

Scaled Limit and Rate of Convergence for the Largest Eigenvalue from the Generalized Cauchy Random Matrix Ensemble

Felix Rubin

University of Zurich

September 03, 2009

Outline

1 Introduction

- Types of Models

2 The Gaussian Unitary Ensemble

- Eigenvalues, Point Processes and Fredholm Determinant
- Scaling of the GUE

3 The Generalized Cauchy Ensemble

- Definition
- Ergodic Measures and Representation Theory

4 Results

- The Convergence in Law of the Rescaled Largest Eigenvalue
- The Painlevé-V Representation of the Scaling Limit
- Idea of Proofs

Introduction

- Interest: Study eigenvalues of (large) random matrices.
 - ▶ In particular the largest eigenvalue.
 - ▶ Source of interesting distributions, occurring in different fields of mathematics and physics.
- Started with Physicists (Wigner) in the 50's.
 - ▶ Model to understand statistical behavior of slow neutron resonances.
 - ▶ Applications in statistical mechanics (eg. limit laws of random growth models, tilting problems).
- 70's: Applications to number theory (Montgomery).

Two Types of Models

Essentially two types of matrix models:

- Choose each entry of a $N \times N$ matrix (independently) according to some given distribution.
 - ▶ eg. GUE, GOE, GSE.
- Choose a compact group and endow it with its normalized Haar measure.
 - ▶ eg. CUE (Dyson Ensemble).

The GUE

Definition (see eg. Mehta, 1991)

A random $N \times N$ Hermitian matrix belongs to the *GUE*, if the diagonal elements x_{jj} and the upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ ($j < k$) are chosen independently with normal densities of the form:

$$\frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} \sim \mathcal{N}\left(0, \frac{1}{2}\right) \text{ (diagonal),}$$

$$\frac{2}{\pi} e^{-2(u_{jk}^2 + v_{jk}^2)} \sim \mathcal{N}\left(0, \frac{1}{4}\right) + i\mathcal{N}\left(0, \frac{1}{4}\right) \text{ (upper triangular)}$$

Joint density function:

$$\rho(X) = \prod_{j=1}^N \frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} \prod_{1 \leq j < k \leq N} \frac{2}{\pi} e^{-2|x_{jk}|^2} = \frac{1}{Z_N} \exp\{-\text{Tr}(X^2)\}.$$

Eigenvalues and Point Processes I

- Eigenvalue distribution?
- Apply basis transformation and integrate out elements independent of the eigenvalues:

Eigenvalue measure on \mathbb{R}^N : If $x_1, \dots, x_N \in \mathbb{R}$,

$$\begin{aligned}
 u_N(x_1, \dots, x_N) &= \frac{1}{Z_N N!} \prod_{1 \leq j < k \leq N} |x_j - x_k|^2 \exp \left(- \sum_{j=1}^N x_j^2 \right) \\
 &= \frac{1}{Z_N N!} \left(\det(p_{j-1}(x_i) e^{(-x_i^2)/2})_{1 \leq i, j \leq N} \right)^2 \\
 &= \frac{1}{N!} \det(K_N(x_i, x_j))_{i, j=1}^N,
 \end{aligned}$$

Eigenvalues and Point Processes II

where

$$\begin{aligned}
 K_N(x, y) &= \sum_{j=0}^{N-1} p_j^H(x) p_j^H(y) e^{-\frac{x^2+y^2}{2}} \\
 &= \text{cst} \cdot e^{-(x^2+y^2)/2} \frac{p_N^H(x) p_{N-1}^H(y) - p_N^H(y) p_{N-1}^H(x)}{x - y}.
 \end{aligned}$$

p_i^H : normalized Hermite polynomial of degree i .

Definition

The n -th correlation function ρ_n is defined by:

$$\begin{aligned}
 \rho_n(x_1, \dots, x_n) &= \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} u_N(x_1, \dots, x_N) dx_{n+1} \cdots dx_N \\
 &= \det(K_N(x_i, x_j))_{i,j=1}^n, \text{ for } n \leq N.
 \end{aligned}$$

Eigenvalues and Point Processes III

The eigenvalue distribution can be viewed as a point process on \mathbb{R} via the application $(x_1, \dots, x_N) \mapsto \sum_{i=1}^N \delta_{x_i}$. Point processes with a correlation function of this determinantal form are called *determinantal point processes*.

