Multitype branching processes conditioned on very late extinction. An example in epidemiology.

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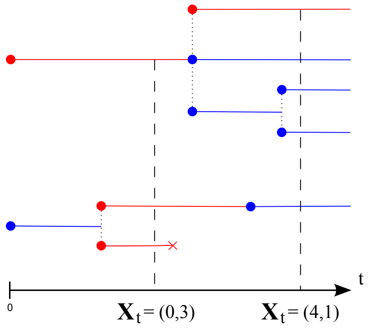
Multitype branching process

- $\mathbb{N}^d$-valued Markov process *(here $d = 2$)*
- offspring distribution $(p_1(\mathbf{k}))_{\mathbf{k}\in\mathbb{N}^d}$, $(p_2(\mathbf{k}))_{\mathbf{k}\in\mathbb{N}^d}$
- exponentially distributed lifetime with parameter α_1, α_2



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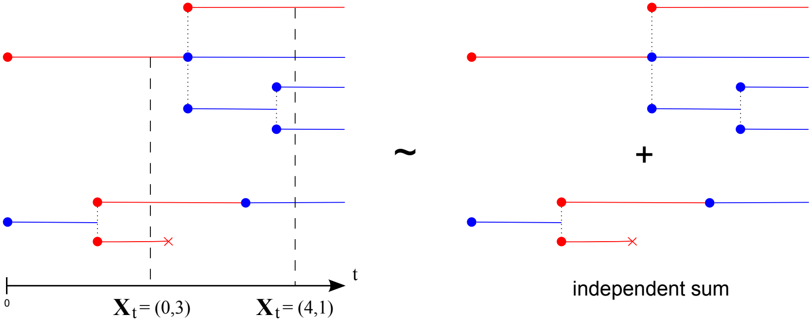

branching property

Extinction of a multitype BP

mean matrix $\mathbf{M}$ with entries

$$m_{ij} := \text{mean number of offsprings of type } j \text{ for a particle of type } i$$

Perron-Frobenius' theorem $\Rightarrow$ if $\mathbf{M}$ is irreducible then the matrix $diag(\alpha_1, \dots, \alpha_d)(\mathbf{M} - \mathbf{I})$ has a unique maximal real eigenvalue ρ and a positive normalized right eigenvector $\boldsymbol{\xi}$

Theorem

$$\rho \leqslant 0 \iff \textit{the process dies out almost surely.}$$

- $\rho < 0$: subcritical process
- $\rho = 0$: critical process
- $\rho > 0$: supercritical process

If $d = 1$, then $\rho \leqslant 0$ is equivalent to $m \leqslant 1$!

Conditioning on very late extinction

Irreducible branching process with $\mathbb{P}$, $\mathbf{M}$, ρ, $\boldsymbol{\xi}$

- We want to work with a process which dies out almost surely:
  - critical or subcritical process
  - supercritical process with positive risk of extinction $\mathbf{q}$, *conditioned on extinction*

Proposition (Jagers, Lagerås, 2008)

A supercritical BP conditioned on extinction is a subcritical BP.

Conditioning on very late extinction

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- We condition this "mortal" BP on very late extinction:

$$\forall t \geqslant 0, \forall B \in \mathcal{F}_t, \quad \mathbb{P}^*(B) := \lim_{\theta \to \infty} \mathbb{P}\left(B \mid \mathbf{X}_{t+\theta} \neq \mathbf{0}, \lim_{s \to \infty} \mathbf{X}_s = \mathbf{0}\right)$$

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Proposition (S. P.)

The conditioned law $\mathbb{P}^*$ is a h-transform of the unconditioned law $\mathbb{P}$

$$d\mathbb{P}^*_{\mathbf{x}}|_{\mathcal{F}_t} = e^{-\widetilde{\rho} t} \frac{\mathbf{q}^{\mathbf{X}_t}}{\mathbf{q}^{\mathbf{x}}} \frac{\mathbf{X}_t . \widetilde{\boldsymbol{\xi}}}{\mathbf{x} . \widetilde{\boldsymbol{\xi}}} \, d\mathbb{P}_{\mathbf{x}}|_{\mathcal{F}_t}, \ \mathbf{x} \in \mathbb{N}^d, \ \mathbf{x} \neq \mathbf{0}.$$

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In particular, if $\mathbb{P}$ is (sub)critical: $d\mathbb{P}^*_{\mathbf{x}}|_{\mathcal{F}_t} = e^{-\rho t} \frac{\mathbf{X}_t.\boldsymbol{\xi}}{\mathbf{x}.\boldsymbol{\xi}}\ d\mathbb{P}_{\mathbf{x}}|_{\mathcal{F}_t}$.

