On a Generalization of the Sherrington-Kirkpatrick Model

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Outline

Definition

Consider the following Hamiltonian on Σ^N where Σ is any finite set:

$$H(\sigma) = \sum_{i<j} \frac{\beta}{\sqrt{N}} g_{ij}(\sigma_i, \sigma_j) + \sum_{i,j} \sqrt{b_i^{(j)}} g_i^{(j)}(\sigma_i) + \sum_{i,j} b_i^{(j)} \Phi^{(j)}(\sigma_i)$$

- the $a_{ij} \geq 0$ and the $b_i^{(j)} \geq 0$, $a_{ii} = b_i^{(i)} = 0$.
- g_{ij} iid copies of a gaussian field with covariance matrix Γ
- $g_i^{(j)}$ iid copies of gaussian field with covariance matrix

$$\mathbb{E} g_i^{(j)}(s) g_i^{(j)}(s') = \Gamma^{(j)}(s, s') := \sum_{t,t'} \Gamma(s, t, s', t') \kappa_j(t, t')$$

- $\Phi^{(j)}(s) := \frac{1}{2}\Big(\sum_t \gamma(s,t)\pi_j(t) - \sum_{t,t'} \Gamma(s,t,s,t')\kappa_j(t,t') \Big)$
- $\big(\pi_j(s)\big)_s$ and $\big(\kappa_j(s,s')\big)_{s,s'}$ are arbitrary numbers (for now).

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- $\left(\pi_j(s)\right)_s$ and $\left(\kappa_j(s, s')\right)_{s,s'}$ are arbitrary numbers (for now).

Generalization

We have three generalizations of the typical SK model:

- Spins can have values in any given finite Set Σ equipped with a probability measure p. The main calamity here is that there is no rule $\sigma_i^2 = 1$ anymore.
- Interactions are multidimensional gaussian fields.
- we can keep track of the fading out of interactions using the a_{ij} and $b_i^{(j)}$. This is the main idea in Talagrand (2009).

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Class of Hamiltonians

We will consider the class $\mathcal{H}$ of Hamiltonians with the constraint:

$$a_{ij} + b_i^{(j)} = c_i^{(j)}$$

where $C = (c_i^{(j)})_{ij}$ is constant.

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TAP

In view of those constraints we define the TAP-like expressions:

$$\Pi_i(s) := \frac{1}{Z} p(s) \exp\left\{ Y_i(s) + \sum_j c_i^{(j)} \Phi^{(j)}(s) \right\}$$

$$Z := \sum_{s \in \Sigma} p(s) \exp\left\{ Y_i(s) + \sum_j c_i^{(j)} \Phi^{(j)}(s) \right\}$$

$$\mathbb{E}\, Y_i(s) Y_i(s') = \sum_j c_i^{(j)} \Gamma(s, s'; \kappa_j)$$

Fixed Point equations

Now π_i and κ_i are defined as solutions to the fixed point equations.

$$\pi_i(s) = \mathbb{E}_Z \Pi_i(s)$$
$$\kappa_i(s, s') = \mathbb{E}_Z \Pi_i(s) \Pi_i(s')$$

That they have solutions is seen readily by Brouwers Fixed Point Theorem. We choose one of them once and for all (it should be unique for high enough temperature).

Gibbs Measure

What do we do with the Hamiltonians? We use them to define a Gibbs measure.

First we introduce the partition function:

$$Z := \sum_{\sigma \in \Sigma^N} \exp\big(H(\sigma)\big) \cdot p^{\otimes N}(\sigma)$$

Then we have

$$P(\sigma) := \tfrac{1}{Z} \exp\big(H(\sigma)\big) \cdot p^{\otimes N}(\sigma),$$

a probability measure on Σ^N.

