Differentiability of martingale driven BSDE and application to hedging in incomplete markets

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Motivation

Risk source: $n$-dimensional SDE

$$dR_s = \sigma(s, R_s) dM_s + b(s, R_s) dC_s$$

with $M$ $d$-dim. continuous local martingale, $d\langle M, M \rangle_s = q_s q_s^* dC_s$

Aim: price and hedge a derivative of the form $F(R_T)$

Correlated financial market: $k \leq d$ assets

$$dS_s = S_s(\beta(s, R_s) dM_s + \alpha(s, R_s) dC_s)$$

Preferences:

$$U(x) = -e^{-\eta x}, \quad 0 < \eta = \text{risk aversion}$$
Value function:

\[ V^F(x, t, r) = \sup_{\lambda} \mathbb{E} \left[ U(x + \sum_{i=1}^{k} \int_t^T \lambda_s^{(i)} \frac{dS_s^{(i)}}{S_s^{(i)}} - F(R^t_r)) \right] \]

What is a BSDE (M=W)?

A BSDE with terminal condition \( B \) and generator \( f \) is an equation of the type

\[ Y_t = B - \int_t^T Z_s dW_s + \int_t^T f(s, Y_s, Z_s) ds. \]

\[ \rightarrow \text{A solution is a pair of adapted processes } (Y, Z). \]
Theorem (Hu et al. ’05, Morlais ’08)

The value function satisfies

\[ V^F(x, t, r) = U(x - Y^F_{t}, t, r) \]

and the optimal strategy \( \pi^F \) is given by

\[ \pi^F_s = Z^F_{s, t}, t, r q^*_s \beta^*(\beta^*)^{-1}(s, R^t_{s}, r) + \frac{1}{\eta} \alpha^*(\beta^*)^{-1}(s, R^t_{s}, r), \]

where \((Y^F_{t}, t, r, Z^F_{t}, t, r)\) is the solution of a certain quadratic BSDE with terminal value \( F(R_T) \).
Indifference pricing and delta hedging

Decomposition:

\[ \pi^F = \pi^0 + \Delta = \text{pure investment} + \text{optimal hedge} \]

Indifference price:

\[ V^F(x - p(t, r), t, r) = V^0(x, t, r) \]

Theorem (Ankirchner et al. ’07)

Let \( M \) be the Brownian motion. The indifference price is given by

\[ p(t, r) = Y^0_{t, t, r} - Y^F_{t, t, r}. \]

The optimal hedge \( \Delta \) can be derived explicitly

\[ \Delta(t, r) = \left[ Z^0_{t, t, r} - Z^F_{t, t, r} \right] \beta^* (\beta \beta^*)^{-1}(t, r) \]

\[ = [-\partial_2 p(t, r) \sigma(t, r)] \beta^* (\beta \beta^*)^{-1}(t, r). \]
Theorem (Ankirchner et al. '07)

Let $f$ be of quadratic growth, i.e. $|f(\cdot, z)| \leq C(1 + |z|^2)$.

Assume that $\sigma$, $b$ are Lipschitz, have uniformly bounded partial derivatives, $f$ is differentiable in $r$, $z$, ...

Then

- **Markov property:**
  
  $$Y_{s}^{t,r} = u(s, R_{s}^{t,r}),$$

- **$Y^{t,r}$ is continuously differentiable in $r$ and Malliavin differentiable:**
  
  $$D_\theta Y_{s}^{t,r} = \partial_2 u(s, R_{s}^{t,r}) D_\theta R_{s}^{t,r}$$

- **Malliavin trace:**
  
  $$Z_{s}^{t,r} = D_{s} Y_{s}^{t,r} = \partial_2 u(s, R_{s}^{t,r}) \sigma(s, R_{s}^{t,r}).$$
Our aim

Brownian setting: If the coefficients $\sigma$, $b$ and $f$ are nice, then

$$Y_{s^{t,r}} = u(s, R_{s^{t,r}})$$
$$Z_{s^{t,r}} = \partial_2 u(s, R_{s^{t,r}}) \sigma(s, R_{s^{t,r}})$$

Question: Does this relation between $Y$ and $Z$ hold in other settings, f.e. in a continuous martingale setting?
What is a martingale driven BSDE?

