

Erdős-Rényi random graphs + forest fires = self organized criticality

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Erdős-Rényi forest fire model

The $\lambda(n) \equiv 0$ case: the Erdős-Rényi model

Phase transition of the E-R graph process

The Smoluchowski coagulation equations

S.O.C. in the forest fire model

The controlled Smoluchowski equation

Erdős-Rényi forest fire model

- ▶ $G(n, t)$ is a continuous time graph-valued Markov chain on n vertices
- ▶ At time t an edge of the graph can be „occupied” or „vacant”
- ▶ We investigate the asymptotic behaviour of the component-size distribution of the random graph consisting of occupied edges as $n \rightarrow \infty$

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- ▶ Each vertex is exposed to a rate $\lambda(n)$ Poisson process of lightnings
- ▶ Fire spreads along the occupied edges and burns them (the number of vertices remains unchanged)

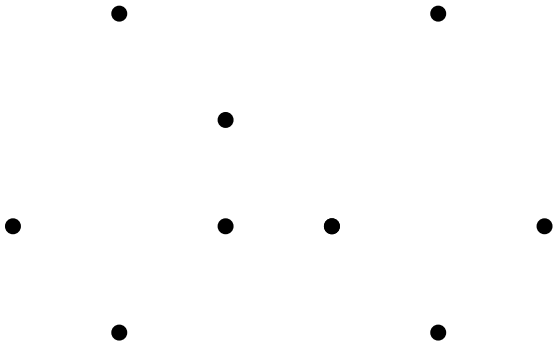
Erdős-Rényi forest fire model

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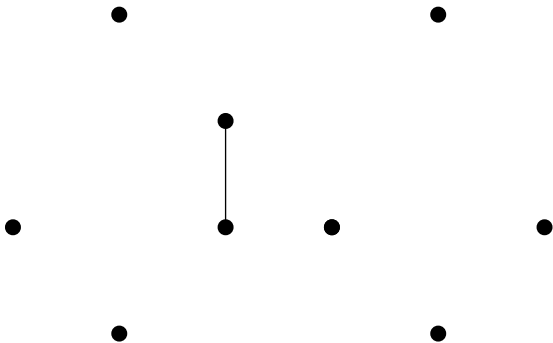
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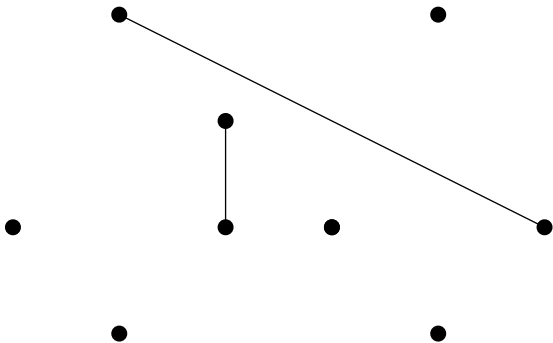
Denote by $\mathcal{V}_k^n(t)$ the total number of vertices contained in components of size k at time t



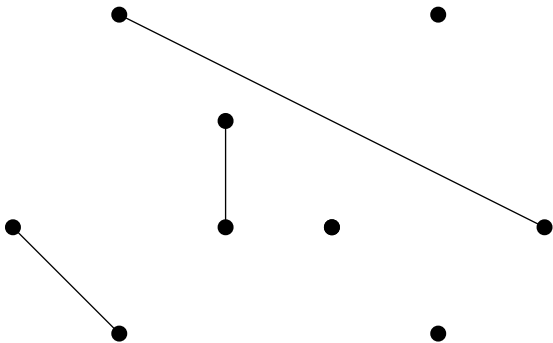
$$\nu_1 = 9 \quad \nu_2 = 0 \quad \nu_3 = 0 \quad \nu_4 = 0$$



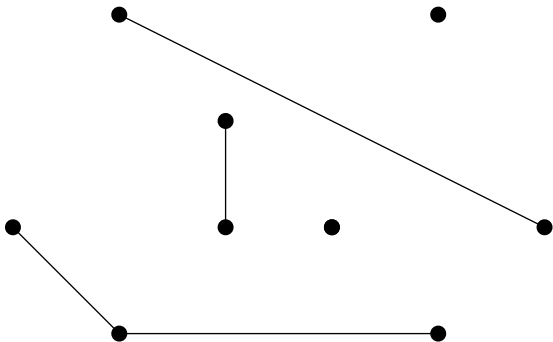
$$\nu_1 = 7 \quad \nu_2 = 2 \quad \nu_3 = 0 \quad \nu_4 = 0$$