If we have a function $f(\sigma^1, \ldots, \sigma^n)$ dependent on n independent replicas we notate the expectation by:

$$\left\langle\, f(\sigma^1, \ldots, \sigma^n) \,\right\rangle := \sum_{\sigma^1 \in \Sigma^N, \ldots, \sigma^n \in \Sigma^N} f(\sigma^1, \ldots, \sigma^n) \cdot P(\sigma^1) \cdots P(\sigma^n).$$

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Free Energy

As usual we will look at the free energy $\log Z$ and show that it converges in the thermodynamic limit $N \to \infty$ to the same as

$$\begin{aligned} p_N := & -\tfrac{1}{4}\sum_{i,j} a_{ij}\big[\sum_{s,t}\gamma(s,t)\pi_i(s)\pi_j(t) \\ & - \sum_{s,t,s',t'}\Gamma(s,t,s',t')\kappa_i(s,s')\kappa_j(t,t')\big] \\ & + \sum_{i\leq N}\mathbb{E}\log\sum_s \Pi_i(s). \end{aligned}$$

Actually we will prove for $\beta \geq 0$ small enough the rate

$$\left|\frac{1}{N}\mathbb{E}\log\sum_\sigma \exp\left(H(\sigma)\right) - p_N\right| \leq O(1)$$

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Examples

- The famous Sherrington-Kirkpatrick model is the example, where $\Sigma = \{-1, 1\}$ and $\Gamma(s, t, s', t') = s \cdot t \cdot s' \cdot t'$. In this case we have:

$$g_{ij}(\sigma_i, \sigma_j) = g_{ij} \cdot \sigma_i \cdot \sigma_j$$

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Smart Path Method

- The most important tool we will use is the Smart Path method, or Guerra's interpolation.
- This is a vehicle to compare the model given by one hamiltonian with one given by another using a 'smart path'.
- Say we have two Hamiltonians H_1 and H_2 and a path H_t between them. Define $\varphi(t) := \log(\sum_\sigma \exp(H_t(\sigma)))$ and use

$$\mathbb{E}|\varphi(1) - \varphi(0)| = \mathbb{E}|\int_0^1 \varphi'(x)\,dx| \leq \sup_x |\varphi'(x)|.$$

- If we can show $|\varphi'(x)| = o(1)$ as $N \to \infty$ uniformely in x this proves that the two values are the same in the thermodynamic limit.
- Usually the smart path consists of fading out some interactions.

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Smart Path Method applied

Consider fading out all interaction, i.e. $a_{ij}(x) := x \cdot a_{ij} = x\frac{\beta^2}{N}$ and $b_i^{(j)}(x) := b_i^{(j)} + (1-x)a_{ij} = (1-x)\frac{\beta^2}{N}$. Then

$$H_x(\sigma) := \sum_{i<j} \sqrt{a_{ij}(x)} g_{ij}(\sigma_i, \sigma_j) + \sum_{i,j} \sqrt{b_i^{(j)}(x)} g_i^{(j)}(\sigma_i) + \sum_{i,j} b_i^{(j)}(x) \Phi^{(j)}(\sigma_i)$$

Let $\varphi(x) := \mathbb{E} \log(\sum_\sigma e^{H_x(\sigma)} \cdot p_{\otimes N}(\sigma))$ and $\nu_x(f) := \mathbb{E} \langle f \rangle_x$. Most important in $x = 0$ the spins are independent, so we have

$$\varphi(0) = \sum_{i \le N} \mathbb{E} \log \sum_s \Pi_i(s)$$

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Differentiation

$$\frac{\partial H_x(\sigma)}{\partial x} = \frac{1}{2}\sum_{i<j}\sqrt{\frac{a_{ij}}{x}}g_{ij}(\sigma_i,\sigma_j) - \frac{1}{2}\sum_{i,j}\frac{a_{ij}}{\sqrt{b_i^{(j)}+(1-x)a_{ij}}}g_i^{(j)}(\sigma_i) - \sum_{i,j}a_{ij}\Phi^{(j)}(\sigma_i)$$

Using simple calculus and the definition of $P(\sigma)$ we get:

$$\varphi'(t) = \mathbb{E}\frac{e^{H_x(\sigma)}}{Z}\frac{\partial H_x(\sigma)}{\partial x} = \nu_x\left(\frac{\partial H_x(\sigma)}{\partial x}\right)$$

$$= \frac{1}{4}\sum_{i,j}a_{ij}\left\{\frac{1}{\sqrt{xa_{ij}}}\nu_x\left[g_{ij}(\sigma_i,\sigma_j)\right] - \frac{1}{\sqrt{b_i^{(j)}+(1-x)a_{ij}}}\nu_x\left[g_i^{(j)}(\sigma_i)\right] - \nu_x\left[\Phi^{(j)}(\sigma_i)\right]\right\}$$

