

Exact asymptotics in the parabolic Anderson model

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The parabolic Anderson model

Consider the following Cauchy-problem:

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = \kappa \Delta u(t, x) + \xi(\omega, x) u(t, x), & (\omega, t, x) \in \Omega \times \mathbb{R}_+ \times \mathbb{Z}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{Z}^d. \end{cases}$$

- $\kappa > 0$ denotes a diffusion constant
- u_0 is a nonnegative function
- $\Delta f(x) := \sum_{y:|y-x|=1} [f(y) - f(x)]$ is the discrete Laplacian
- $\xi: \Omega \times \mathbb{Z}^d \rightarrow \mathbb{R}$ is a random potential/medium.

Feynman-Kac Representation

Feynman-Kac representation

$$u(t, x) = \mathbf{E}_x \exp \left\{ \int_0^t \xi(X_s) ds \right\} u_0(X_t).$$

Here X denotes a simple symmetric random walk with generator $\kappa\Delta$.

The Setting

Statistical moments

$$\langle u(t, 0)^p \rangle, \quad p \in \mathbb{N}.$$

$\langle \cdot \rangle$ denotes expectation with respect to the medium.

- Let $u_0 \equiv 1$.
- Let $\{\xi(x), x \in \mathbb{Z}^d\}$ be a field of i.i.d. nonnegative random variables
- Let $\bar{F}(h) := \mathbb{P}(\xi(0) > h)$ denote the tail of $\xi(0)$
- Let $H(t) := \log \langle e^{t\xi(0)} \rangle$ be the cumulant generating function of $\xi(0)$
- We assume

$$H(t) < \infty \text{ for all } t \geq 0.$$

$$e^{H(pt) - 2d\kappa pt} \leq \langle u(t, 0)^p \rangle \leq e^{H(pt)} \quad (\text{Gärtner, Molchanov, 1990})$$

Assumptions

Assumption (F): If $x \neq y$ then

$$\mathbb{P} \left(\frac{\xi(x) + \xi(y)}{2} > h - c \right) = o(\bar{F}(h)), \quad h \rightarrow \infty, \quad \text{for all } c > 0.$$

This implies

Assumption (\bar{F}):

$$(\bar{F}(h - c))^2 = o(\bar{F}(h)), \quad h \rightarrow \infty, \quad \text{for all } c > 0.$$

Special case: Weibull distribution

$$\bar{F}(h) = \exp\{-h^\gamma\}, \quad \gamma > 1.$$

Boundary Conditions

$$\begin{cases} \frac{\partial}{\partial t} u_R(t, x) = \kappa \Delta u_R(t, x) + \xi(x) u_R(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{T}_R^d, \\ u_R(0, x) = 1, & x \in \mathbb{T}_R^d. \end{cases}$$

- Let \mathbb{T}_R^d be the d -dimensional centered lattice cube of length $2[R] + 1$
- Let u_R^f denote the solution with free boundary conditions
- Let u_R^0 denote the solution with zero boundary conditions.
- Let $R_t := t \log^2 t$.

Lemma (Gärtner, Molchanov, 1997)

Almost surely,

$$u_{R_t}^*(t, 0) \sim u(t, 0), \quad t \rightarrow \infty.$$

Spectral Representation

- $\mathcal{H} := \kappa\Delta + \xi$ on $\ell^2(\mathbb{T}_R^d)$
- Eigenvalues $\lambda_1^{R,0} > \lambda_2^{R,0} \geq \dots$; $\lambda_1^{R,f} > \lambda_2^{R,f} \geq \dots$
- Corresponding orthonormal basis of eigenfunctions $e_1^{R,0}, e_2^{R,0}, \dots$; $e_1^{R,f}, e_2^{R,f}, \dots$

Spectral representation

$$u_R^*(t, x) = \sum_{k=1}^{|\mathbb{T}_R^d|} e^{\lambda_k^{R,*} t} \left(e_k^{R,*}, \mathbb{1} \right) e_k^{R,*}(x).$$

Lemma

Almost surely,

$$\max \left(\lambda_1^{R,f}, \lambda_1^{R_t \setminus R,f} \right) \geq \lambda_1^{R_t,*} \geq \lambda_1^{R,0}.$$

Principal Eigenfunctions

Probabilistic representation of $e_1^{R,*}$ (if $\max_{x \in \mathbb{T}_R^d} \xi(x) = \xi(0)$)

$$\begin{cases} \kappa \Delta e_1^{R,f}(x) + (\xi(x) - \lambda_1^{R,f}) e_1^{R,f}(x) = 0, & x \in \mathbb{T}_R^d \setminus \{0\}, \\ e_1^{R,f}(0) = 1. \end{cases}$$

Hence

$$e_1^{R,f}(x) = \mathbf{E}_x e^{\int_0^{\tau_0} (\xi(X_s) - \lambda_1^{R,f}) ds}.$$

Analogously

$$e_1^{R,0}(x) = \mathbf{E}_x e^{\int_0^{\tau_0} (\xi(X_s) - \lambda_1^{R,f}) ds} \mathbb{1}_{\tau_0 < \tau_{(\mathbb{T}_R^d)^c}}.$$

Theorem for Moments

Theorem (Gärtner, S.)