Fredholm Determinant and Law of the Largest Eigenvalue

Problem: Describe probabilities of the form

$$E(k, J) = P[\text{there are exactly } k \text{ eigenvalues in } J].$$

- $K_N(x, y)$ is the kernel of a trace class integral operator K_N on $L^2(\mathbb{R})$:

$$K_N f(x) = \int_{\mathbb{R}} K_N(x, y) f(y) dy, \quad \text{for } f \in L^2(\mathbb{R}).$$

- Fredholm determinant ($\phi \in L^\infty(J)$):

$$\det(I + K_N \phi)|_J = \sum_{n \geq 0} \frac{1}{n!} \int_{J^n} \prod_{j=1}^n \phi(x_j) \rho_n(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

Now:

- $E(k, J) = \frac{(-1)^k}{k!} \frac{d^k}{d\lambda^k} \det(I - \lambda K_N|_J)|_{\lambda=1}$.
- Law of the largest eigenvalue $\lambda_1(N)$:

$$P[\lambda_1(N) \leq t] = E(0, (t, \infty)) = \det(I - K_N)|_{(t, \infty)}.$$

GUE: Airy Kernel and Painlevé-II

Scale around the largest eigenvalue $\lambda_1(N)$ of the GUE, to obtain for $N \rightarrow \infty$:

$$P \left[\lambda_1(N) \leq \sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}} \right] \longrightarrow F_{TW}(s) = \det(\text{Id} - K_{\text{Airy}})|_{L_2(s, \infty)},$$

where
$$F_{TW}(s) = \exp \left(- \int_s^\infty (x - s) q^2(x) dx \right).$$

q is the solution of a Painlevé-II equation $q'' = sq + 2q^3$ with boundary condition $q(s) \sim \text{Ai}(s)$ for $s \rightarrow \infty$.

(see Tracy, Widom 1994)

The GCyE I

Setting: Unitary group $U(N)$ with *Haar measure* μ_N , complex parameter s , $\Re(s) \geq -1/2$.

- Deformed Haar measure: $\text{cst} \cdot \det((I - U)^{\bar{s}}) \det((I - U^*)^s) \mu_N(dU)$.
- Eigenvalue measure on $\mathbb{S}^N/S(N)$:

$$\text{cst} \cdot \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^N (1 - e^{i\theta_j})^{\bar{s}} (1 - e^{-i\theta_j})^s.$$

Cayley transform: $U(N) \longleftrightarrow H(N)$, via $X \in H(N) \mapsto \frac{X+i}{X-i} \in U(N)$.

- *Generalized Cauchy measure* on $H(N)$:

$$\text{cst} \cdot \det((1 + iX)^{-s-N}) \det((1 - iX)^{-\bar{s}-N}) \prod_{1 \leq j < k \leq N} dX_{jk} \prod_{i=1}^N dX_{ii}.$$

The GCyE II

- Eigenvalue measure on $\mathbb{R}^N/S(N)$:

$$m_N^s = \text{cst} \cdot \prod_{1 \leq j < k \leq N} (x_j - x_k)^2 \prod_{j=1}^N w_H(x_j),$$

where $w_H(x_j) = (1 + ix_j)^{-s-N}(1 - ix_j)^{-\bar{s}-N}$.

⇒ As in the GUE case, this eigenvalue process is determinantal.