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Integration by parts

We attack the terms $\nu_x[g(\ldots)]$ using the formula:

$$\mathbb{E} g \cdot F(g_1, \ldots, g_k) = \sum_{i=1}^{k} \operatorname{cov}(g, g_i) \cdot \mathbb{E} \frac{\partial F}{\partial g_i}(g_1, \ldots, g_k)$$

The problem are the implicit terms $P(\sigma) = \frac{e^{H(\sigma)}}{\sum_{\sigma'} e^{H(\sigma')}}$ in the expectation. They contain multiple references to the $g(\ldots)$ and we have to keep track of this. It will introduce another replica.

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Integration by parts applied

We have therefore

$$\begin{aligned}\frac{1}{\sqrt{xa_{ij}}}\nu_x\left[g_{ij}(\sigma_i,\sigma_j)\right] &= \frac{1}{\sqrt{xa_{ij}}}\sum_{\sigma}\mathbb{E}\,g_{ij}(\sigma_i,\sigma_j)\frac{\exp\left(H_x(\sigma)\right)}{\sum_{\sigma'}\exp\left(H_x(\sigma')\right)}\cdot p^{\otimes N}(\sigma)\\ &= \frac{\sqrt{xa_{ij}}}{\sqrt{xa_{ij}}}\sum_{\sigma}\gamma(\sigma_i,\sigma_j)\mathbb{E}\,P_x(\sigma)\\ &\qquad -\sum_{\sigma}\sum_{\sigma'}\Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j')\mathbb{E}\,P_x(\sigma)P_x(\sigma')\\ &= \nu_x\left[\gamma(\sigma_i,\sigma_j)-\Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j')\right]\end{aligned}$$

and analogous:

$$\frac{1}{\sqrt{b_i^{(j)}+(1-x)a_{ij}}}\nu_x\left[g_i^{(j)}(\sigma_i)\right] = \nu_x\left[\gamma^{(j)}(\sigma_i)-\Gamma^{(j)}(\sigma_i,{\sigma_i}')\right]$$

So we have:

$$\varphi'(x) = \frac{1}{4}\sum_{i,j} a_{ij}\nu_x\Big[\gamma(\sigma_i,\sigma_j) - \Gamma(\sigma_i,\sigma_j,\sigma_i{}',\sigma_j{}') \\ - \gamma^{(j)}(\sigma_i) - \gamma^{(i)}(\sigma_j) + \Gamma^{(j)}(\sigma_i,\sigma_i{}') + \Gamma^{(i)}(\sigma_j,\sigma_j{}') \\ - \Phi^{(j)}(\sigma_i) - \Phi^{(i)}(\sigma_j)\Big]$$

One interesting 'coincidence' is

$$\gamma^{(j)}(s) + \Phi^{(j)}(s) = \sum_{t,t'}\Gamma(s,t,s,t')\kappa_j(t,t') + \sum_t \gamma(s,t)\pi_j(t) \\ - \sum_{t,t'}\Gamma(s,t,s,t')\kappa_j(t,t') \\ = \sum_t \gamma(s,t)\pi_j(t)$$

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Let $\gamma(s,\pi_j) := \sum_t \gamma(s,t)\pi_j(t)$ and
$\Gamma(s,s';\kappa_j) := \sum_{t,t'} \Gamma(s,t,s,t')\kappa_j(t,t')$. Then:

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$$- \Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j') + \Gamma(\sigma_i,\sigma_i';\kappa_j) + \Gamma(\sigma_j,\sigma_j';\kappa_i)\Big]$$

$$= \frac{1}{4}\sum_{ij} a_{ij}\nu_x\Big[\sum_{s,t}\gamma(s,t)\big\{\delta_{\sigma_i}(s)\delta_{\sigma_j}(t) - \delta_{\sigma_i}(s)\pi_j(t) - \pi_i(s)\delta_{\sigma_j}(t)\big\}$$

$$- \sum_{s,t,s',t'}\Gamma(s,t,s',t')\big\{\delta_{\sigma_i,\sigma_i'}(s,s')\delta_{\sigma_j,\sigma_j'}(t,t')$$

$$- \delta_{\sigma_i,\sigma_i'}(s,s')\kappa_j(t,t') - \kappa_i(s,s')\delta_{\sigma_j,\sigma_j'}(t,t')\big\}\Big]$$

In order to factorize this we subtract the first term in p.

