \\
\mathcal{I}(X_1/x_1, 0, J_1/x_1) &= \mathrm{ideal}\{y^2\}, & \mathcal{I}(X_1/x_1, 0, J_2/x_1) &= \mathrm{ideal}\{y^3\}, \\
\mathcal{P}(X_1/x_1, 0, J_1/x_1) &= \mathrm{span}\{1,y\}, & \mathcal{P}(X_1/x_1, 0, J_2/x_1) &= \mathrm{span}\{1,y,y^2\}.
\end{aligned}$$

The reader should check that $\mathcal{P}(X_1,0,J_i) = p_x\mathcal{P}(X_1 \setminus x_1, 0, J_i \setminus x_1) \oplus \mathcal{P}(X_1/x_1, 0, J_i/x_1)$ holds for $i = 1$ and $i = 2$.

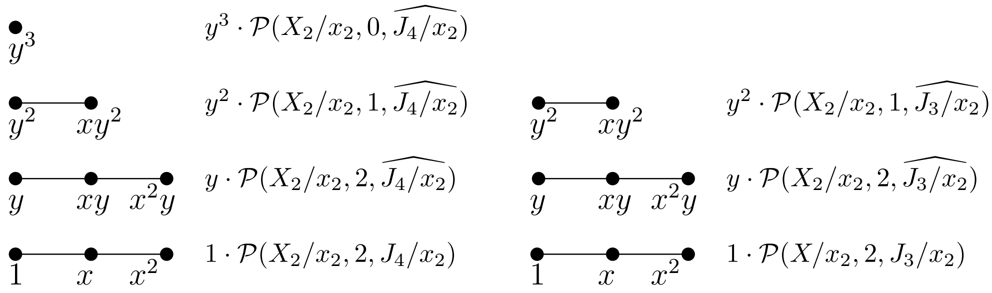

FIGURE 4. On the left, $\mathcal{P}(X_2, 2, J_4)$ and on the right $\mathcal{P}(X_2, 2, J_3)$. For both spaces, the decompositions corresponding to Proposition 5.2 are shown.

7.3. Recursion for the Hilbert series.

EXAMPLE 7.1. This is an example for the description of $\mathcal{P}$-spaces in Proposition 3.12 and for the decomposition of $\mathcal{P}$-spaces that appears in the proof of Proposition 5.2:

$$X_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (x_1, x_2) \qquad J_3 := \{X_2\} \qquad J_4 := \{X_2, (x_1)\}$$

$$\mathcal{I}(X_2, 2, J_3) = \text{ideal}\{x^3, y^3, x^2y^2\} \qquad \widehat{J_3/x_2} = \widehat{J_4/x_2} = \{(x_1)\} \tag{III.59}$$

$$\mathcal{P}(X_2, 2, J_3) = \text{span}\{1, x, y,\ x^2, xy, y^2,\ x^2y, xy^2\} \tag{III.60}$$

$$= \text{span}\{1, x, x^2\} \oplus y\,\text{span}\{1, x, x^2\} \oplus y^2\,\text{span}\{1, x\} \tag{III.61}$$

$$\mathcal{I}(X_2, 2, J_4) = \text{ideal}\{x^3, y^4, x^2y^2, xy^3\} \tag{III.62}$$

$$\mathcal{P}(X_2, 2, J_4) = \text{span}\{1, x, y,\ x^2, xy, y^2,\ x^2y, xy^2, y^3\} \tag{III.63}$$

$$= \text{span}\{1, x, x^2\} \oplus y\,\text{span}\{1, x, x^2\} \oplus y^2\,\text{span}\{1, x\} \oplus y^3\,\text{span}\{1\}$$

For a graphical description of the decomposition, see Figure 4.

7.4. Problems in the semi-internal case.

EXAMPLE 7.2 (No canonical basis for internal spaces). In Section 5.3, we defined the set of semi-internal bases $\mathbb{B}_-(X, J)$. Now consider $\tilde{\mathcal{B}}_-(X, J) := \{p_{X \setminus (B \cup E(B))} : B \in \mathbb{B}_-(X, J)\}$. This example shows that even in the internal case, where $\tilde{\mathcal{B}}_-(X, J)$ has the right cardinality, this set is in general not contained in $\ker \mathcal{I}(X, -1, J)$.

$$X_3 := \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} = (x_1, x_2, x_3, x_4, x_5) \tag{III.64}$$

$$\mathbb{B}_-(X_3, \{X_3\}) = \{(x_1, x_2, x_3), (x_1, x_2, x_4)\}$$

$$\mathcal{I}(X_3, -1, \{X_3\}) = \text{ideal}\{x^2, y, z\} \tag{III.65}$$

$$\mathcal{P}(X_3, -1, \{X_3\}) = \ker \mathcal{I}(X_3, -1, \{X_3\}) = \text{span}\{1, x\} \tag{III.66}$$

$$E((x_1, x_2, x_3)) = (x_4, x_5) \qquad E((x_1, x_2, x_4)) = (x_5) \tag{III.67}$$

$$\tilde{\mathcal{B}}_-(X_3, \{X_3\}) = \{1, x + z\} \not\subseteq \text{span}\{1, x\}$$

Example 7.3. This example shows why there is an additional condition on the ideal J in the Main Theorem for $k = -1$. We use the matrix X_1 defined at the beginning of this section and the ideal $J_5 := \{X_1, \{x_1\}\}$. Note that J does not contain all hyperplanes in X. Then $J_5 \setminus x_1 = \{X_1 \setminus x_1\}$ and $J_5/x_1 = \{X_1/x_1, \bar{\emptyset}\}$. This implies $S(X_1 \setminus x_1, -1, J_5 \setminus x_1) = \emptyset$, $S(X_1, -1, J_5) = \{1\}$ and $S(X_1/x_1, -1, J_5/x_1) = \{1, y\}$. Hence, the map $\mathrm{Sym}(\pi_{x_1}) : \mathcal{P}(X_5, -1, J) \to \mathcal{P}(X_5/x_1, -1, J_5/x_1)$ is not surjective.