Interpretation of the conditioned process

- Conditioning on *extinction* preserves the branching property
  → modifies the offspring distribution of the BP
- Conditioning on *very late extinction* does not!
  → adds an external structure to the unconditioned BP

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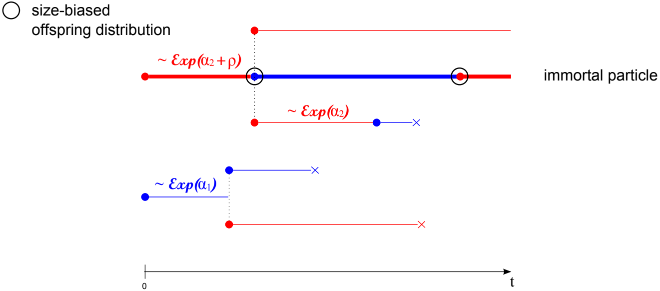
Proposition (S. P.)

conditioned process $\sim$ *unconditioned process* $+$ *immortal particle*



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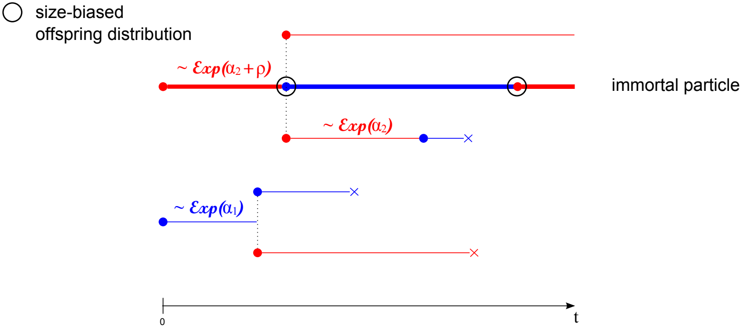

- immortal particle remains of type i for a time $\sim Exp(\alpha_i + \rho)$
- produces $\mathbf{k}$ offsprings according to the *size-biased* distribution

$$q_i(\mathbf{k}) := \frac{\alpha_i}{(\alpha_i+\rho)\xi_i}\mathbf{k}.\boldsymbol{\xi}\, p_i(\mathbf{k})$$

- mutates to type j with probability $\frac{k_j \xi_j}{\mathbf{k}.\boldsymbol{\xi}}$

Investigation of other limits

- A rescaled BP converges to the multitype Feller diffusion solution of

$$dX_{t,i} = \sigma_i \sqrt{X_{t,i}}\, dB_{t,i} + \sum_{j=1}^{d} c_{ji} X_{t,j}\, dt, \quad i = 1 \dots d.$$

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- What about the *conditioned* BP? Does it converge to the *conditioned* Feller process solution of

$$dX_{t,i} = \sigma_i \sqrt{X_{t,i}}\, dB_{t,i} + \sum_{j=1}^{d} c_{ji} X_{t,j}\, dt + \frac{1}{\mathbf{X}_t \boldsymbol{.} \boldsymbol{\xi}} \sigma_i^2 \xi_i X_{t,i}\, dt?$$

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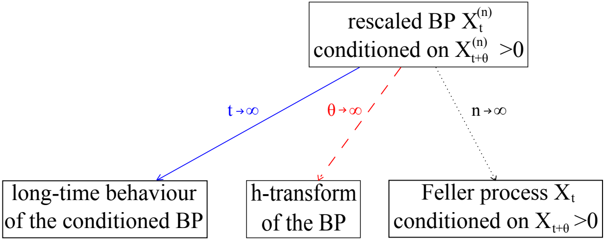
- What about the long-time behavior of the conditioned processes?
- Three kinds of limits for $\mathbb{P}^{(n)}\left(\mathbf{X}_t \in \,.\,|\, \mathbf{X}_{t+\theta} \neq \mathbf{0}, \lim_{s\to\infty} \mathbf{X}_s = \mathbf{0}\right)$
  - scaling limit $n \to \infty$
  - conditioning on very late extinction $\theta \to \infty$
  - asymptotic behaviour $t \to \infty$