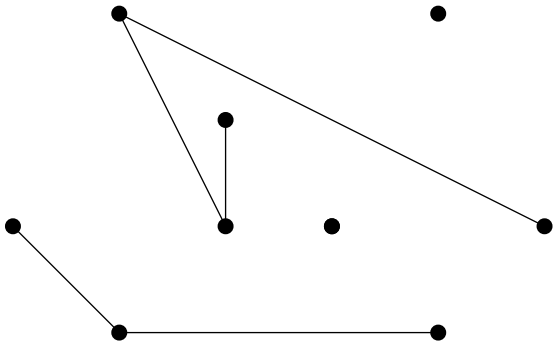
$$\nu_1 = 5 \quad \nu_2 = 4 \quad \nu_3 = 0 \quad \nu_4 = 0$$



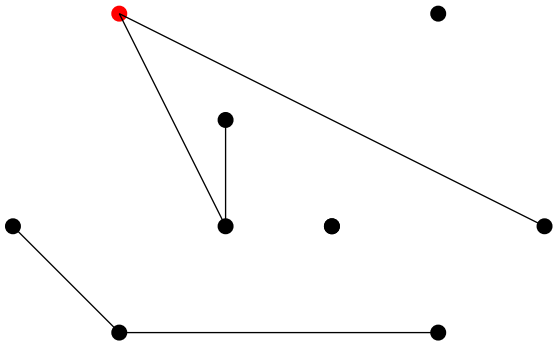
$$\nu_1 = 3 \quad \nu_2 = 6 \quad \nu_3 = 0 \quad \nu_4 = 0$$



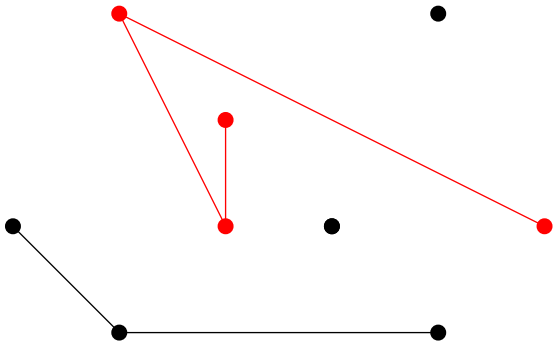
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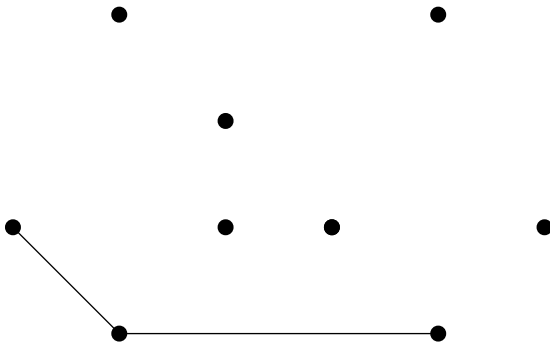
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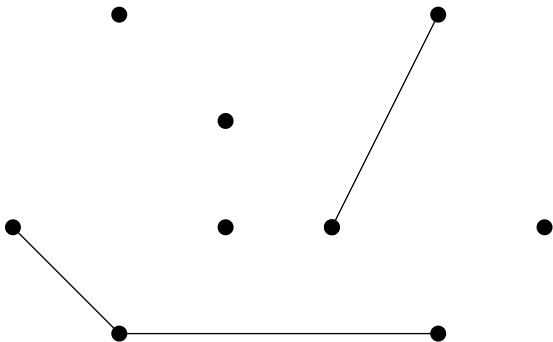
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The **phase transition** of the Erdős-Rényi model can be treated in a different way:

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$$\sum_{k=1}^n v_k^n(t) \equiv 1$$

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More generally: $v_k(t) = \frac{k^{k-1}}{k!} e^{-kt} t^{k-1}$

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$$\dot{v}_k = -k v_k + \frac{k}{2} \sum_{l=1}^{k-1} v_l v_{k-l}$$