The Kernel and its Scaling Limit

- *The kernel:* $K_N(x, y) = \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x - y}$,
 - ▶ where $\phi(x) = \text{cst} \cdot \sqrt{w_H(x)} p_N(x)$, and $\psi(x) = \text{cst} \cdot \sqrt{w_H(x)} p_{N-1}(x)$, and
 - ▶ $p_m(x) = (x - i)^m {}_2F_1[-m, s + N - m, 2\Re(s) + 2N - 2m; 2/(1 + iX)]$ are the *pseudo Jacobi polynomials*.
 - ▶ The *Gauss Hypergeometric function*: ${}_2F_1[a, b, c; x] = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n n!} x^n$, and $(a)_n = a \cdot \dots \cdot (a + n - 1)$.
- *The scaling limit:* $NK_N(Nx, Ny) \longrightarrow K_\infty(x, y)$, when $N \rightarrow \infty$,
 - ▶ where $K_\infty(x, y) = \text{cst} \cdot \frac{\tilde{P}(x)Q(y) - Q(x)\tilde{P}(y)}{x - y}$, and
 - ▶ $\tilde{P}(x) = |2/x|^{\Re(s)} e^{-i/x + \pi \Im(s) \text{Sgn}(x)/2} {}_1F_1[s, 2\Re(s) + 1; 2i/x]$, and $Q(x) = (2/x) |2/x|^{\Re(s)} e^{-i/x + \pi \Im(s) \text{Sgn}(x)/2} {}_1F_1[s + 1, 2\Re(s) + 2; 2i/x]$.
 - ▶ The *Confluent Hypergeometric function*: ${}_1F_1[r, q; x] = \sum_{n \geq 0} \frac{(r)_n}{(q)_n n!} x^n$.

(see Borodin, Olshanski 2001)

U -invariant Measures on H , Ergodicity I

Definition

Assume there is a group acting on a Borel space. A probability measure on the Borel space which is invariant under this action is called *ergodic*, if any invariant set has either measure 0 or 1.

- If the group is compact and the action continuous, the ergodic measures are exactly the orbital measures. I.e. invariant measures supported by orbits.
- Consider the group $U = \bigcup_N U(N)$ and the space H of infinite Hermitian matrices. U acts on H by conjugation. m_N^s are consistent with natural projections and define a two parameter family of U -invariant probability measures m^s on H .

U -invariant Measures on H , Ergodicity II

- U -invariant probability measures can be written as continual convex combinations of ergodic measures (*spectral decomposition*). This decomposition is determined by a *spectral measure* on the space of point configurations $Conf(\mathbb{R} \setminus \{0\})$.

⇒ The spectral measure of m^s defines a determinantal point process on $\mathbb{R} \setminus \{0\}$ with kernel K_∞ .

(see Borodin, Olshanski 2001)

Irreducible Spherical Representation of $U \ltimes H$

- Set $G(N) = U(N) \ltimes H(N)$, $U(N)$ acting on $H(N)$ (additive group) by conjugation. Also, $G = \varinjlim G(N)$.
- A unitary representation T of G is *spherical* if it possesses a cyclic unit vector ξ which is invariant wrt the subgroup $U \subset G$.
- One to one correspondence of equivalence classes of spherical representation (T, ξ) and U -invariant probability measures on H . I.e. irreducible spherical representations are parametrized by ergodic measures on H .

(see Borodin, Olshanski 2001)

Convergence of the Rescaled Largest Eigenvalue I

Set $F_N(t) = \det(I - K_N)|_{(Nt, \infty)}$ the Fredholm determinant of the rescaled kernel $NK_N(Nx, Ny)$, respectively $F_\infty(t) = \det(I - K_\infty)_{(t, \infty)}$, where $t > 0$.

Theorem (NNR, 2009)

For s such that $\Re(s) > -1/2$, and $t > 0$, F_N and F_∞ are in $\mathcal{C}^3(\mathbb{R}_+^, \mathbb{R})$ and for $p \in \{0, 1, 2, 3\}$, the p -th derivative (wrt t) of F_N converges pointwise to the p -th derivative of F_∞ .*

Convergence of the Rescaled Largest Eigenvalue II

Corollary (NNR, 2009)

The law of $\lambda_1(N)/N$ converges to the distribution of the largest point of the determinantal process on \mathbb{R}^ described by the limiting kernel $K_\infty(x, y)$:*

$$P \left[\frac{\lambda_1(N)}{N} \leq x_0 \right] = \det(I - K_N)|_{(Nx_0, \infty)} \longrightarrow \det(I - K_\infty)|_{(x_0, \infty)}, \text{ as } N \rightarrow \infty,$$

for any $x_0 > 0$.

