

Maximum of a Rough Path from Its Signature

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Definition

Let $1 \leq p < 2$ a real number, and V a Banach space. Let $X : J \rightarrow V$ be a continuous path. Then the p -variation of X on $[0, T]$ is defined by

$$\|X\|_{p,[0,T]} = \left[\sup_{D \subset [0,T]} \sum_{j=0}^{r-1} \|X_{t_j} - X_{t_{j+1}}\|^p \right]^{\frac{1}{p}}$$

where $D = \{t_0, t_1, \dots, t_r\}$ is a finite partition of $[0, T]$, and $0 = t_0 < t_1 < \dots < t_{r-1} < t_r = T$.

Definition

$X : [0, T] \rightarrow V$ is a path with finite p -variation, $1 \leq p < 2$. The p -Length of X , which we denote by $l_p : \Delta_T \rightarrow \mathbb{R}$ is a real valued function, defined as

$$l_p(s, t) = \|X\|_{p,[s,t]}^p$$

where $\Delta_T := \{(s, t) \mid 0 \leq s \leq t \leq T\}$

Definition

Suppose $X : [0, T] \rightarrow V$ is a path of finite p -variation, $1 \leq p < 2$. Then the Signature of X : $S(X) : [0, T] \rightarrow T((V))$, ($T((V)) = V \oplus V^{\otimes 2} \oplus \dots \oplus V^{\otimes n} \oplus \dots$) is defined as

$$S(X_{0,t}) = (1, X_{0,t}^1, X_{0,t}^2, \dots, X_{0,t}^k, \dots)$$

Here $X_{0,t}^k \in V^{\otimes k}$, and $X_{0,t}^k$ is called the Iterated Integral of X :

$$X_{0,t}^k = \int \dots \int_{0 < u_1 < \dots < u_k < t} dX_{u_1} \otimes \dots \otimes dX_{u_k}$$

Definition

If $e = (e_{i_1}^*, e_{i_2}^*, \dots, e_{i_k}^*) \in (V^*)^{\otimes k}$, $X : [0, T] \rightarrow V$ is a path of finite p -variation, $1 \leq p < 2$. Define k th Coordinate Iterated Integral of X with respect to e as

$$\varphi_e^X(t) = \int \dots \int_{s < u_1 < \dots < u_k < t} e_{i_1}^*(dX_{u_1}) \dots e_{i_k}^*(dX_{u_k})$$

Definition

Suppose X is a path of finite p -variation, $1 \leq p < 2$. Reparametrize X by its p -Length, i.e. let $\widehat{X}_{l_p(0,t)} = X_t$.

$$\|\widehat{X}_{l_p(0,t_2)} - \widehat{X}_{l_p(0,t_1)}\|^p \leq l_p(t_1, t_2) \leq l_p(0, t_2) - l_p(0, t_1)$$

$$\|\widehat{X}_{l_p^1} - \widehat{X}_{l_p^2}\| \leq (l_p^1 - l_p^2)^{\frac{1}{p}} \text{ with } l_p^1 = l_p(0, t_1), l_p^2 = l_p(0, t_2)$$

- \widehat{X}_{l_p} is non-constant on any subinterval.
- Reparametrization does not change the path, nor does it change the Signature of the path.

$$\begin{aligned} (\widehat{X}_{l_p(0,t)})^k &= \int \cdots \int_{0 < l_p^k < \cdots < l_p^1 < l_p(0,t)} d\widehat{X}_{l_p^1} \otimes \cdots \otimes d\widehat{X}_{l_p^k} \\ &= \int \cdots \int_{0 < u_1 < \cdots < u_k < t} dX_{u_1} \otimes \cdots \otimes dX_{u_k} \end{aligned}$$

Theorem

If f is α -Hölder continuous on interval J , with $0 < \alpha \leq 1$, $c > 0$:
 $|f(u) - f(v)| \leq c|u - v|^\alpha$, for any $u, v \in J$. Denote $M := \max |f|_J$, then

$$\left| M - \left(\int_J f^{2n}(s) ds \right)^{\frac{1}{2n}} \right| \leq M \left[\frac{\ln 2n}{2\alpha n} + o\left(\frac{\ln 2n}{2\alpha n}\right) \right] \quad \text{as } n \rightarrow \infty$$

And, there exists f , such that $\left| M - \left(\int_J f^{2n}(s) ds \right)^{\frac{1}{2n}} \right| = M \left[\frac{\ln 2n}{2\alpha n} + o\left(\frac{\ln n}{n}\right) \right]$.