The three S-sets appearing in this example span the corresponding kernels. However, our proof of the Main Theorem fails here, since

$$\ker \mathcal{I}(X_1, -1, J_5) \neq p_{x_1} \ker \mathcal{I}(X_1 \setminus x_1, -1, J_5 \setminus x_1) \oplus \ker \mathcal{I}(X_1/x_1, -1, J_5/x_1),$$

i. e. Proposition 3.18 does not hold.

Example 7.4. This example shows why our proof of the Main Theorem in general does not work in the case $J = \{X\}$ (cf. Remark 3.21). It demonstrates that $\mathrm{Sym}(\pi_x) : S(X, -1, \{X\}) \to S(X/x, -1, \{X/x\})$ is in general not surjective.

Consider the following matrix:

$$X_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7).$$

The corresponding internal $\mathcal{P}$-space and ideal are:

$$\mathcal{I}(X_4, -1, \{X_4\}) = \mathrm{ideal}\{x^3,\, y^3,\, z^2,\, (x-y)^3,\, (x-z)^2, (x-y-z)^2\} \text{ and}$$
$$\mathcal{P}(X_4, -1, \{X_4\}) = \mathrm{span}\{1,\, x,\, y,\, z,\, xy+yz,\, xy+y^2,\, x^2+xz,\, x^2y+xy^2+xyz+y^2z\}.$$

By deletion and contraction of x_7 we obtain

$$\mathcal{I}(X_4 \setminus x_7, -1, X_4 \setminus x_7) = \mathrm{ideal}\{x,\, y,\, z\}, \tag{III.68}$$
$$\mathcal{P}(X_4 \setminus x_7, -1, X_4 \setminus x_7) = \mathrm{span}\{1\}, \tag{III.69}$$
$$\mathcal{I}(X_4/x_7, -1, \{X_4/x_7\}) = \mathrm{ideal}\{x^3, y^3, (x-y)^3\}, \tag{III.70}$$
$$\text{and } \mathcal{P}(X_4/x_7, -1, X_4/x_7) = \mathrm{span}\{1,\, x,\, y,\, x^2,\, xy,\, y^2,\, x^2y+xy^2\}. \tag{III.71}$$

The Main Theorem and Proposition 3.18 both hold in this example.

$p_{\bar{x}_5}p_{\bar{x}_6} \in S(X_4/x_7, -1, \{X_4/x_7\})$, but $p_{x_5}p_{x_6} \notin S(X_4, -1, \{X_4\})$. No element of $S(X_4, -1, \{X_4\})$ is projected to $p_{\bar{x}_5}p_{\bar{x}_6}$. Hence, our proof of the Main Theorem does not work in this case.

Acknowledgements for this chapter. The author would like to thank Olga Holtz for fruitful discussions of this work, Federico Ardila who suggested the possibility of the results in Section 6, and Andrew Berget who made some valuable remarks.

CHAPTER IV

Matroid Polynomials and Mason's Conjecture

We will show that f-vectors of matroid complexes of realisable matroids are log-concave. This was conjectured by Mason in 1972. Our proof uses the recent result by Huh and Katz who proved that the coefficients of the characteristic polynomial of a realisable matroid form a log-concave sequence. In addition, we will give an example which shows that the analogous statement for arithmetic matroids does not hold.

We will discuss the relationship between log-concavity of f-vectors and log-concavity of h-vectors and show that various graph and matroid polynomials can be obtained from the Hilbert series of the internal and central zonotopal spaces.

1. Introduction

Let $M = (E, \Delta)$ be a matroid of rank r. E denotes the ground set and $\Delta \subseteq 2^E$ denotes the matroid complex, *i. e.* the abstract simplicial complex of independent sets. Let $f = (f_0, \ldots, f_r)$ be the f-vector of Δ, *i. e.* f_i is the number of sets of cardinality i in Δ. Dominic Welsh conjectured in 1969 [**90**] that the f-vector of a matroid complex is *unimodal*, *i. e.* there exists $j \in \{0, 1, \ldots, r\}$ s. t. $f_0 \le f_1 \le \ldots \le f_j \ge \ldots \ge f_r$. Three successive strengthenings of this conjecture were proposed by John Mason in 1972 [**72**]. The weakest of them is *log-concavity* of the f-vector, *i. e.*

$$f_i^2 \ge f_{i-1} f_{i+1} \text{ for } i = 1, \ldots, r-1. \tag{IV.1}$$

Since then, these conjectures have received considerable attention. See for example [**15, 27, 44, 52, 60, 61, 71, 81, 82, 89, 92**]. Carolyn Mahoney proved log-concavity for cycle matroids of outerplanar graphs in 1985 [**71**]. David Wagner [**89**] describes further partial results, several stronger variants of Mason's conjecture, and other sequences of integers that are associated with a matroid and that are conjectured to be log-concave. Log-concave sequences arising in combinatorics have been studied by many authors. For an overview, see the surveys by Francesco Brenti and Richard Stanley [**11, 83**].

Recall that a matroid is *realisable* if it is equivalent to a matroid whose ground set is a multiset of vectors in a vector space over some field $\mathbb{K}$ and whose independent sets are the linearly independent subsets of this multiset. The main result in this chapter is the following theorem.