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&- \Gamma(\sigma_i,\sigma_j,\sigma_i{}',\sigma_j{}') + \Gamma(\sigma_i,\sigma_i{}';\kappa_j) + \Gamma(\sigma_j,\sigma_j{}';\kappa_i)\Big] \\
= \frac{1}{4}\sum_{ij} a_{ij}\nu_x\Big[&\sum_{s,t}\gamma(s,t)\big\{\delta_{\sigma_i}(s)\delta_{\sigma_j}(t) - \delta_{\sigma_i}(s)\pi_j(t) - \pi_i(s)\delta_{\sigma_j}(t)\big\} \\
&- \sum_{s,t,s',t'}\Gamma(s,t,s',t')\big\{\delta_{\sigma_i,\sigma_i{}'}(s,s')\delta_{\sigma_j,\sigma_j{}'}(t,t') \\
&\qquad - \delta_{\sigma_i,\sigma_i{}'}(s,s')\kappa_j(t,t') - \kappa_i(s,s')\delta_{\sigma_j,\sigma_j{}'}(t,t')\big\}\Big]
\end{aligned}$$

In order to factorize this we subtract the first term in p.

$$\varphi'(x) + \tfrac{1}{4}\sum_{i,j} a_{ij}\left[\gamma(\pi_i,\pi_j) - \Gamma(\kappa_i;\kappa_j)\right]$$

$$= \frac{1}{4}\sum_{ij} a_{ij}\nu_x\Bigg[\sum_{s,t}\gamma(s,t)\cdot[\delta_{\sigma_i}-\pi_i](s)\cdot[\delta_{\sigma_j}-\pi_j](t)$$

$$- \sum_{s,t,s',t'}\Gamma(s,t,s',t')\cdot[\delta_{\sigma_j,\sigma_j{}'}-\kappa_j](t,t')\cdot[\delta_{\sigma_i,\sigma_i{}'}-\kappa_i](s,s')\Bigg]$$

$$= \frac{1}{4}\sum_{s,t}\gamma(s,t)\sum_i \nu_x\Bigg[[\delta_{\sigma_i}-\pi_i](s)\sum_j a_{ij}[\delta_{\sigma_j}-\pi_j](t)\Bigg]$$

$$- \frac{1}{4}\sum_{s,t,s',t'}\Gamma(s,t,s',t')\sum_i$$

$$\nu_x\Bigg[[\delta_{\sigma_i,\sigma_i{}'}-\kappa_i](s,s')\sum_j a_{ij}[\delta_{\sigma_j,\sigma_j{}'}-\kappa_j](t,t')\Bigg]$$

We will have to study the two terms in this last expression.

$$
\begin{aligned}
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Outline

Upper bound

Let $M = (m_{ij})_{ij} := \sum_{n\geq 0}(LC)^n$ and $w_i := \sum_j m_{ij}\big(\|c^{(j)}\|_2\big)$.

Lemma

For each $i \leq N$ and for each Hamiltonian $H \in \mathcal{H}$ we have:

$$\Big|\nu_H\Big[\big(\delta_{\sigma_i}(s) - \pi_i(s)\big)\sum_j a_{ij}\big(\delta_{\sigma_j}(t) - \pi_i(t)\big)\Big]\Big| \leq Lw_i^2$$

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Because $w_i^2 = O(\frac{1}{N})$ if β small enough, this implies our statement about the free energy.

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Cavity

For now we take $i = N$.

In order to create the cavity we fade out all interactions with σ_N.

We want to stay in $\mathcal{H}$, so define $a_{ij}(x)$, $b_i^{(j)}(x)$, $x \in [0,1]$:

$$a_{ij}(x) := \begin{cases} a_{ij} & i < j < N \\ xa_{ij} & i < j = N \end{cases},$$

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What shall we look at?