Theorem

If f is a Lipschitz function on interval J : $|f(u) - f(v)| \leq c|u - v|$, for any $u, v \in J$. $M := \max |f|_J$, and $\frac{c}{M} \geq 1$. Then

$$\left| \left(\int_J f^{2n}(s) ds \right)^{\frac{1}{2n}} - M \right| \leq \frac{M}{2N} \left| \ln(2n + 1) + \ln \frac{c}{M} \right|$$

Proposition: If $M := \max |f|_J$ is fixed, the error of approximation increases when c increases. That is, the speed of convergence is slower when the value of f changes dramatically in a small interval.

Example

Let $f_k(t) = e^{-kt}$, $k \geq 1$. Then $\frac{c}{M} = k$ for $f_k(t)$ on $t \in [0, 1]$. For some fixed $\epsilon > 0$, if we want

$$\left| \left(\int_J f_k(s)^{2n} ds \right)^{\frac{1}{2n}} - 1 \right| = \left| \left(\frac{1 - e^{-2nk}}{2nk} \right)^{\frac{1}{2n}} - 1 \right| < \epsilon$$

It is equivalent to

$$2nk(1 - \epsilon)^{2n} + e^{-2nk} < 1$$

if it holds, we must have

$$2n(1 - \epsilon)^{2n} < \frac{1}{k}$$

So, as $k \rightarrow \infty$, $n \rightarrow \infty$.

Proposition: The Convergence is quicker when the mass of f is more evenly distributed.

- If f is a real-valued function on J , and $J_1 \subset J$

$$M - \left(\int_{J_1} f(s)^{2n} ds \right)^{\frac{1}{2n}} \geq M - \left(\int_J f^{2n}(s) ds \right)^{\frac{1}{2n}}$$

- If f is a Non-negative real-valued function on J , then for any $a \geq 0$

$$M + a - \left(\int_J [f(s) + a]^{2n} ds \right)^{\frac{1}{2n}} \geq M - \left(\int_J f^{2n}(s) ds \right)^{\frac{1}{2n}}$$

Theorem

If f is a real-valued function on J . Then for any given n

$$\min_{a \in \mathbb{R}} \theta_n(a) = \theta_n\left(-\frac{\text{ess sup } f_J + \text{ess inf } f_J}{2}\right)$$

where $\theta_n(a) := \text{ess sup}(f(s) + a)_J - \left(\int_J [f(s) + a]^{2n} ds \right)^{\frac{1}{2n}}$

Definition

$$\theta_n(a) := \text{ess sup}(f(s) + a)_J - \left(\int_J [f(s) + a]^{2n} ds \right)^{\frac{1}{2n}}$$

Corollary

For any fixed n , $-\frac{\text{ess sup } f_J + \text{ess inf } f_J}{2}$ is a globally (strict) minimum point of $\theta_n(a)$, i.e.

$$\theta_n(a) > \theta_n\left(-\frac{\text{ess sup } f_J + \text{ess inf } f_J}{2}\right) \text{ if } a \neq -\frac{\text{ess sup } f_J + \text{ess inf } f_J}{2}$$

- If this minimum point is found, $\text{ess sup } f_J + \text{ess inf } f_J$ is known.
Moreover

$$\text{ess sup}\left(f - \frac{\text{ess sup } f_J + \text{ess inf } f_J}{2}\right) = \frac{\text{ess sup } f_J - \text{ess inf } f_J}{2}$$

Therefore, we could extract $\text{ess sup } f_J$ and $\text{ess inf } f_J$ from its $2n$ -norm, especially the diameter of f on J : $\text{ess sup } f_J - \text{ess inf } f_J$, which, however, is not obvious from the definition of $2n$ -norm.

Approximation Using Signature

Theorem

Suppose f is a continuous real-valued function, and $|f| \leq M$. If

$$\int_J \left(\frac{(M + |K|)S_N\left(\frac{f-K}{M+|K|}\right) + f + K}{2} - K \right)^2 dt > \frac{(M + |K|)^4 m(J)}{4(2N + 1)^2}$$

then $\max f_J > K$, where $S_N(x) := \frac{2}{\pi} + \frac{4}{\pi} \sum_{r=1}^N \frac{(-1)^{r+1}}{4r^2-1} T_{2r}(x)$ is the partial sum of Chebyshev series of $|x|$.

- Suppose $X : J \rightarrow R^n$ is a path of finite p -variation $1 \leq p < 2$, and $\varphi_e^X(s, t)$ is a Coordinate Iterated Integral function of X , then $\varphi_e^X(s, t)$ is continuous on $\Delta_T := \{(s, t) | 0 \leq s \leq t \leq T\}$, thus bounded on compact interval J .
- Polynomial of φ_e^X is a linear combination of higher ordered Coordinate Iterated Integrals of X .