Deducing the Smoluchowski equation with martingales

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If $k \neq l$ then

$$L\mathcal{W}_{k,l}^n(t) = \lim_{dt \rightarrow 0_+} \frac{1}{dt} \mathbf{E}(\mathcal{W}_{k,l}^n(t + dt) - \mathcal{W}_{k,l}^n(t) \mid \mathcal{F}_t) =$$

$$\mathcal{C}_k^n(t)\mathcal{C}_l^n(t)\frac{1}{n}kl = \mathcal{V}_k^n(t)\mathcal{V}_l^n(t)\frac{1}{n} = (n\mathcal{V}_k^n(t))(n\mathcal{V}_l^n(t))\frac{1}{n} = n\mathcal{V}_k^n(t)\mathcal{V}_l^n(t)$$

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If $k = l$ then

$$\lim_{dt \rightarrow 0_+} \frac{1}{dt} \mathbf{E}(\mathcal{W}_{k,k}^n(t + dt) - \mathcal{W}_{k,k}^n(t) \mid \mathcal{F}_t) =$$

$$\frac{\mathcal{C}_k^n(t)(\mathcal{C}_k^n(t) - 1)}{2} \frac{1}{n} k^2 \approx \frac{1}{2} n \nu_k^n(t)^2$$

$\mathcal{W}_{k,l}^n(t)$ is counting the number of times when a component of size k coagulated with a component of size l from before t

\mathcal{V}_k^n is the total weight of the components of size k

$$d\mathcal{V}_k = k \cdot \sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} d\mathcal{W}_{l,k-l} - k \cdot \sum_{l=1}^n d\mathcal{W}_{l,k}$$

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Thus

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is a martingale with small jumps, thus $\mathbf{Var}(M^n(t)) = \mathcal{O}\left(\frac{1}{n}\right)$.

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How to solve these differential equations?

PDE

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$$V(t, x) := \sum_{k=1}^{\infty} v_k(t) e^{-kx}$$

Laplace-transform/generating function

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The transformed equation:

$$\partial_t V(t, x) = \partial_x V(t, x) - \frac{1}{2} \partial_x V^2(t, x)$$

Burgers-equation

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This equation can be solved explicitly, eg. with the **method of characteristics**, or with the following method:

Solution of the Burgers equation

$$\partial_t V(t, x) = \partial_x V(t, x) - V(t, x) \partial_x V(t, x)$$

Denote by $X(t, v)$ the inverse function of $V(t, x)$ in the second variable $V(t, X(t, v)) = v$.

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The transformed PDE for $X(t, v)$ is

$$\partial_t X(t, v) = v - 1$$

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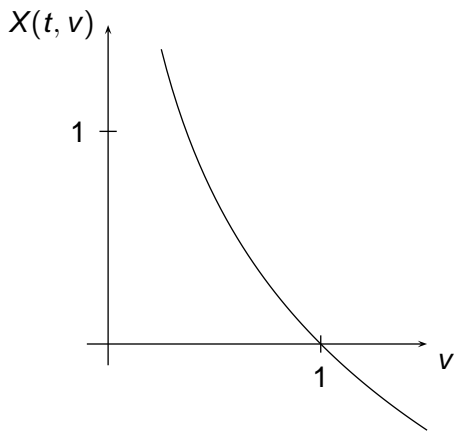
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The transformed PDE for $X(t, v)$ is

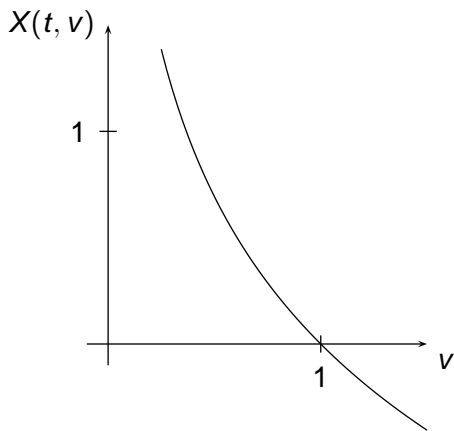
$$\partial_t X(t, v) = v - 1$$

Solution:

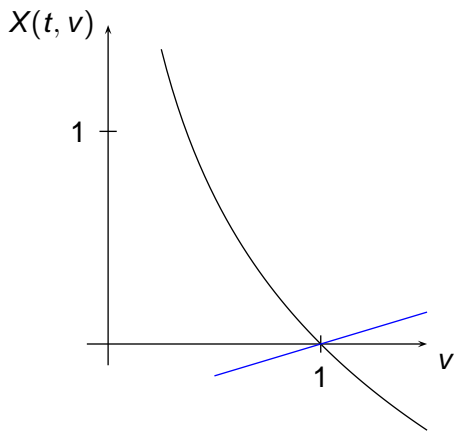
$$X(t, v) = X(0, v) + t \cdot (v - 1)$$



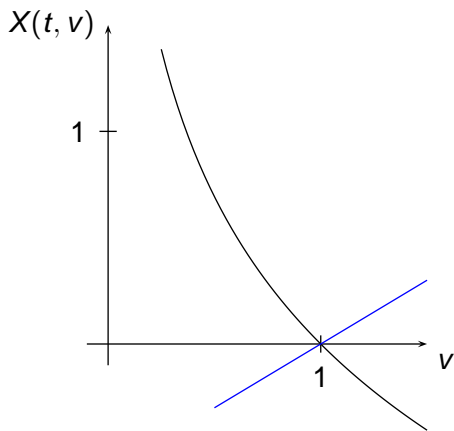
$$V(0, x) = e^{-x} \quad X(0, v) = -\log(v)$$



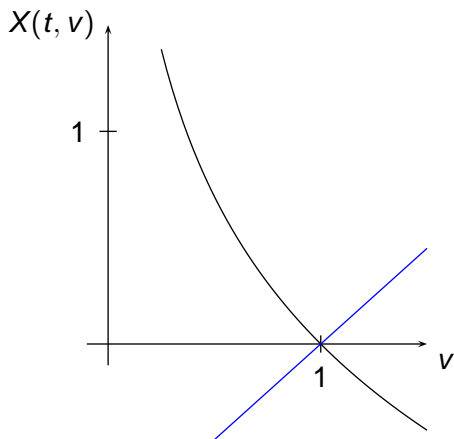
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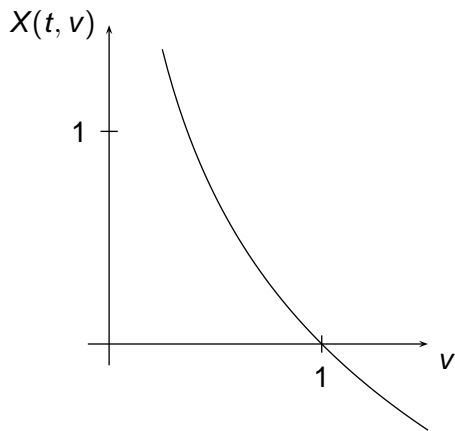
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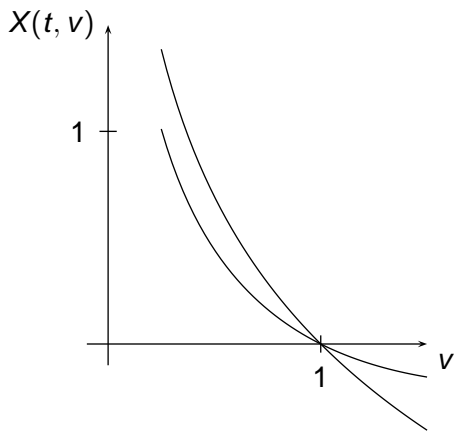
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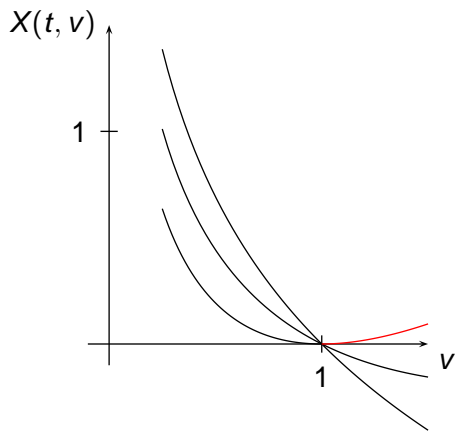
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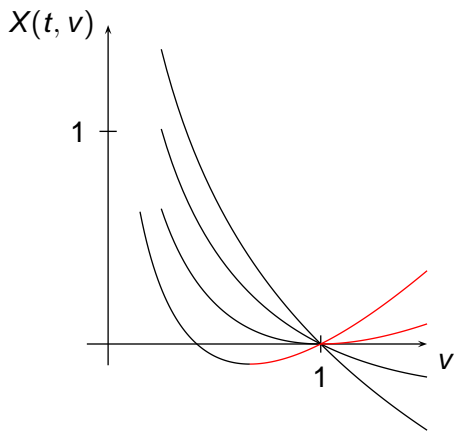
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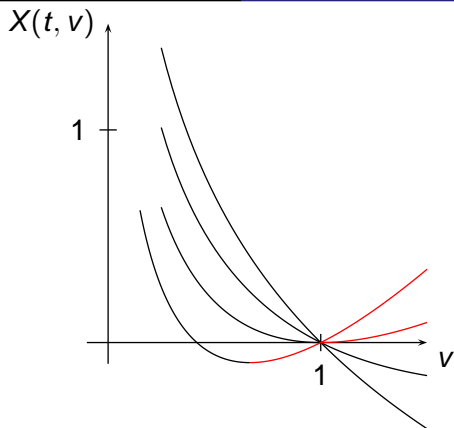
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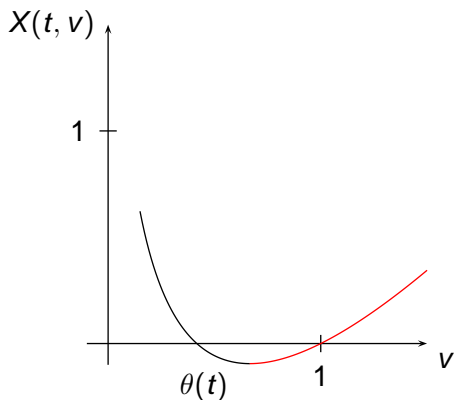
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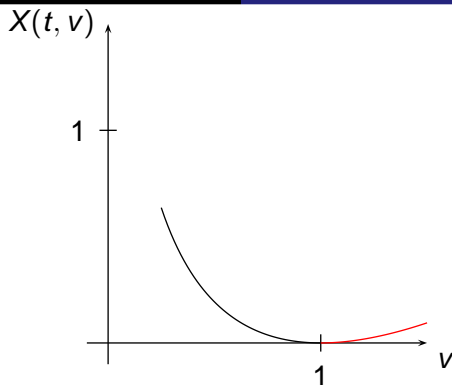
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if $t \neq 1$ then $\min_v X(t, v) = v_{min} < 0$, so $V(t, x)$ can be extended analytically to the domain $(v_{min}, +\infty)$, thus v_k decays exponentially.