Theorem 1.1. *The f-vector of the matroid complex of a realisable matroid is log-concave.*

The strongest of Mason's three conjectures [**72**] is ultra-log-concavity, *i. e.* the conjecture that the following inequalities hold:

$$\frac{f_i^2}{\binom{f_1}{i}^2} \geq \frac{f_{i-1}}{\binom{f_1}{i-1}} \frac{f_{i+1}}{\binom{f_1}{i+1}} \text{ for } i = 1, \ldots, r-1. \tag{IV.2}$$

This conjecture was one of the main topics of a workshop at AIM in 2011[3].

Finding inequalities satisfied by f-vectors of matroid complexes is interesting because it is a step towards the classification of f-vectors and h-vectors of matroid complexes. In this context, it is also interesting to know that the convex hull of the set of f-vectors of matroid complexes on N elements is a simplex whose vertices are f-vectors of uniform matroids [**63**].

Johnson, Kontoyiannis, and Madiman [**60**] have shown that a stronger version of Theorem 1.1 would imply a bound on the entropy of the cardinality of a random independent set in a matroid. Our log-concavity results might also help to prove statements about coefficients and zeroes of various graph polynomials in the future. A possible application to the theory of network reliability is explained in Section 6.

This chapter is based on [**66**].

1.1. Outline of this chapter. We will introduce the f-polynomial and the characteristic polynomial of a matroid in Section 2. Recently, June Huh and Eric Katz [**58**] proved that the characteristic polynomial of a realisable matroid is log-concave (a univariate polynomial is log-concave if its coefficients form a log-concave sequence). In Section 3 we will establish a connection between the characteristic polynomial and the f-polynomial. In conjunction with the result by Katz and Huh, this implies log-concavity of the f-polynomial of realisable matroids. In Section 4 we will discuss connections between (strict) log-concavity of h-vectors and f-vectors and the matroid operation thickening.

In Section 5 we will show how the f-polynomial and the characteristic polynomial can be obtained from the Hilbert series of the internal and central zonotopal spaces. In Section 6 we will explain the relationship between various other graph and matroid polynomials and zonotopal algebra. In Section 7 we will give an example which shows that the f-polynomial and the characteristic polynomial of an arithmetic matroid are in general not log-concave.

2. Matroid polynomials

In this section we will review the definitions of some matroid polynomials. For more information on matroids, see Subsection I.3.1.

Recall that we denote by $M = (E, \Delta)$ a matroid of rank r. Let rk denote the rank function of M. The *Tutte polynomial* [**18**] of M is defined as

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{r - \mathrm{rk}(A)} (y-1)^{|A| - \mathrm{rk}(A)}. \tag{IV.3}$$

[3]Workshop on *Stability, hyperbolicity, and zero localization of functions*, December 5 to December 9, 2011 at the American Institute of Mathematics, Palo Alto, California. Organised by Petter Brändén, George Csordas, Olga Holtz, and Mikhail Tyaglov.
`http://www.aimath.org/ARCC/workshops/hyperbolicpoly.html`

An important specialisation of the Tutte polynomial is the *characteristic polynomial*

$$\chi_M(q) = (-1)^r T_M(1-q,0) = \sum_{A\subseteq E}(-1)^{|A|}q^{r-\mathrm{rk}(A)}. \tag{IV.4}$$

The *reduced characteristic polynomial* is defined as

$$\bar{\chi}_M(q) = \frac{1}{q-1}\chi_M(q). \tag{IV.5}$$

Note that since $E \neq \emptyset$, $\chi_M(q)$ vanishes for $q = 1$, so $\bar{\chi}_M(q)$ is indeed a polynomial. Huh and Katz proved the following theorem, generalising an earlier result by Huh [**57**].

THEOREM 2.1 ([**58**]). *If M is a realisable matroid, then the coefficients of its reduced characteristic polynomial $\bar{\chi}_M(q)$ form a log-concave sequence.*

It is easy to see that log-concavity of $\bar{\chi}_M(q)$ implies log-concavity of $\chi_M(q)$. We are interested in the *f-polynomial* of the matroid given by

$$f_M(q) = T_M(1+q,1) = \sum_{A\in\Delta} q^{r-\mathrm{rk}(A)} = \sum_{i=0}^{r} f_i q^{r-i}. \tag{IV.6}$$

3. Free (co-)extensions

In this section we will introduce free (co-)extensions of matroids. This will help us to establish a connection between the characteristic polynomial and the f-polynomial. In conjunction with Theorem 2.1, this connection implies log-concavity of the f-polynomial of realisable matroids.

DEFINITION 3.1. Let $M = (E,\Delta)$ be a matroid of rank r and let $e \notin E$. The *free extension* of M (by e) is the matroid $M + e = (E \cup \{e\}, \Delta + e)$, where

$$\Delta + e := \Delta \cup \{(I \cup \{e\}) : I \in \Delta \text{ and } |I| \leq r-1\}. \tag{IV.7}$$

Several properties of the free extension are described in [**16**, 7.3.3. Proposition].

REMARK 3.2. If M is realised over the field $\mathbb{K}$ by the list of vectors $X \subseteq \mathbb{K}^r$, then $M + e$ is realised by the list (X, x), where $x \in \mathbb{K}^r$ is a vector that is not contained in any (linear) hyperplane spanned by the vectors in X. If $\mathbb{K}$ is a finite field, such a vector might not exist. However, if M is realisable over the field $\mathbb{K}$, it is also realisable over the infinite field $\mathbb{K}(t)$ of rational functions in t with coefficients in $\mathbb{K}$.