Convergence of the Rescaled Largest Eigenvalue III

Theorem (NNR, 2009)

For all $x_0 > 0$, and for $x > x_0$,

$$\left| P \left[\frac{\lambda_1(N)}{N} \leq x \right] - \det(I - K_\infty)|_{(x, \infty)} \right| \leq \frac{1}{N} C(x_0, s),$$

where $C(x_0, s)$ is a constant depending only on x_0 and s .

The Painlevé-V equation

Define $\theta_\infty(\tau) = \tau \frac{d \log \det(I - K_\infty)|_{(\tau^{-1}, \infty)}}{d\tau}$, $\tau > 0$.

Theorem (NNR, 2009)

Let s be such that $\Re(s) > -1/2$. then, θ_∞ is well defined and is a solution of the Painlevé-V equation on \mathbb{R}_+^* :

$$-\tau^2(\theta''(\tau))^2 = [2(\tau\theta'(\tau) - \theta(\tau)) + (\theta'(\tau))^2 + i(\bar{s} - s)\theta'(\tau)]^2 - (\theta'(\tau))^2(\theta'(\tau) - 2is)(\theta'(\tau)) + 2i\bar{s}.$$

This implies the result of Jimbo, Miwa, Mōri and Sato (1980) that the sine kernel (case $s = 0$) satisfies the Painlevé-V equation with $s = 0$.

Scaling Limit I

Lemma (NNR, 2009)

Let $x, y \in \mathbb{R}^*$ and let $\Re s > -1/2$. Then $K_{[N]}$ and K_∞ are C^∞ in $(\mathbb{R}^*)^2$ and for all $p, q \in \mathbb{N}_0$,

$$(\text{Sgn}(xy))^N \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x, y) \xrightarrow{N \rightarrow \infty} \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_\infty(x, y).$$

Moreover, for any $x_0 > 0$, and $|x|, |y| \geq x_0 > 0$:

$$\left| \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x, y) \right| \leq \frac{C(x_0, s, p, q)}{|x|^{\Re s + p + 1} |y|^{\Re s + q + 1}}.$$

- $\rho_{k,N}(x_1, \dots, x_k) = \det(K_{[N]}(x_i, x_j))_{i,j=1}^k$ and its scaling limit $\rho_{k,\infty}$ are in C^3 , and $\rho_{k,N}$ and its derivatives converge pointwise to the corresponding derivatives of $\rho_{k,\infty}$.

Scaling Limit II

- For all $k, p \in \{0, 1, 2, 3\}$, $N \in \mathbb{N} \cup \{\infty\}$ and $t \geq x_0$,

$$\sup_{t \geq x_0} \left| \frac{d^p}{dt^p} \frac{(-1)^k}{k!} \int_{(t, \infty)^k} \rho_{k,N}(x_1, \dots, x_k) dx_1 \cdots dx_k \right| \leq \frac{C(x_0)^k}{(k-1)!}.$$

- F_N and F_∞ are well defined and \mathcal{C}^3 (dominated convergence).
- Again by dominated convergence, F_N and its partial derivatives converge pointwise to F_∞ and its partial derivatives.

Painlevé-V

- Show that $F_\infty(t) > 0, \forall t > 0$. Then, θ_∞ is well defined.
- $F_N(t) > 0, \forall t > 0$ (as $P[\lambda_1(N) \leq Nt] > 0$). Thus,

$$\theta_N(\tau) = \tau \frac{d}{d\tau} \log(F_N(\tau^{-1}))$$

is well defined.

- The derivatives of θ_N converge pointwise to the derivatives of θ_∞
- By Forrester, Witte (2002), θ_N satisfies a Painlevé-VI equation. This equation is of the form

$$\sum_{k=0}^m N^{-k} \frac{P_k(\tau, \theta_N(\tau), \theta'_N(\tau), \theta''_N(\tau))}{\tau^q} = 0,$$

where m and q are universal integers and the P_k 's are polynomials which are independent of N . Moreover,

$P_0(\tau, \theta_N(\tau), \theta'_N(\tau), \theta''_N(\tau))\tau^{-q}$ corresponds to the σ -form of the Painlevé-V equation.