

Differentiability of martingale driven BSDE and application to hedging in incomplete markets

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IRTG Stochastic Models of Complex Processes



Summer School
Chorin, September 2009

Motivation

Risk source: n -dimensional SDE

$$dR_s = \sigma(s, R_s)dM_s + b(s, R_s)dC_s$$

with M d -dim. continuous local martingale, $d\langle M, M \rangle_s = q_s q_s^* dC_s$

Aim: price and hedge a derivative of the form $F(R_T)$

Correlated financial market: $k \leq d$ assets

$$dS_s = S_s(\beta(s, R_s)dM_s + \alpha(s, R_s)dC_s)$$

Preferences:

$$U(x) = -e^{-\eta x}, \quad 0 < \eta = \text{risk aversion}$$

Value function:

$$V^F(x, t, r) = \sup_{\lambda} \mathbb{E} \left[U \left(x + \sum_{i=1}^k \int_t^T \lambda_s^{(i)} \frac{dS_s^{(i)}}{S_s^{(i)}} - F(R_T^{t,r}) \right) \right]$$

What is a BSDE (M=W)?

A BSDE with terminal condition B and generator f is an equation of the type

$$Y_t = B - \int_t^T Z_s dW_s + \int_t^T f(s, Y_s, Z_s) ds.$$

→ A solution is a pair of *adapted* processes (Y, Z) .

Theorem (Hu et al. '05, Morlais '08)

The value function satisfies

$$V^F(x, t, r) = U(x - Y_t^{F,t,r})$$

and the optimal strategy π^F is given by

$$\pi_s^F = Z_s^{F,t,r} q_s^* \beta^* (\beta \beta^*)^{-1}(s, R_s^{t,r}) + \frac{1}{\eta} \alpha^* (\beta \beta^*)^{-1}(s, R_s^{t,r}),$$

where $(Y^{F,t,r}, Z^{F,t,r})$ is the solution of a certain quadratic BSDE with terminal value $F(R_T)$.

Indifference pricing and delta hedging

Decomposition:

$$\pi^F = \pi^0 + \Delta = \text{pure investment} + \text{optimal hedge}$$

Indifference price: $V^F(x - p(t, r), t, r) = V^0(x, t, r)$

Theorem (Ankirchner et al. '07)

Let M be the Brownian motion. The indifference price is given by

$$p(t, r) = Y_t^{0,t,r} - Y_t^{F,t,r}.$$

The optimal hedge Δ can be derived explicitly

$$\begin{aligned}\Delta(t, r) &= \left[Z_t^{0,t,r} - Z_t^{F,t,r} \right] \beta^*(\beta\beta^*)^{-1}(t, r) \\ &= \left[-\partial_2 p(t, r) \sigma(t, r) \right] \beta^*(\beta\beta^*)^{-1}(t, r).\end{aligned}$$

Background - The Brownian Case ($M=W$)

Theorem (Ankirchner et al. '07)

Let f be of quadratic growth, i.e. $|f(\cdot, z)| \leq C(1 + |z|^2)$.

Assume that σ , b are Lipschitz, have uniformly bounded partial derivatives, f is differentiable in r , z , ...

Then

- ▶ Markov property:

$$Y_s^{t,r} = u(s, R_s^{t,r}),$$

- ▶ $Y^{t,r}$ is continuously differentiable in r and Malliavin differentiable:

$$D_\theta Y_s^{t,r} = \partial_2 u(s, R_s^{t,r}) D_\theta R_s^{t,r}$$

- ▶ Malliavin trace:

$$Z_s^{t,r} = D_s Y_s^{t,r} = \partial_2 u(s, R_s^{t,r}) \sigma(s, R_s^{t,r}).$$

Brownian setting: If the coefficients σ , b and f are nice, then

$$\begin{aligned}Y_s^{t,r} &= u(s, R_s^{t,r}) \\Z_s^{t,r} &= \partial_2 u(s, R_s^{t,r}) \sigma(s, R_s^{t,r})\end{aligned}$$

Question: Does this relation between Y and Z hold in other settings, f.e. in a continuous martingale setting?