Recall that the *dual matroid* $M^* = (E, \Delta^*)$ is given by

$$\Delta^* = \{A : \mathrm{rk}(E \setminus A) = r\}. \tag{IV.8}$$

The dual matroid has rank $r^* = |E| - r$ and its rank function is given by $\mathrm{rk}^*(A) = |A| + \mathrm{rk}(E \setminus A) - r$. The Tutte polynomial satisfies $T_M(x,y) = T_{M^*}(y,x)$. We will use the *free coextension* $M \times e$ of a matroid M which is defined as

$$M \times e := (M^* + e)^*. \tag{IV.9}$$

Equivalently, the free coextension of M is the extension by a non-loop e which is contained in every dependent flat [**77**, Section 7.3].

PROPOSITION 3.3. *Let M be a matroid of rank r and let $M \times e$ denote its free coextension. Then,*

$$(-1)^{r+1}\chi_{M\times e}(-q) = (1+q)f_M(q). \tag{IV.10}$$

PROOF. For the proof of this statement, we use the fact that both the characteristic polynomial and the f-polynomial are evaluations of the Tutte polynomial. Note that the matroid $M \times e$ has rank $r+1$. To simplify notation, the rank functions of M^* and $M^* + e$ are both denoted by rk^*.

$$(-1)^{r+1}\chi_{M\times e}(-q) = T_{M\times e}(1+q,0) = T_{M^*+e}(0,1+q) \tag{IV.11}$$

$$= \sum_{A\subseteq E\cup\{e\}} (-1)^{r^*-\mathrm{rk}^*(A)} q^{|A|-\mathrm{rk}^*(A)} \tag{IV.12}$$

$$= \sum_{A\subseteq E} \Big((-1)^{r^*-\mathrm{rk}^*(A)} q^{|A|-\mathrm{rk}^*(A)} + (-1)^{r^*-\mathrm{rk}^*(A\cup e)} q^{|A|+1-\mathrm{rk}^*(A\cup e)} \Big) \tag{IV.13}$$

$$= (1+q) \sum_{\substack{A\subseteq E\\ \mathrm{rk}^*(A)=r^*}} q^{|A|-r^*} = (1+q)T_{M^*}(1,1+q) \tag{IV.14}$$

$$= (1+q)T_M(1+q,1) = (1+q)f_M(q) \tag{IV.15}$$

(IV.14) is equal to (IV.13) because $\mathrm{rk}^*(A) < r^*$ implies that $\mathrm{rk}^*(A\cup e) = \mathrm{rk}^*(A) + 1$. For those A, the summands vanish. □

REMARK 3.4. Proposition 3.3 appeared implicitly in an article by Tom Brylawski on (reduced) broken-circuit complexes [**19**].

In Section 5 we will give another proof of Proposition 3.3 for matroids that are realisable over a field of characteristic zero. This proof uses zonotopal algebra.

PROOF OF THEOREM 1.1. Combine Proposition 3.3 and Theorem 2.1. Bear in mind that free coextensions of realisable matroids are realisable (cf. Remark 3.2). □

EXAMPLE 3.5. We consider the uniform matroid $U_{2,6}$, *i. e.* the matroid on six elements where every set of cardinality at most two is independent. Note that $U_{2,6} \times e = (U_{4,6} + e)^* = U_{4,7}^* = U_{3,7}$.

$$f_{U_{2,6}}(q) = q^2 + 6q + 15$$

$$(-1)^3\chi_{U_{3,7}}(-q) = q^3 + 7q^2 + 21q + 15 = (q+1)f_{U_{2,6}}(q)$$

4. h-vectors, f-vectors, and strict log-concavity

This section contains some results on connections between (strict) log-concavity of h-vectors and f-vectors and the matroid operation thickening. In Subsection 4.1 we will show that log-concavity of h-vectors implies strict log-concavity of f-vectors. In Subsection 4.2 we will show that strict log-concavity of f-vectors implies strict log-concavity of h-vectors of certain thickenings of a matroid. In Subsection 4.3, we will discuss possible locations of the modes of f-vectors.

As one might expect, a sequence of real numbers is called *strictly log-concave* if it is log-concave and all inequalities are strict.

4.1. h-vectors and strict log-concavity.

In this subsection we will show that log-concavity of h-vectors implies strict log-concavity of f-vectors. The former was shown very recently by June Huh in the case of matroids that are realisable over a field of characteristic zero [**56**].

The fact that f-vectors of a large class of matroid complexes are strictly log-concave indicates that they might satisfy even stronger inequalities as Mason conjectured.

DEFINITION 4.1. Let M be a matroid of rank r. Its *h-vector* $(h_0, \ldots, h_r)$ consists of the coefficients of the *h-polynomial* defined by the equation $h_M(q) = \sum_{i=0}^{r} h_i q^{r-i} = f_M(q-1)$, *i. e.*

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{r-i}{j-i} f_i \quad \text{for } i = 0, \ldots, r. \tag{IV.16}$$

It is well-known that log-concavity of h-vectors implies log-concavity of f-vectors (see [**11**, Corollary 8.4], [**17**, Proposition 6.13], [**27**]). In fact, it implies even strict log-concavity of f-vectors. This is a consequence of the following lemma.

LEMMA 4.2. *Let $a_0, \ldots, a_r$ be non-negative integers and $a_0 \neq 0$. Suppose that the polynomial $a(q) = \sum_{i=0}^{r} a_i q^{r-i}$ is log-concave. Then, the polynomial $b(q) = \sum_{i=0}^{r} b_i q^{r-i} = a(q+1)$ is* strictly *log-concave.*

PROOF. Our proof is inspired by Dawson's proof in [**27**]. For $0 \leq k \leq r$, we define $a^k(q) = \sum_{i=0}^{k} a_i q^{k-i}$ and $b^k(q) = \sum_{i=0}^{k} b_{i,k} q^{k-i} = a^k(q+1)$.