- $M$ continuous $d$-dim. martingale, $(\mathcal{F}_t)$ cont. and complete and thus every martingale is of the form $Z \cdot M + L$
- $d\langle M, M \rangle_t = q_t q^*_t dC_t$
- $B$ $\mathcal{F}_T$-measurable r.v.
- $f: \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ Borel-measurable function

A BSDE with terminal value $B$ and generator $f$ is an equation

$$Y_t = B - \int_t^T Z_s dM_s - \int_t^T dL_s + \frac{\eta}{2} \int_t^T d\langle L, L \rangle_s$$
$$+ \int_t^T f(s, Y_s, Z_s) dC_s.$$

A solution is a triple of adapted processes $(Y, Z, L)$ such that the above equation makes sense.
Our Setting - Martingale driven FBSDE

- $M$ continuous $d$-dim. martingale

For $(x, m) \in \mathbb{R}^n \times \mathbb{R}^d$ and $t \in [0, T]$ we consider

$$X^{x,m}_t = x + \int_0^t \sigma(s, X^{x,m}_s, M^m_s) dM_s + \int_0^t b(s, X^{x,m}_s, M^m_s) dC_s$$

with solution processes $(X^{x,m}, Y^{x,m}, Z^{x,m}, L^{x,m})$. 

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Differentiability of martingale driven BSDE
Our Setting - Martingale driven FBSDE

- \(M\) continuous \(d\)-dim. martingale
- \(F\) is a bounded function
- \(f\) is quadratic in \(z\)

For \((x, m) \in \mathbb{R}^n \times \mathbb{R}^d\) and \(t \in [0, T]\) we consider

\[
X_t^{x, m} = x + \int_0^t \sigma(s, X_s^{x, m}, M_s^m) dM_s + \int_0^t b(s, X_s^{x, m}, M_s^m) dC_s
\]

\[
Y_t^{x, m} = F(X_T^{x, m}) - \int_t^T Z_r^{x, m} dM_r - \int_t^T dL_r^{x, m} + \frac{\eta}{2} \int_t^T d\langle L_r^{x, m}, L_r^{x, m} \rangle_r
\]

\[+ \int_t^T f(r, X_r^{x, m}, M_r^m, Y_r^{x, m}, Z_r^{x, m} q^*_r) dC_r\]

with solution processes \((X^{x, m}, Y^{x, m}, Z^{x, m}, L^{x, m})\).
The Markov property

Theorem (IRR ’09)

Let $M_{t,m}$ be a strong Markov process. Then there exist deterministic functions $u$ and $v$ such that for $s \in [t, T]$

$$Y_{s}^{t,x,m} = u(s, X_{s}^{t,x,m}, M_{s}^{t,m}), \quad Z_{s}^{t,x,m} = v(s, X_{s}^{t,x,m}, M_{s}^{t,m}).$$

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Differentiability of martingale driven BSDE
Differentiability of $Y$

**Difficulty:** Presence of $\langle L^x,m, L^x,m \rangle$ in

$$Y^x,m_t = F(X^x,m_T) - \int_t^T Z^x,m_r dM_r - \int_t^T dL^x,m_r + \frac{\eta}{2} \int_t^T d\langle L^x,m, L^x,m \rangle_r$$

$$+ \int_t^T f(r, X^x,m_r, M^m_r, Y^x,m_r, Z^x,m_r q^*_r) dC_r$$

Introduce assumption (MRP):
There exists a square-integrable martingale $N$ with $\langle M, N \rangle = 0$ and such that $(M, N)$ satisfies the martingale representation property.

$$\implies \exists \text{ process } U^x,m \text{ such that } L^x,m = U^x,m \cdot N$$
Theorem (IRR '09)