The size of the giant component: $\theta(t) = V(t, 0)$, thus
 $X(t, \theta(t)) = 0$



At the critical time: $\partial_v X(t=1, v=1) = 0$, but $\partial_{vv} X(t=1, v=1) > 0$, so $|V(t=1, x) - 1| \asymp \sqrt{x}$, so it follows from **Tauberian theory** that $v_k(t=1) \asymp k^{-\frac{3}{2}}$

S.O.C. in the forest fire model

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There are deterministic functions $(v_k(t))_{k=1}^{\infty}$, such that $v_k^n(t) \rightarrow v_k(t)$ if $\frac{1}{n} \ll \lambda(n) \ll 1$.

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- ▶ The functions $v_k(\cdot)$ are independent of the exact decay rate of $\lambda(n)$, we do not have to fine-tune the model to see permanent criticality in the limit: **S.O.C.**

About the proof

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The critical controlled Smoluchowski equations:

- ▶ For $k \geq 2$ we have the usual $\dot{v}_k = -kv_k + \frac{k}{2} \sum_{l=1}^{k-1} v_l v_{k-l}$

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The critical controlled Smoluchowski equation

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$$\partial_t V(t, x) = \partial_x V(t, x) - \frac{1}{2} \partial_x V^2(t, x) + \varphi(t) e^{-x}$$

$$V(t, 0) \equiv 1 \quad \iff \quad \sum_{k=1}^{\infty} v_k(t) \equiv 1 \quad \iff \quad \theta(t) \equiv 0$$

Critical controlled Burgers equation

Thank you for your attention

Frequently Asked Questions

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Conjecture:

If $\lambda(n) = n^{-\alpha}$ where $0 < \alpha < 1$, then for $t > 1$ the expected size of the largest component is n^β , where

$$\beta = \begin{cases} 2\alpha & \text{if } \alpha \leq \frac{1}{3} \\ \frac{1+\alpha}{2} & \text{if } \alpha \geq \frac{1}{3} \end{cases}$$