The polynomials $a^k(q)$ are by construction log-concave. We show by induction over k that this implies log-concavity of the polynomials $b^k(q)$. This is sufficient since $b(q) = b^r(q)$.

For $k \leq 1$, nothing needs to be shown. For $k = 2$, we need to check one inequality:

$$\begin{aligned} b_1^2 &= (a_1 + 2a_0)^2 = a_1^2 + 4a_0a_1 + 4a_0^2 && \text{(IV.17)} \\ &\geq a_0a_2 + 4a_0a_1 + 4a_0^2 > a_0(a_2 + a_1 + a_0) = b_0b_2. && \text{(IV.18)} \end{aligned}$$

Now let $k \geq 3$. Note that

$$b^{k+1}(q) = a^{k+1}(q+1) = (q+1)a^k(q+1) + a_{k+1} = (q+1)b^k(q) + a_{k+1}.$$

This polynomial is strictly log-concave if $(q+1)(qb^k(q) + a_{k+1}) = q((q+1)b^k(q) + a_{k+1}) + a_{k+1}$ is, since setting the q^0 coefficient to zero followed by a division by q preserves strict log-concavity.

It is an easy exercise to show that multiplication by $(q+1)$ preserves strict log-concavity of a polynomial in q. Hence, it is sufficient to prove that $(qb^k(q) + a_{k+1})$ is strictly log-concave. By induction, we only need to check the inequality involving the term a_{k+1}, *i. e.* $b_{k,k}^2 > b_{k-1,k}a_{k+1}$:

$$b_{k,k}^2 - b_{k-1,k}a_{k+1} = (a_0 + \ldots + a_k)^2 - \sum_{j=0}^{k-1}(k-j)a_j a_{k+1} \tag{IV.19}$$

$$\geq (a_0 + \ldots + a_k)^2 - \sum_{j=0}^{k-1}\sum_{i=1}^{k-j} a_{j+i}a_{k+1-i} \tag{IV.20}$$

$$= \sum_{i+j\leq k} a_i a_j \geq a_0^2 \geq 1. \tag{IV.21}$$

To see that (IV.19) is greater than (IV.20), note that log-concavity of the a_j implies $a_j a_{k+1} \leq a_{j+i}a_{k+1-i}$ for $1 \leq i \leq k-j$. □

In a very recent preprint, June Huh proved the following result about h-vectors of matroids that was conjectured by Jeremy Dawson in [**27**].

THEOREM 4.3 ([**56**]). *The h-vector of a matroid complex of a matroid that is realisable over a field of characteristic zero is log-concave.*

Combining this theorem with Lemma 4.2, we obtain the following Corollary that slightly strengthens Theorem 1.1 in the case of matroids that are realisable over a field of characteristic zero.

COROLLARY 4.4. *The f-vector of a matroid complex of a matroid that is realisable over a field of characteristic zero is strictly log-concave.*

4.2. Thickenings. In this section we will introduce the matroid operation k-fold thickening and we show that the f-vector of a "sufficiently thick" matroid is strictly log-concave if and only its h-vector is.

DEFINITION 4.5. Let $M = (E, \Delta)$ be a matroid and let k be a positive integer. We define the *k-fold thickening* M^k of M to be the matroid on the ground set $E \times \{1, \ldots, k\}$ whose matroid complex is given by

$$\Delta^k = \{I \subseteq E \times \{1, \ldots, k\} : \pi_E(I) \in \Delta \text{ and } |\pi_E(I)| = |I|\}. \tag{IV.22}$$

In this definition, $\pi_E : E \times \{1, \ldots, k\} \to E$ denotes the projection to E.

REMARK 4.6. If M is realised by a list of vectors X, M^k is realised by the list X^k that contains k copies of every element of X.

PROPOSITION 4.7. *Let $M = (E, \Delta)$ be a matroid of rank r and let f_1 denote the number of elements in E that are not loops. Suppose that the f-vector of M is strictly log-concave. Then there exists an integer $k_0 \leq (f_1 r)^{3r}$ s. t. for all $k \geq k_0$, the h-vector of M^k, the k-fold thickening of M, is strictly log-concave.*

Put differently, for "sufficiently thick" matroids, the f-vector is strictly log-concave if and only if the h-vector is strictly log-concave.

REMARK 4.8. We expect that a careful analysis will yield an upper bound on k_0 that is a lot stronger.

REMARK 4.9. One should note that Proposition 4.7 holds for arbitrary matroids and even for other classes of simplicial complexes that have positive h-vectors and that are closed under k-fold thickening.

PROOF OF PROPOSITION 4.7. First, we observe the following connection between the f-polynomials of M and M^k:

$$f_{M^k}(q) = \sum_{i=0}^{r} k^i f_i q^{r-i} = k^r f_M\left(\frac{q}{k}\right). \tag{IV.23}$$

Let $(f_0, \ldots, f_r)$ denote the f-vector of M and let $(h'_0, \ldots, h'_r)$ denote the h-vector of M^k. By (IV.16), $h'_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{r-i}{j-i} k^i f_i$. Hence,

$$(h'_j)^2 = \left(\sum_{i=0}^{j} (-1)^{j-i} \binom{r-i}{j-i} k^i f_i\right)^2 = k^{2j} f_j^2 + o(k^{2j}) \tag{IV.24}$$

$$\begin{aligned} h'_{j-1}h'_{j+1} &= \left(\sum_{i=0}^{j-1} (-1)^{j-i} \binom{r-i}{j-i-1} k^i f_i\right)\left(\sum_{i=0}^{j+1} (-1)^{j-i} \binom{r-i}{j-i+1} k^i f_i\right) \\ &= k^{2j} f_{j-1} f_{j+1} + o(k^{2j}). \end{aligned} \tag{IV.25}$$

Thus, for large k, $(h'_j)^2 > h'_{j-1}h'_{j+1}$ is equivalent to $f_j^2 > f_{j-1}f_{j+1}$. The latter inequality holds by assumption.