What is a martingale driven BSDE?

- ▶ M continuous d -dim. martingale, (\mathcal{F}_t) cont. and complete and thus every martingale is of the form $Z \cdot M + L$
- ▶ $d\langle M, M \rangle_t = q_t q_t^* dC_t$
- ▶ B \mathcal{F}_T -measurable r.v.
- ▶ $f: \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ Borel-measurable function

A BSDE with terminal value B and generator f is an equation

$$Y_t = B - \int_t^T Z_s dM_s - \int_t^T dL_s + \frac{\eta}{2} \int_t^T d\langle L, L \rangle_s + \int_t^T f(s, Y_s, Z_s) dC_s.$$

A **solution** is a triple of adapted processes (Y, Z, L) such that the above equation makes sense.

Our Setting - Martingale driven FBSDE

- ▶ M continuous d -dim. martingale

For $(x, m) \in \mathbb{R}^n \times \mathbb{R}^d$ and $t \in [0, T]$ we consider

$$X_t^{x,m} = x + \int_0^t \sigma(s, X_s^{x,m}, M_s^m) dM_s + \int_0^t b(s, X_s^{x,m}, M_s^m) dC_s$$

with solution processes $(X^{x,m}, Y^{x,m}, Z^{x,m}, L^{x,m})$.

Our Setting - Martingale driven FBSDE

- ▶ M continuous d -dim. martingale
- ▶ F is a bounded function
- ▶ f is quadratic in z

For $(x, m) \in \mathbb{R}^n \times \mathbb{R}^d$ and $t \in [0, T]$ we consider

$$X_t^{x,m} = x + \int_0^t \sigma(s, X_s^{x,m}, M_s^m) dM_s + \int_0^t b(s, X_s^{x,m}, M_s^m) dC_s$$

$$Y_t^{x,m} = F(X_T^{x,m}) - \int_t^T Z_r^{x,m} dM_r - \int_t^T dL_r^{x,m} + \frac{\eta}{2} \int_t^T d\langle L^{x,m}, L^{x,m} \rangle_r \\ + \int_t^T f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m}, q_r^*) dC_r$$

with solution processes $(X^{x,m}, Y^{x,m}, Z^{x,m}, L^{x,m})$.

The Markov property

Theorem (IRR '09)

Let $M^{t,m}$ be a strong Markov process.

Then there exist deterministic functions u and v such that for $s \in [t, T]$

$$Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m}), \quad Z_s^{t,x,m} = v(s, X_s^{t,x,m}, M_s^{t,m}).$$

Difficulty: Presence of $\langle L^{x,m}, L^{x,m} \rangle$ in

$$Y_t^{x,m} = F(X_T^{x,m}) - \int_t^T Z_r^{x,m} dM_r - \int_t^T dL_r^{x,m} + \frac{\eta}{2} \int_t^T d\langle L^{x,m}, L^{x,m} \rangle_r \\ + \int_t^T f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m} q_r^*) dC_r$$

Introduce assumption **(MRP)**:

There exists a square-integrable martingale N with $\langle M, N \rangle = 0$ and such that (M, N) satisfies the martingale representation property.

$\implies \exists$ process $U^{x,m}$ such that $L^{x,m} = U^{x,m} \cdot N$

Theorem (IRR '09)

Let (MRP) be satisfied, $\partial_i f$ of linear growth in z, \dots

Then there exists a modification $(Y^{x,m}, Z^{x,m}, U^{x,m})$ such that

- ▶ $Y^{x,m}$ is continuously differentiable in x and m ,
- ▶ and there exist processes $\partial_x Z^{x,m}$, $\partial_m Z^{x,m}$ and $\partial_x U^{x,m}$, $\partial_m U^{x,m}$ such that the derivatives

$$(\partial_x Y^{x,m}, \partial_x Z^{x,m}, \partial_x U^{x,m}) \text{ and } (\partial_m Y^{x,m}, \partial_m Z^{x,m}, \partial_m U^{x,m})$$

solve BSDEs.