Definition

$S_N(x) := \frac{2}{\pi} + \frac{4}{\pi} \sum_{r=1}^N \frac{(-1)^{r+1}}{4r^2-1} T_{2r}(x)$, where T_{2r} , $r \geq 1$, are the first kind Chebyshev Polynomials.

Theorem

Suppose $X: J \rightarrow \mathbb{R}^n$ is an n -dimensional path of finite p -variation, $1 \leq p < 2$. e is an unit constant n -dimensional vector. If $0 < m \leq |X \cdot e| \leq M$, then there exists a series of polynomials S_N^{m+M} , s.t.

$$\left\| S_N^{m+M}(X \cdot e) - |X \cdot e| \right\|_{p,J} \leq \frac{C(\frac{\pi}{2} - \arccos \frac{m}{m+M})}{N} \|X\|_{p,J}$$

where $S_N^{m+M}(f) = (m+M)S_N(\frac{f}{m+M})(x)$

$$C(\theta) := \frac{1}{\pi \sin(\frac{1}{2}\theta)} + \frac{1}{2\pi} \int_{\theta}^{\pi} \left| \frac{\cos \frac{1}{2}t \cos[(N + \frac{1}{2})t]}{\sin^2 \frac{1}{2}t} \right| dt$$

Extract Abstract Value of x from Power Series

Definition

$$\text{Let } T_N(x) := \sum_{i=0}^N \frac{x^{2i}}{i!}.$$

Lemma

Suppose f is a real-valued function, then

$$\left| |f| - \sqrt{\ln[T_N(f)]} \right| \leq \max\{\delta_N^1(f), \delta_N^2(f)\} := \delta_N(f)$$

$$\delta_N^1(f) : = \sqrt{\ln[T_N(f)] - \ln\left[1 - e\left(\frac{ef^2}{N+1}\right)^{N+1}\right]} - \sqrt{\ln[T_N(f)]}$$

$$\delta_N^2(f) : = \sqrt{\ln[T_N(f)]} - \sqrt{\ln[T_N(f)] - \ln\left[1 + e\left(\frac{ef^2}{N+1}\right)^{N+1}\right]}$$

Lemma

For any fixed $N \geq 7$, $\delta_N(x)$ is increasing on $[0, \sqrt{\frac{N+1}{e}}]$.

Theorem

Suppose f is a real-valued function, and $N \geq 7$. If $|f(J)| \leq a \leq \sqrt{\frac{N+1}{e}}$, then on J

$$0 \leq |f| - \sqrt{\ln[T_N(f)]} \leq \delta_N(a)$$

Corollary

Suppose f is a real-valued function, $N \geq 7$, if for some $K \in \mathbb{R}$, $|f(J) - K| \leq a \leq \sqrt{\frac{N+1}{e}}$, and

$$\int_J (\sqrt{\ln[T_N(f - K)]} + f - K)^2 dt > m(J) [\delta_N(a)]^2$$

then $\max f_J > K$.

Definition

$$\lambda_N(x) := \max\left\{\frac{\delta_N^2(x)}{\sqrt{\ln[T_N(x)]}}, \frac{\delta_N^1(x)}{\sqrt{\ln[T_N(x)]}}\right\}$$

Lemma

There exists $x_\lambda^N \geq 0$, s.t. $\lambda_N(x)$ is increasing on $[x_\lambda^N, \sqrt{\frac{N+1}{2e}}]$, where $\lim_{N \rightarrow \infty} x_\lambda^N = 0$.

Theorem

$X: J \rightarrow R^n$ is a n -dimensional path of finite p -variation, $1 \leq p < 2$, e is a n -dimensional constant unit vector, then for any fixed $N \geq 2$, if

$$x_\lambda^N \leq |X \cdot e| \leq \sqrt{\frac{N+1}{2e}}$$

$$\left\| |X \cdot e| - \sqrt{\ln[T_N(X \cdot e)]} \right\|_{p,J} \leq \lambda_N\left(\sqrt{\frac{N+1}{2e}}\right) \|X\|_{p,J}$$

For all $f : J \rightarrow R$, is a continuous function, $V_f : R \rightarrow \overline{R^+}$ is defined as

$$V_f(a) := \int_{\{f \geq a\}} (f(t) - a) dt$$

Then V_f has the following properties:

- non-negative.
- non-increasing.
- differentiable with the derivative $-m(f \geq a)$, which is bounded by $m(J)$.
- convex.
- $\text{ess sup } f_J$ is the smallest point where $V_f = 0$.