For the upper bound on k_0, note that Ed Swartz proved in [**87**] that

$$f_i \leq \sum_{j=0}^{i} \binom{r-j}{r-i} \left(\binom{r-1}{j} h_r + \binom{r-1}{j-1}\right). \tag{IV.26}$$

h_r can be bounded above by the following argument: the h-vector of a matroid complex is the h-vector of a multicomplex [**84**, Theorem II.3.3]. It follows directly from (IV.16) that $h_1 = f_1 - r$. Hence, $h_r \leq \binom{f_1-1}{r-1}$. Thus, we can deduce from (IV.26) that $f_i \leq r^{2i} f_1^r$. Comparing this with (IV.24) and (IV.25) implies the upper bound. □

REMARK 4.10. Jason Brown and Charles Colbourn showed that every matroid has a thickening s. t. its h-polynomial has only real zeroes [**15**]. This implies that it is log-concave. Here, thickening denotes an operation where additional copies of some elements of the ground set are added. In contrast to the k-fold thickening, the number of additional copies can be different for every element.

4.3. Modes of f-vectors. For a unimodal sequence $f_0, \ldots, f_r$, it is interesting to find the location of its *modes*, *i. e.* the element(s) where the maximum of the sequence is attained.

REMARK 4.11. The index of the smallest mode of the f-vector of a rank r matroid is at least $\lfloor r/2 \rfloor$. In fact, the first half of the f-vector of every matroid is strictly monotonically increasing [**8**, 7.5.1. Proposition]. The minimum $\lfloor r/2 \rfloor$ is attained by the uniform matroid $U_{r,r}$. Some matroids have monotonically increasing f-vectors. It follows from (IV.23) that for an arbitrary matroid M and sufficiently large k, the f-vector of the k-fold thickening of M is strictly monotonically increasing.

5. Zonotopal algebra and matroid polynomials

Recall that the Hilbert series of the internal, central, and external zonotopal spaces are evaluations of the Tutte polynomial (cf. Section I.4). In this section we will show that this implies that various matroid and graph polynomials are evaluations of the Tutte polynomial. In addition, we will give a proof of Proposition 3.3 that uses zonotopal algebra.

While this section and the following do not contain any new results, we will point out some connections between combinatorics and zonotopal algebra that might be useful in the future.

The two zonotopal spaces that are of interest to us now are the central space $\mathcal{P}(X)$ and the internal space $\mathcal{P}_-(X)$. Since we are only interested in the Hilbert series, we sometimes just call them the central and the internal space. Recall that if X is a list of rank r with N elements, their Hilbert series are

$$\operatorname{Hilb}(\mathcal{P}(X), q) = q^{N-r} T_{\mathfrak{M}(X)}(1, \frac{1}{q}) \tag{IV.27}$$

$$\text{and } \operatorname{Hilb}(\mathcal{P}_-(X), q) = q^{N-r} T_{\mathfrak{M}(X)}(0, \frac{1}{q}). \tag{IV.28}$$

Let $X^* \in \mathbb{K}^{(N-r)\times r}$ denote a list of vectors realising the matroid dual to the matroid realised by X. In the central case, we obtain

$$q^r \operatorname{Hilb}(\mathcal{P}(X^*), \frac{1}{q}) = T_{\mathfrak{M}(X)}(q, 1) \tag{IV.29}$$

by dualising and by reversing the order of the coefficients. In the internal case, we obtain

$$q^r \operatorname{Hilb}(\mathcal{P}_-(X^*), \frac{1}{q}) = T_{\mathfrak{M}(X)}(q, 0) \tag{IV.30}$$

by dualising and by reversing the order of the coefficients. By comparing (IV.29) and (IV.30) with the definitions in Section 2 we obtain the following result.

Proposition 5.1. *Let $X \subseteq \mathbb{K}^r$ be a list of vectors spanning $\mathbb{K}^r$. Then,*

$$f_X(q) = T_{\mathfrak{M}(X)}(q+1, 1) = (q+1)^r \operatorname{Hilb}(\mathcal{P}(X^*), \frac{1}{q+1})$$

$$\textit{and } (-1)^r \chi_X(-q) = T_{\mathfrak{M}(X)}(q+1, 0) = (q+1)^r \operatorname{Hilb}(\mathcal{P}_-(X^*), \frac{1}{q+1}).$$

Remark 5.2. Proposition 5.1 can be restated as follows: the Hilbert series of the internal space is equal to the h-polynomial of the *broken-circuit complex* [**19**] of $\mathfrak{M}(X^*)$ and the Hilbert series of the central space equals the h-polynomial of the *matroid complex* of $\mathfrak{M}(X^*)$ (cf. [**8**] (7.12) and (7.15)).

Remark 5.3. The sum of the entries of the h-vector of the broken circuit complex (resp. the dimension of the internal space of the dual matroid) is called the *Möbius invariant*. Recently, De Loera, Sturmfels and Vinzant have shown that the degree of the central curve in linear programming is the Möbius invariant of a certain matroid related to the linear program [**41**].

Example 5.4. Let $X = ((1,0),(0,1),(1,1))$. X realises the uniform matroid $U_{2,3}$ and $X^* = (1,1,1)$.