The representation formula

Theorem (IRR '09)

If

- ▶ $Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m})$ and
- ▶ $Y^{t,x,m}$ continuously differentiable in (x, m) .

Then

$$\begin{aligned} Z_s^{t,x,m} &= \partial_2 u(s, X_s^{t,x,m}, M_s^{t,m}) \sigma(s, X_s^{t,x,m}, M_s^{t,m}) \\ &\quad + \partial_3 u(s, X_s^{t,x,m}, M_s^{t,m}). \end{aligned}$$

Hedging with stochastic correlation [Ankirchner,Heyne '09]

- ▶ $F(R_T)$ = derivative of risk R with payoff function F
- ▶ S = correlated traded asset

$$dS_t = S_t(\mu_S dt + \sigma_S dW_t^1)$$

$$dR_t = R_t(\mu_R dt + \sigma_R(\rho_t dW_t^1 + \sqrt{1 - \rho_t^2} dW_t^2)),$$

where ρ is the **stochastic correlation** with dynamics

$$d\rho_t = a(\rho_t)dt + g(\rho_t)(\gamma dW_t^1 + \delta dW_t^2 + \sqrt{1 - \gamma^2 - \delta^2} dW_t^3).$$

Aim: Find a local risk minimizing hedge for $F(R_T)$.

FS decomposition and BSDEs

- ▶ Standard method for deriving the local risk minimizing strategy π is based on FS decomposition

$$F(R_T) = C + \int_0^T \pi_u dS_u + L_T.$$

- ▶ Let (Y, Z) be the solution of the linear BSDE

$$Y_t = F(R_T) - \int_t^T Z_u dW_u - \int_t^T Z_u^1 \frac{\mu_S}{\sigma_S} du.$$

Then FS decomposition of $F(R_T)$ is given by

$$F(R_T) = Y_0 + \int_0^T \frac{Z_u^1}{\sigma_S S_u} dS_u + \int_0^T Z_u^2 dW_u^2 + \int_0^T Z_u^3 dW_u^3.$$

Optimal hedge

Hedging strategy $\pi = \frac{Z^1}{\sigma_S}$ \rightsquigarrow Explicit description of π ?

The solution of

$$Y_t = F(R_T) - \int_t^T Z_u dW_u - \int_t^T Z_u^1 \frac{\mu_S}{\sigma_S} du.$$

can be described in terms of

$$u(t, x, v) = E^Q [h(R_T^{t,x,v})], \text{ i.e. } Y_t = u(t, R_t, \rho_t).$$

Difficulty: $x \mapsto u(t, x, v)$ is only locally Lipschitz continuous
 \rightsquigarrow no (direct) access to the chain rule of Malliavin Calculus

With the representation formula

$$Z_t = \sigma(t, R_t, \rho_t)^* \begin{pmatrix} \partial_2 u(t, R_t, \rho_t) \\ \partial_3 u(t, R_t, \rho_t) \end{pmatrix}.$$

- ▶ *Pricing and hedging based on non-tradable underlyings*
S. Ankirchner, P. Imkeller, G. Dos Reis, to appear in Math. Finance
- ▶ *Differentiability of quadratic BSDE generated by continuous martingales and hedging in incomplete markets*
P. Imkeller, A. Réveillac, A. Richter, 2009, arXiv:0907.0941v1
- ▶ *Quadratic BSDEs driven by continuous martingales and applications to the utility maximization problem*
M.-A. Morlais, Finance Stoch., 2009
- ▶ *Cross hedging with stochastic correlation*
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Thank you for your attention!