We will need the calculation a bit more apt. Let $\alpha_1, \ldots, \alpha_N \geq 0$ be a sequence. Then:

$$F(\sigma) := \big(\delta_{\sigma_N}(\hat{s}) - \pi_N(\hat{s})\big) \sum_{i \neq N} \alpha_i \big(\delta_{\sigma_i}(\hat{s}) - \pi_i(\hat{s})\big)$$

$$G(\sigma^1, \sigma^2) := \big(\delta_{\sigma^1_N, \sigma^2_N}(\hat{s}, \hat{s}') - \kappa_N(\hat{s}, \hat{s}')\big) \sum_{i \neq N} \alpha_i \big(\delta_{\sigma^1_i, \sigma^2_i}(\hat{s}, \hat{s}') - \kappa_i(\hat{s}, \hat{s}')\big)$$

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Isolated Spin

It is clear that:

$$\varphi(0) = \alpha_N \nu_0 \left[\left(\delta_{\sigma_N}(\hat{t}) - \pi_N(\hat{t}) \right)^2 \right] + \nu_0 \left[\left(\delta_{\sigma_N}(\hat{t}) - \pi_N(\hat{t}) \right) \sum_{i<N} \alpha_i \left(\delta_{\sigma_i}(\hat{s}) - \pi_i(\hat{s}) \right) \right]$$

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At $x = 0$ the N-th spin is decoupled from the others and is distributed by κ_N. Therefore in both cases the last summand vanishes and we have $|\varphi(0)| \leq \alpha_N$ and $|\psi(0)| \leq \alpha_N$.

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Inner differentiation

On the other hand after the same procedure we had before we get again two additional replicas:

$$\varphi'(x) = \nu_x \left[F(\sigma^1) \frac{1}{2} \sum_{i<N} a_{iN} \left\{ v(1) - v(2) - 2v(1,2) + 2v(2,3) \right\} \right]$$

$$\psi'(x) = \nu_x \left[G(\sigma^1, \sigma^2) \Big\{ \frac{1}{2} \sum_{i<N} a_{iN} \Big(v(1) + v(2) - 2v(3) \right.$$

$$\left. + 2v(1,2) - 4v(1,3) - 4v(2,3) + 6v(3,4) \Big) \Big\} \right]$$

$$v(l) := \gamma(\sigma_i^l, \sigma_N^l) - \gamma(\sigma_i^l, \pi_N) - \gamma(\sigma_N^l, \pi_i)$$

$$v(l, l') := \Gamma(\sigma_i^l, \sigma_N^l, \sigma_i^{l'}, \sigma_N^{l'}) - \Gamma^{(N)}(\sigma_i^l, \sigma_i^{l'}) - \Gamma^{(i)}(\sigma_N^l, \sigma_N^{l'})$$

Again we can factorize the result. We use the following notation, where dependence in s, t, s', t' and in i are implicit:

$$\bar{v}(l) := \sum_{i<N} a_{iN}\left[\left(\delta_{\sigma_i^l}(s) - \pi_i(s)\right)\cdot\left(\delta_{\sigma_N^l}(s) - \pi_N(t)\right)\right]$$

$$\bar{v}(l,l') := \sum_{i<N} a_{iN}\left[\left(\delta_{\sigma_i^l\sigma_i^{l'}}(s,s') - \kappa_i(s,s')\right)\cdot\left(\delta_{\sigma_N^l\sigma_N^{l'}}(s,s') - \kappa_N(s,s')\right)\right]$$

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$$\varphi'(x) = \frac{1}{2}\sum_{s,t}\gamma(s,t)\nu_x\left[F(\sigma)\cdot\left(\bar{v}(1)-\bar{v}(2)\right)\right]$$

$$-\frac{2}{2}\sum_{s,t,s',t'}\Gamma(s,t,s',t')\nu_x\left[F(\sigma)\cdot\left(\bar{v}(1,2)-\bar{v}(2,3)\right)\right]$$