The Tutte polynomial is $T_{\mathfrak{M}(X)}(x, y) = x^2 + x + y$.

$$\mathcal{P}(X^*) = \operatorname{span}\{1,\, s,\, s^2\} \qquad \mathcal{P}_-(X^*) = \operatorname{span}\{1,\, s\}$$
$$q^2 \operatorname{Hilb}(\mathcal{P}(X^*), 1/q) = q^2 + q + 1 \qquad q^2 \operatorname{Hilb}(\mathcal{P}_-(X^*), 1/q) = q^2 + q$$
$$f_{\mathfrak{M}(X)}(q) = q^2 + 3q + 3 \qquad \chi_{\mathfrak{M}(X)}(-q) = q^2 + 3q + 2$$

PROPOSITION 5.5. *Let $\mathbb{K}$ be some field and let $X \subseteq \mathbb{K}^r$ be a list of vectors spanning $\mathbb{K}^r$. Let $x \in \mathbb{K}^r$ be generic, i. e. x is not contained in any (linear) hyperplane spanned by the vectors in X. Then*

$$\mathcal{P}_-(X, x) = \mathcal{P}(X). \tag{IV.31}$$

PROOF. Recall that $\mathcal{P}_-(X, x) = \bigcap_{y \in (X,x)} \mathcal{P}((X, x) \setminus y)$. This implies that $\mathcal{P}(X)$ contains $\mathcal{P}_-(X, x)$. Equality can be established by a dimension argument: in [**54**], it is shown that the dimension of $\mathcal{P}(X)$ is equal to the cardinality of the set $\mathbb{B}(X)$ of bases that can be selected from X and that the dimension of $\mathcal{P}_-(X)$ equals the number of internal bases in X, *i. e.* bases that have no internally active elements. It can easily be seen that $B \subseteq (X, x)$ is an internal basis if and only if B is a basis and $x \notin B$. □

REMARK 5.6. Proposition 5.1 and Proposition 5.5 imply Proposition 3.3 for realisable matroids. This is how the author (re-)discovered the connection between the characteristic polynomial and the f-polynomial. The author believes that in the future, zonotopal algebra will help to solve further problems in matroid theory.

QUESTION 5.7. We have seen that for $\mathcal{P}_\bullet(X) \in \{\mathcal{P}_-(X), \mathcal{P}(X)\}$, the coefficients of the polynomial $(q + 1)^{N-r} \operatorname{Hilb}(\mathcal{P}_\bullet(X), 1/(q + 1))$
(a) have a combinatorial interpretation and
(b) form a log-concave sequence.
For which other zonotopal spaces does this hold?

6. Graph polynomials and zonotopal algebra

In this section we will present some graph polynomials that are related to internal and central zonotopal spaces. In all cases, the connection is made via the Tutte polynomial. Even though this connection is rather straightforward, it has never been stated explicitly in the literature. A good survey on graph polynomials that are related to the Tutte polynomial is [**47**] by Joanna Ellis-Monaghan and Criel Merino.

Let $G = (V, E)$ be a graph, possibly with multiple edges and loops. Let $\mathfrak{M}(G)$ denote the *cycle matroid* of G, *i. e.* the matroid on the ground set E whose bases are the spanning trees of the graph G. If $\kappa(G)$ denotes the number of connected components of G, then $\mathfrak{M}(G)$ has rank $\operatorname{rk}(\mathfrak{M}(G)) = |V| - \kappa(G)$. Let $X(G)$ denote a reduced oriented incidence matrix of G. Note that $X(G)$ realises the matroid $\mathfrak{M}(G)$.

6.1. Chromatic and flow polynomials.

The chromatic polynomial and the flow polynomial of a graph are related to the internal space $\mathcal{P}_-(X)$.

The chromatic polynomial χ_G of G evaluated at $q \in \mathbb{N}$ equals the number of proper colourings of the graph G with q colours. The chromatic polynomial is equal to the characteristic polynomial of $\mathfrak{M}(G)$ up to a factor:

$$\chi_G(q) = (-1)^{\mathrm{rk}(\mathfrak{M}(G))} q^{\kappa(G)} T_{\mathfrak{M}(G)}(1-q, 0). \tag{IV.32}$$

Hence,

$$(-1)^{\mathrm{rk}(\mathfrak{M}(G))}\chi_G(-q) = (q+1)^{\mathrm{rk}(\mathfrak{M}(G))} q^{\kappa(G)} \operatorname{Hilb}(\mathcal{P}_-(X(G)^*), \frac{1}{q+1}).$$

Let $\vec{E}$ denote an orientation of the edges of G and let $q \geq 2$. A *nowhere-zero q-flow* is an assignment $E \to \{1, \dots, q-1\}$ s. t. for each vertex, the sum over the incoming edges equals the sum over the outgoing edges modulo q. The function $\phi_G(q)$ which counts the number of nowhere zero q-flows is a polynomial and independent of the orientation $\vec{E}$:

$$\phi_G(q) = (-1)^{|E|-\mathrm{rk}(\mathfrak{M}(G))} T_{\mathfrak{M}(G)}(0, 1-q). \tag{IV.33}$$

Hence, $\phi_G(q)$ is equal to the characteristic polynomial of the dual matroid. This implies that by the result of Huh and Katz, the coefficients of $\phi_G(q)$ form a log-concave sequence. Furthermore,

$$\phi_G(q) = (q-1)^{|E|-\mathrm{rk}(\mathfrak{M}(G))} \operatorname{Hilb}(\mathcal{P}_-(X(G)), 1/(1-q)). \tag{IV.34}$$

6.2. Chip-firing games, shellings, and reliability. Three graph and matroid polynomials are related to the central space $\mathcal{P}(X)$: the critical configuration polynomial, the shelling polynomial, and the reliability polynomial.

