Skorohod reflection of Brownian paths and BES$^3$

Bálint Vető

Budapest University of Technology and Economics
http://www.math.bme.hu/~vetob

Sep 3, 2009
joint work with Bálint Tóth

Outline

1. Definition
2. Result
3. Proof
4. Outlook
Let \( b, x : [0, T) \to \mathbb{R} \) be continuous functions with \( x(0) \geq b(0) \).

1. \( \exists \ x_b^\uparrow : [0, T) \to \mathbb{R} \) with
   - \( x_b^\uparrow(0) = x(0) \),
   - \( x_b^\uparrow - b \) non-negative,
   - \( x_b^\uparrow - x \) non-decreasing,
   - \( x_b^\uparrow - x \) increases only when \( x_b^\uparrow = b \).

2. \( t \mapsto x_b^\uparrow(t) \) is given by
   \[
   x_b^\uparrow(t) = x(t) + \sup_{0 \leq s \leq t} (x(s) - b(s))_.
   \]

3. The map \( (b(\cdot), x(\cdot)) \mapsto (b(\cdot), x_b^\uparrow(\cdot)) \) is continuous in supremum distance.

The function \( x_b^\uparrow \) is the *upwards Skorohod-reflection* of \( x \) on \( b \).

Similarly, \( y_b^\downarrow(t) = y(t) - \sup_{0 \leq s \leq t} (y(s) - b(s))_+ \).
Explanation

\[ x_0(t) \]

\[ x(t) \]

\[ b \equiv 0 \]

\[ X_{B\uparrow}(t) \]

\[ B(t) \]

\[ Y_{B\downarrow}(t) \]
Let $B(t)$, $X(t)$ and $Y(t)$ be independent standard 1d Brownian motions starting from 0.
For each realization of the Brownian motions, $X_{B\uparrow}$ and $Y_{B\downarrow}$ can be defined.

$$ Z(t) := X_{B\uparrow}(t) - Y_{B\downarrow}(t). $$

**Theorem (B. Tóth, B. V., 2007)**

The process $2^{-1/2} Z(t)$ is $\text{BES}^3$, that is a standard 3d Bessel process,

$$ dZ(t) = 2 \frac{1}{Z(t)} \, dt + \sqrt{2} \, dW(t), \quad Z(0) = 0. $$

**Special case of Jon Warren’s theorem (2007)**
We have a simpler proof.
Dual square lattices:

\[ \mathcal{L} := \{(t, x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is even}\} \]

\[ \mathcal{L}^* := \{(t, x) \in \mathbb{Z} \times \mathbb{Z} : t + x \text{ is odd}\} \]

Lattice walk:

\[ y : [0, T] \cap \mathbb{Z} \to \mathbb{Z} \text{ if the consecutive points of} \]

\[ (0, y(0)), (1, y(1)), \ldots, (T, y(T)) \text{ are edges in } \mathcal{L} \text{ or } \mathcal{L}^*. \]
Discrete Skorohod reflection

Let \( b, x : [0, T] \cap \mathbb{Z} \rightarrow \mathbb{Z} \) be walks in \( \mathcal{L} \) and \( \mathcal{L}^* \) with \( x(0) \geq b(0) \).

1. \( \exists x_{b\uparrow} : [0, T] \cap \mathbb{Z} \rightarrow \mathbb{Z} \) in \( \mathcal{L}^* \) with the following properties:
   - \( x_{b\uparrow}(0) = x(0) \),
   - \( x_{b\uparrow} - b \) non-negative,
   - \( x_{b\uparrow} - x \) non-decreasing,
   - \( x_{b\uparrow} - x \) increases only when \( x_{b\uparrow} = b + 1 \).

2. The function \( t \mapsto x_{b\uparrow}(t) \) can be expressed as
   \[
   x_{b\uparrow}(t) = x(t) + \sup_{s \in [0,t] \cap \mathbb{Z}} (x(s) - b(s) - 1)_-.
   \]

The walk \( x_{b\uparrow} \) is the discrete upwards Skorohod reflection of \( x \) on \( b \).