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$$\psi'(x) = \frac{1}{2}\sum_{s,t} \gamma(s,t)\nu_x\left[G(\sigma,\sigma') \cdot \left(\bar{v}(1) + \bar{v}(2) - 2\bar{v}(2)\right)\right]$$

$$- \sum_{s,t,s',t'} \Gamma(s,t,s',t')\nu_x\left[G(\sigma,\sigma') \cdot \left(\bar{v}(1,2) - 2\bar{v}(1,3)\right.\right.$$

$$\left.\left. - 2\bar{v}(2,3) + 3\bar{v}(3,4)\right)\right]$$

Using Cauchy Inequality

We want to use Cauchy inequality to estimate this quantity and therefore define:

$$U(\alpha) := \max_{s} \sup_{H} \sqrt{\nu_H \left\{ \left(\sum_i \alpha_i [\delta_{\sigma_i}(s) - \pi_i(s)] \right)^2 \right\}}$$

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$$|\nu_x [F(\sigma) \cdot \bar{v}(I)]| \leq U(\alpha) \cdot U(a_{\bullet N})$$
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Wrap Up

We have now:

$$|\varphi'(x)| \leq 2 \cdot K \cdot U(\alpha) \cdot \Big(U(a_{\bullet N}) + V(a_{\bullet N}) \Big)$$

$$|\psi'(x)| \leq 8 \cdot K \cdot V(\alpha) \cdot \Big(U(a_{\bullet N}) + V(a_{\bullet N}) \Big)$$

where $K := \sum_{s,t,s',t'} |\Gamma(s,t,s',t')|$.

Summerizing we get:

$$|\varphi(1)| \leq \alpha_N + 2 \cdot K \cdot U(\alpha) \cdot \big(U(a_{\bullet N}) + V(a_{\bullet N}) \big)$$

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We could have isolated every spin. . .

Now forget our assumption $i = N$. Let thus:

$$F_i(\sigma) := \big(\delta_{\sigma_i}(\hat{s}) - \pi_i(\hat{s})\big) \sum_{j \neq i} \alpha_j \big(\delta_{\sigma_j}(\hat{s}) - \pi_j(\hat{s})\big)$$

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The calculations would have been the same:

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Observe that those inequations are uniform in $\hat{s}, \hat{s}'$.

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Final inequality

Now $U(\alpha)^2 = \sum_i \alpha_i \varphi_i(1)$ and $V(\alpha)^2 = \sum_i \alpha_i \psi_i(1)$.

$$U(\alpha)^2 \leq \sum_i \alpha_i^2 + 2 \cdot K \cdot U(\alpha) \sum_i |\alpha_i| \Big(U(a_{\bullet i}) + V(a_{\bullet i}) \Big)$$

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And using $x^2 \leq Ax + B \Rightarrow x \leq A + \sqrt{B}$ we get the uniform recursive inequality:

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Now $U(\alpha)^2 = \sum_i \alpha_i \varphi_i(1)$ and $V(\alpha)^2 = \sum_i \alpha_i \psi_i(1)$.

$$U(\alpha)^2 \leq \sum_i \alpha_i^2 + 2 \cdot K \cdot U(\alpha) \sum_i |\alpha_i| \Big(U(a_{\bullet i}) + V(a_{\bullet i}) \Big)$$

$$V(\alpha)^2 \leq \sum_i \alpha_i^2 + 8 \cdot K \cdot V(\alpha) \sum_i |\alpha_i| \Big(U(a_{\bullet i}) + V(a_{\bullet i}) \Big)$$

And using $x^2 \leq Ax + B \Rightarrow x \leq A + \sqrt{B}$ we get the uniform recursive inequality:

$$\max\{U(\alpha), V(\alpha)\} \leq \sqrt{\sum_i \alpha_i^2} + 8 \cdot K \cdot \sum_i |\alpha_i| \Big(U(a_{\bullet i}) + V(a_{\bullet i}) \Big)$$

Recursive inequality

So if we set:

$$v_i := \frac{1}{2}\left(\sup U(\beta) + \sup V(\beta) \right),$$

where the suprema are over all sequences β, s.t. $0 \leq \beta_j \leq c_j^{(i)}$, we get recursively:

$$v_i \leq \sum_j m_{ij}\left(\|a_{\bullet j}\|_2\right) =: w_i$$

The Lemma then follows from applying this and finishes our proof.

Recursive inequality

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