The *critical configuration polynomial* $P_G(q) := T_{\mathfrak{M}(G)}(1, q)$ is related to chip-firing games played on the graph G. Its q^i coefficient equals the number of critical configurations of level i in the chip-firing game played on the graph G. The polynomial $h_M(q) := T_M(q, 1)$ that we defined in Definition 4.1 is also called the *shelling polynomial* of the matroid M. This polynomial encodes certain combinatorial properties of shellings of the matroid complex of M.

By (IV.27), the shelling polynomial $h_M(q)$ and the critical configuration polynomial $P_G(q)$ are evaluations of the Hilbert series of the central $\mathcal{P}$-space of $X(G)$ resp. of a realisation of M^*. For further information on these two polynomials, see [**8**] and [**47**, Sections 6.4 and 6.6].

Let $G = (V, E)$ be a connected graph on n vertices. Let $R_G(p)$ denote the probability that G is connected if each edge is independently removed with probability p. The function $R_G(p)$ is a polynomial [**15**]. It is called *reliability polynomial* of G and it can be be expressed in the following way:

$$R_G(p) = (1-p)^{n-1} \sum_{i=0}^{|E|-n+1} h_i p^i \tag{IV.35}$$

$$= (1-p)^{n-1} p^{|E|-n+1} T_G(1, \frac{1}{p}). \tag{IV.36}$$

The h_i denote the coefficients of the h-polynomial of the cycle matroid of G. The relationship between the h-vector and the reliability polynomial implies that bounds for the h-vector (*e. g.* Huh's result [**56**] resp. Theorem 4.3) might have some real-world applications in determining the reliability of a network. Brown and Colbourn [**15**, p. 117] state that if log-concavity of the h-vector *"holds for matroids arising in reliability problems, it would imply stronger constraints on the relation between coefficients in the h-vector than does Stanley's conditions. These conditions can be incorporated in the Ball-Provan strategy for computing reliability bounds and, hence, would lead to an efficient bounding technique of the reliability polynomial."*

Example 6.1. Let G be the complete graph on three vertices. Its cycle matroid is realised by the matrix X in Example 5.4. Recall that the Tutte polynomial of this matroid is $T_G(x,y) = x^2 + x + y$. Hence

$$\begin{aligned} &\chi_G(q) = q^3 - 3q^2 + 2q, && \phi_G(q) = q - 1, \\ &h_G(q) = q^2 + q + 1, && P_G(q) = q + 2, \quad \text{and } R_G(p) = (1-p)^2(1+2p). \end{aligned}$$

7. Arithmetic matroids and log-concavity

It is interesting to find out which properties of matroids have a suitable analogue for arithmetic matroids. In this section we will give an example which shows that the f-polynomial and the characteristic polynomial of an arithmetic matroid (cf. Section I.6) are in general not log-concave.

The f-polynomial and the characteristic polynomial of a matroid are specialisations of the Tutte polynomial. The f-polynomial and the characteristic polynomial of an arithmetic matroid are defined to be the same specialisations of the arithmetic Tutte polynomial. Let $(\mathfrak{M}, m)$ be an arithmetic matroid of rank r on the ground set E. Then we define

$$f_{(\mathfrak{M},m)}(q) := \sum_{\substack{A \subseteq E \\ A \text{ independent}}} m(A) q^{r-|A|} = M_{(\mathfrak{M},m)}(q+1, 1) \tag{IV.37}$$

$$\text{and } \chi_{(\mathfrak{M},m)}(q) := \sum_{A \subseteq E} (-1)^{|A|} m(A) q^{r - \operatorname{rk}(A)} = (-1)^r M_{(\mathfrak{M},m)}(1-q, 0).$$

Incidentally, if $(\mathfrak{M}, m)$ is realised by the list X, $\chi_{(\mathfrak{M},m)}(q)$ is the characteristic polynomial of the toric arrangement defined by the list X just as the characteristic polynomial of a realisable matroid is the characteristic polynomial of the hyperplane arrangement defined by a realisation of the matroid.

It is a natural question to ask whether $f_{(\mathfrak{M},m)}$ and $\chi_{(\mathfrak{M},m)}(q)$ are log-concave. In general this is false as we can see from the following example.

Example 7.1. Let e_i denote the ith unit vector in $\mathbb{R}^r$. Let $\alpha \in \mathbb{Z}$ and let

$$X := (e_1, e_2, \ldots, e_{r-1}, e_1 + \ldots + e_{r-1} + \alpha e_d) \tag{IV.38}$$

We consider the arithmetic matroid $(\mathfrak{M}_X, m)$ defined by the list X. X is a basis for $\mathbb{R}^r$ and all strict sublists of X have multiplicity one. The multiplicity of X is α. Hence,

$$M_{(\mathfrak{M},m)}(x,y) = \alpha + \sum_{i=1}^{r} \binom{r}{i} (x-1)^i, \tag{IV.39}$$

$$f_{(\mathfrak{M},m)}(q) = q^r + \binom{r}{1} q^{r-1} + \ldots + \binom{r}{r-1} q^1 + \alpha, \tag{IV.40}$$

$$\text{and } \chi_{(\mathfrak{M},m)}(q) = q^r - \binom{r}{1} q^{r-1} + \ldots \pm \binom{r}{r-1} q^1 \mp \alpha. \tag{IV.41}$$

For sufficiently large α and $r \geq 2$, the polynomials $f_{(\mathfrak{M},m)}(q)$ and $\chi_{(\mathfrak{M},m)}(q)$ are not log-concave and for $r \geq 4$, they are not even unimodal.

A special case of Example 7.1 is mentioned in [**22**, Section 8]. It was suggested to the authors of that paper by the author of this thesis.

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