Let (MRP) be satisfied, $\partial_i f$ of linear growth in $z,...$

Then there exists a modification $(Y^{x,m}, Z^{x,m}, U^{x,m})$ such that

- $Y^{x,m}$ is continuously differentiable in $x$ and $m$,
- and there exist processes $\partial_x Z^{x,m}, \partial_m Z^{x,m}$ and $\partial_x U^{x,m}, \partial_m U^{x,m}$ such that the derivatives

$$ (\partial_x Y^{x,m}, \partial_x Z^{x,m}, \partial_x U^{x,m}) \text{ and } (\partial_m Y^{x,m}, \partial_m Z^{x,m}, \partial_m U^{x,m}) $$

solve BSDEs.
The representation formula

**Theorem (IRR '09)**

If

- $Y^{t,x,m}_s = u(s, X^{t,x,m}_s, M^{t,m}_s)$ and
- $Y^{t,x,m}$ continuously differentiable in $(x, m)$.

Then

$$Z^{t,x,m}_s = \partial_2 u(s, X^{t,x,m}_s, M^{t,m}_s) \sigma(s, X^{t,x,m}_s, M^{t,m}_s)$$

$$+ \partial_3 u(s, X^{t,x,m}_s, M^{t,m}_s).$$

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Differentiability of martingale driven BSDE
Hedging with stochastic correlation [Ankirchner, Heyne '09]

- $F(R_T) =$ derivative of risk $R$ with payoff function $F$
- $S =$ correlated traded asset

\[
dS_t = S_t (\mu_S dt + \sigma_S dW^1_t)
\]
\[
dR_t = R_t (\mu_R dt + \sigma_R (\rho_t dW^1_t + \sqrt{1-\rho_t^2} dW^2_t)),
\]

where $\rho$ is the stochastic correlation with dynamics

\[
d\rho_t = a(\rho_t) dt + g(\rho_t) (\gamma dW^1_t + \delta dW^2_t + \sqrt{1-\gamma^2-\delta^2} dW^3_t).
\]

**Aim:** Find a local risk minimizing hedge for $F(R_T)$. 
FS decomposition and BSDEs

- Standard method for deriving the local risk minimizing strategy $\pi$ is based on FS decomposition

\[ F(R_T) = C + \int_0^T \pi_u dS_u + L_T. \]

- Let $(Y, Z)$ be the solution of the linear BSDE

\[ Y_t = F(R_T) - \int_t^T Z_u dW_u - \int_t^T Z_u \frac{\mu_S}{\sigma_S} du. \]

Then FS decomposition of $F(R_T)$ is given by

\[ F(R_T) = Y_0 + \int_0^T \frac{Z_u^1}{\sigma_S S_u} dS_u + \int_0^T Z_u^2 dW_u^2 + \int_0^T Z_u^3 dW_u^3. \]
Optimal hedge

Hedging strategy $\pi = \frac{Z^1}{\sigma_S S} \rightsquigarrow$ Explicit description of $\pi$?

The solution of

$$Y_t = F(R_T) - \int_t^T Z_u dW_u - \int_t^T Z^1_u \frac{\mu_S}{\sigma_S} du.$$ 

can be described in terms of

$$u(t, x, \nu) = E^Q \left[ h(R^t_{T, x, \nu}) \right], \text{ i.e. } Y_t = u(t, R_t, \rho_t).$$

**Difficulty:** $x \mapsto u(t, x, \nu)$ is only locally Lipschitz continuous $\rightsquigarrow$ no (direct) access to the chain rule of Malliavin Calculus

With the representation formula

$$Z_t = \sigma(t, R_t, \rho_t)^* \begin{pmatrix} \partial_2 u(t, R_t, \rho_t) \\ \partial_3 u(t, R_t, \rho_t) \end{pmatrix}.$$
Literature

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- *Quadratic BSDEs driven by continuous martingales and applications to the utility maximization problem*
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- *Cross hedging with stochastic correlation*
  S. Ankirchner, G. Heyne, 2009

Thank you for your attention!