Let Assumption (F) be satisfied. Then for every $R > 0$, $p \in \mathbb{N}$ and $0 < \underline{C} < 1 < \overline{C} < \infty$ there exist $c_0, t_0 > 0$ such that for every $c > c_0$ and $t > t_0$,

$$\begin{aligned} & \underline{C} p t \int_c^\infty e^{pth} \mathbb{P} \left(\lambda_1^{R,0}(\xi) > h \left| \max_{y \in \mathbb{T}_R^d \setminus \{0\}} \xi(y) \leq h - c \right. \right) dh \\ & \leq \langle u(t, 0)^p \rangle \\ & \leq \overline{C} p t \int_c^\infty e^{pth} \mathbb{P} \left(\lambda_1^{R,f}(\xi) > h \left| \max_{y \in \mathbb{T}_R^d \setminus \{0\}} \xi(y) \leq h - c \right. \right) dh. \end{aligned}$$

Generalized Theorem

The result can be generalized to expressions such as

$$\left\langle \prod_{i=1}^p f_i(u(t_i(t), 0)) \right\rangle,$$

with monotonically increasing $f_i, t_i \in C_1$ where the f_i are regularly varying and

$$\max_{1 \leq j \leq p} e^{t_j(t)a} = o\left(\min_{1 \leq i \leq p} \left\langle f_i\left(e^{t_i(t)\xi}\right) \right\rangle\right), \quad \text{for all } a \geq 0, t \rightarrow \infty.$$

Proof (Idea)

Upper bound.

1.

$$\langle u_{R_t}^f(t, 0)^p \mathbb{1}_{\xi^{(1)} - \xi^{(2)} \leq c} \rangle \stackrel{(F)}{=} o(\langle u_{R_t}^f(t, 0)^p \rangle), \quad \text{for all } c > 0$$

2.

$$e_1^{R_t, f}(x) \leq Ce^{-|x|K(c)}, \quad \lim_{c \rightarrow \infty} K(c) = \infty$$

3.

$$\langle u_{R_t}^f(t, 0)^p \rangle \leq \frac{1}{|\mathbb{T}_{R_t}^d|} \left\langle e^{pt\lambda_1^{R, f}} \left| \xi_{R_t}^{(1)} = \xi(0) \right. \right\rangle, \quad t \rightarrow \infty$$

4.

$$\begin{aligned} & \left\langle e^{pt\lambda_1^{R, f}} \left| \xi_{R_t}^{(1)} = \xi(0) \right. \right\rangle \\ & \approx |\mathbb{T}_{R_t}^d| \int_c^\infty e^{pth} \mathbb{P} \left(\lambda_1^{R, f}(\xi) > h \left| \max_{y \in \mathbb{T}_R^d \setminus \{0\}} \xi(y) \leq h - c \right. \right) dh. \end{aligned}$$

Weibull Tails

$$\bar{F}(h) = \exp\{-h^\gamma\}, \quad \gamma > 1$$

$$\begin{aligned} & \mathbb{P} \left(\lambda_1^{R,f}(\xi) > h \mid \max_{y \in \mathbb{T}_R^d \setminus \{0\}} \xi(y) \leq h - c \right) \\ & \sim \exp \left\{ -h^\gamma + 2d\kappa^2 \gamma h^{\gamma-2} + \mathcal{O} \left(h^{\gamma-1-(3+\gamma)/\gamma} \right) \right\}, \quad h \rightarrow \infty. \end{aligned}$$

$$1 < \gamma < 2: \quad \langle u(t, 0)^p \rangle$$

$$\sim \exp \left\{ (\gamma - 1) \left(\frac{p}{\gamma} t \right)^{\frac{\gamma}{\gamma-1}} - 2d\kappa p t + \log p t + \frac{1}{2} \log \frac{2\pi}{\gamma(\gamma - 1) \left(\frac{p}{\gamma} t \right)^{\frac{\gamma}{\gamma-1}}} \right\}.$$

Ageing

Heuristics

A system ages if the time it spends in a certain state increases with the age of the system.

Our Definition

Let f be monotonically increasing to infinity and let

$$C_f(s) := \lim_{t \rightarrow \infty} \text{Corr}(f(u(t, 0)), f(u(t + s(t), 0))), \quad \lim_{t \rightarrow \infty} s(t) = \infty.$$

The system ages if C_f is not constant.

Ageing Theorem

Let $\langle u(0, t) \rangle = e^{h(t)+l(t)+k(t)}$, l a linear function, $l = o(h)$ and $k = o(l)$.

Lemma (Gärtner, S.)

Let $\{\xi(x), x \in \mathbb{Z}^d\}$ be a field of i.i.d. random variables such that $H(t) < \infty$ for all $t \geq 0$. Furthermore, let $k \in C^2$ such that $\lim_{t \rightarrow \infty} k''(t)$ exists.

Then the PAM ages for $f = \text{id}$ iff $\lim_{t \rightarrow \infty} h''(t) = 0$.

Theorem (Gärtner, S.)

Let $\xi(0) \sim \text{Weibull}(\gamma)$, $\gamma > 1$. Then the PAM ages for regularly varying $f \in C_1$ iff $\gamma > 2$.

Some Further Challenges

- $\langle f(u(t, x)) \rangle$ if f is slowly varying
- “quenched” ageing
- $u_0 = \delta_0$