Pólya's Urn and its Perspective on Point Processes

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Model events in time ($\mathbb{R}_+$) or objects space ($\mathbb{R}^d$).

Definition (Poisson Process)

There exists exactly one point process with

1. for each bounded, measurable B, $N_B \sim Poi\big(\rho(B)\big)$
2. $B_1, \ldots B_n$ disjoint, then $N_{B_1}, \ldots N_{B_n}$ independent

This point process is the *Poisson process* with *intensity measure* ρ, $\mathbf{P}_\rho$.

Example: $\rho = a\lambda$, $a > 0$.

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1. simple iff intensity measure ρ non-atomic
2. conditioned on $\{N_B = n\}$, location of points in B is *iid.*

$$\mathbf{P}_\rho(\varphi | N_B = n) = \rho(B)^{-n} \int_B \cdots \int_B \varphi(\delta_{x_1} + \ldots + \delta_{x_n}) \rho(\mathrm{d}x_n) \cdots \rho(\mathrm{d}x_1)$$

3. local representation

$$\mathbf{P}_{\rho,B}(\varphi) = \mathrm{e}^{-\rho(B)} \sum_{n \geq 1} \frac{1}{n!} \times \int_B \cdots \int_B \varphi(\delta_{x_1} + \ldots + \delta_{x_n}) \rho(\mathrm{d}x_n) \cdots \rho(\mathrm{d}x_1)$$

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   for each $k \geq 1$, $\mathbf{P}_\rho = \underbrace{\mathbf{P}_{\rho/k} * \ldots * \mathbf{P}_{\rho/k}}_{k \text{ times}}$
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Motivation and main object

- relax *iid*: replace by Pólya-like dynamic

Questions concern:

- local representation
- independent increments
- infinite divisibility
- fidi distribution
- typical point

Pólya's Urn

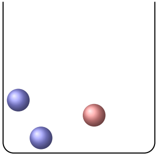
Marbles of two colours, after each draw additional marble.

Hoppe (1984): Pólya-like Urn

Introduction of new colours by a special marble.

Blackwell/MacQueen (1973)

Extension to a continuum of colours.



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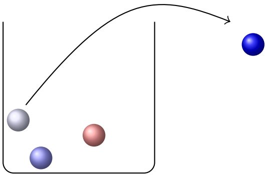

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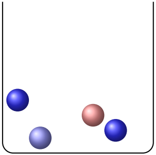

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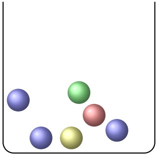

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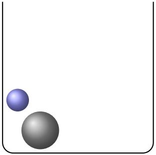

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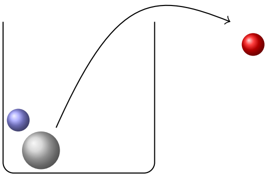

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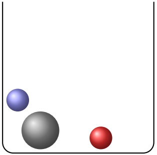

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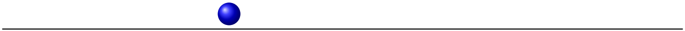

The Pólya Sum Process

Generalisations of Pólya's Urn Scheme

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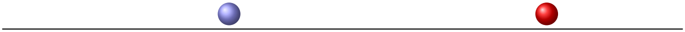

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Inductive construction:

1. realise x_1 according to ρ, $\mu_0 := 0$
2. set $\mu_k := \mu_{k-1} + \delta_{x_k}$
3. realise x_{k+1} according to $\rho + \mu_k$

$$\mathsf{S}_{z,\rho}(\varphi | N_B = n) = \rho(B)^{-[n]} \times \int_B \cdots \int_B \varphi(\mu_n)(\rho + \mu_{n-1})(\mathrm{d}x_n) \cdots (\rho + \mu_1)(\mathrm{d}x_2)\rho(\mathrm{d}x_1)$$

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$$a^{[n]} := (a + n - 1) \cdots (a + 1)a$$

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Local representation of the Pólya Sum Process

Let $z \in (0,1)$, ρ a locally bounded measure

$$\mathsf{S}_{z,\rho,B}(\varphi) = (1-z)^{\rho(B)} \sum_{n\geq 0} \frac{z^n}{n!} \times \int_B \cdots \int_B \varphi(\mu_n)\big(\rho+\mu_{n-1}\big)(\mathrm{d}x_n)\cdots\big(\rho+\mu_1\big)(\mathrm{d}x_2)\rho(\mathrm{d}x_1)$$

(Zessin 09)

Let $0 < z < 1$. Then

$$\mathsf{S}_{z,\rho}(N_B = k) = (1-z)^{\rho(B)} \frac{z^k}{k!} \rho(B)^{[k]}.$$

If $B_1, \ldots, B_m$ disjoint, the $N_{B_1}, \ldots, N_{B_m}$ are independent.

$\rho(B) = 1$, then $N_B \sim Geom(z)$
$\rho(B) \in \mathbb{N}$, then N_B negative binomial distribution

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1. $S_{z,\rho}$ is *not* simple

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3. locations of j-towers:
   Poisson process with intensity measure $\frac{z^j}{j}\rho$

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Pólya Sum Process

- further properties
- models and applications

Papangelou Processes

- replace $\rho + \mu$ by more general $\eta(\mu, \cdot)$
  interacting gases, Cox processes