Similarly, \( y_{b\downarrow}(t) = y(t) - \sup_{s \in [0,t] \cap \mathbb{Z}} (y(s) - b(s) + 1)_+ \).
Example

$\text{Skorohod reflection of Brownian paths and } \text{BES}^3$
Proof of Theorem

$M(t)$ symmetric random lattice walk on $\mathcal{L}$ with $M(0) = 0$.

$U(t)$ and $L(t)$ symmetric random lattice walk on $\mathcal{L}^*$ with $U(0) = 1$ and $L(0) = -1$ independently.

$$\left( \frac{M(nt)}{\sqrt{n}}, \frac{U(nt)}{\sqrt{n}}, \frac{L(nt)}{\sqrt{n}} \right) \overset{d}{\Rightarrow} (B(t), X(t), Y(t)).$$

By Donsker’s invariance principle

$$\left( \frac{M(nt)}{\sqrt{n}}, \frac{U_M(nt)}{\sqrt{n}}, \frac{L_M(nt)}{\sqrt{n}} \right) \overset{d}{\Rightarrow} (B(t), X_B(t), Y_B(t)).$$

$$D(n) := \frac{1}{2} (U_M(n) - L_M(n))$$

Enough to show that

$$\frac{\sqrt{2}D(nt)}{\sqrt{n}} \overset{d}{\Rightarrow} \text{BES}^3.$$
$D(n)$ is Markovian with transition matrix

$$\widetilde{P}_{xy} = \frac{y}{x} \cdot \begin{cases} 
\frac{1}{2} & \text{if } y = x \\
\frac{1}{4} & \text{if } |y - x| = 1 \\
0 & \text{otherwise.}
\end{cases}$$

Heuristic calculations with $dt = \frac{1}{A}$ yield

$$E \left( \frac{dD(At)}{\sqrt{A}} \right) = \frac{1}{2} \frac{\sqrt{A}}{D(At)} \, dt \quad \text{Var} \left( \frac{dD(At)}{\sqrt{A}} \right) = \left( \frac{1}{2} + o(1) \right) \, dt$$

Therefore

$$\frac{2D(At)}{\sqrt{A}} \xrightarrow{d} Z(t)$$

with

$$dZ(t) = 2 \frac{1}{Z(t)} \, dt + \sqrt{2} \, dW(t).$$

**Remark:** given $D(n)$, the position of $M(n)$ is discrete uniform between $L_{M(1)}(n)$ and $U_{M(1)}$. 
Let $\xi_n$ be a birth and death process, which is a Markov chain on $\{0, 1, 2, \ldots\}$ with transition matrix $Q$. Suppose that $\xi_n$ is a martingale, i.e. $\sum_y yQ_{xy} = x$ for all $x$. Let $\tau_0 := \min\{k \geq 0 : \xi_k = 0\}$. Then

$$P\left(\xi_{n+1} = y \mid \xi_n = x, \tau_0 = \infty\right) = \tilde{Q}_{xy} := \frac{y}{x}Q_{xy}.$$

1. If the transition matrix of $\xi_n$ is

$$P_{xy} = \begin{cases} 
1/2 & \text{if } y = x \\
1/4 & \text{if } |y - x| = 1
\end{cases}$$

then $\xi_{At}/\sqrt{A} \overset{d}{\Rightarrow} \text{BES}^1$ and

$$\left(\xi_{At}/\sqrt{A} \mid \tau_0 = \infty\right) \overset{d}{=} D(At)/\sqrt{A} \overset{d}{\Rightarrow} \text{BES}^3.$$ $1 + 3 = 4$

2. If $U_n$ is a critical branching process, then $U_{At}/\sqrt{A} \overset{d}{\Rightarrow} \text{BESQ}^0$

and $\left(U_{At}/\sqrt{A} \mid \tau_0 = \infty\right) \overset{d}{\Rightarrow} \text{BESQ}^4.$ $0 + 4 = 4$
Thank you for your attention!