rescaled BP $X_t^{(n)}$ conditioned on $X_{t+\theta}^{(n)} > 0$

$t \to \infty$

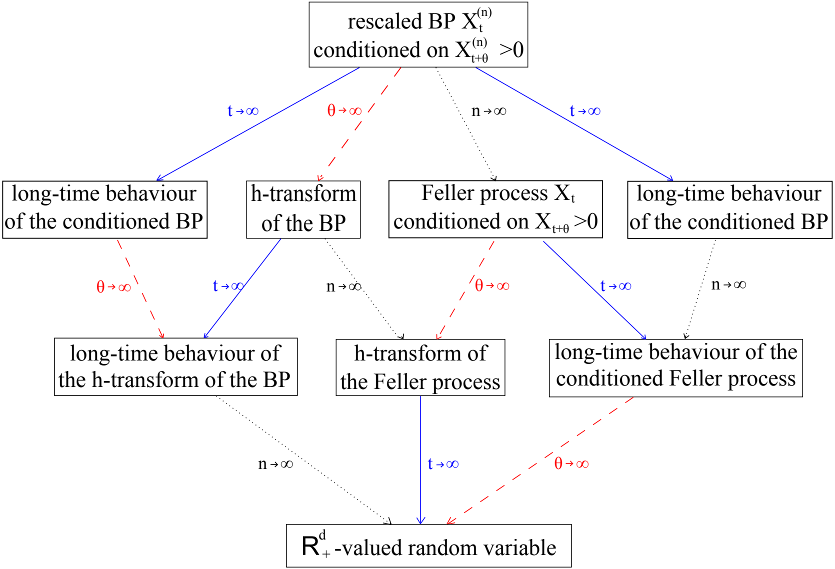
$\theta \to \infty$

$n \to \infty$

long-time behaviour of the conditioned BP

h-transform of the BP

Feller process X_t conditioned on $X_{t+\theta} > 0$



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long-time behaviour of the conditioned Feller process

$\mathbb{R}_+^d$-valued random variable

Conditioning and rescaling

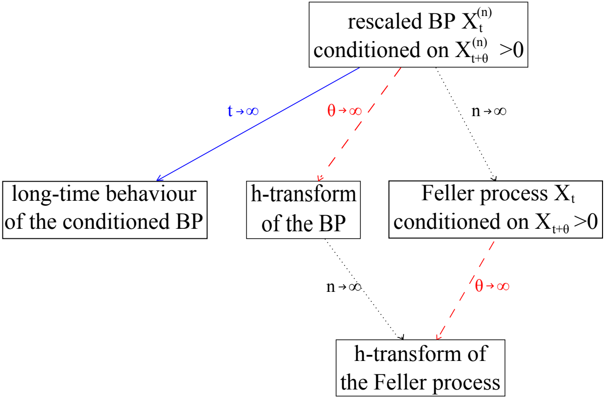
Theorem (S. P.)

Let $\mathbf{C}$ *be an irreducible mutation matrix. Assume that the offspring distribution* $p_i^{(n)}(\mathbf{j})$ *of the rescaled BP satisfies*

- *mean matrix* $\mathbf{M}^{(n)} = \mathbf{I} + \frac{1}{n}\mathbf{C} + o(\frac{1}{n})$
- *second order moments* $\sum_{\mathbf{j}\in\mathbb{N}^d} p_i^{(n)}(\mathbf{j})(j_i - 1)^2 = \sigma_i^2 + o(1)$
- $\lim_{N\to\infty} \sup_{n\in\mathbb{N}} \sum_{\|\mathbf{j}\|>N} \|\mathbf{j}\|^2 p_i^{(n)}(\mathbf{j}) = 0$
- $\sup_{n\in\mathbb{N}} \sum_{\mathbf{j}\in\mathbb{N}^d} p_i^{(n)}(\mathbf{j}) j_l^2 j_k < \infty$

Then, provided the weak convergence of the initial distributions, the following diagram is commutative.

$$\begin{tikzcd}
\textit{rescaled BP} \arrow[r, dashed] \arrow[d, Rightarrow] & \textit{conditioned rescaled BP} \arrow[d, Rightarrow] \\
\textit{Feller diffusion} \arrow[r, dashed] & \textit{conditioned Feller diffusion}
\end{tikzcd}$$



rescaled BP $X_t^{(n)}$ conditioned on $X_{t+\theta}^{(n)} > 0$

long-time behaviour of the conditioned BP

h-transform of the BP

Feller process X_t conditioned on $X_{t+\theta} > 0$

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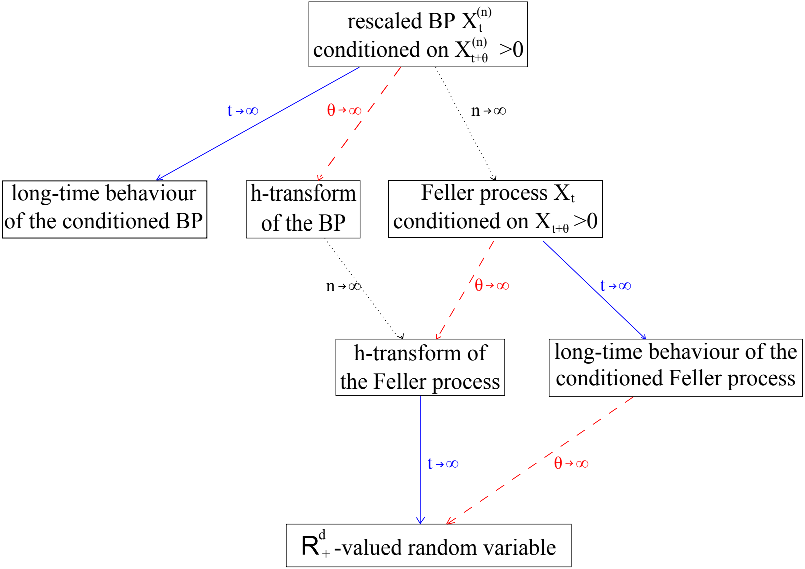

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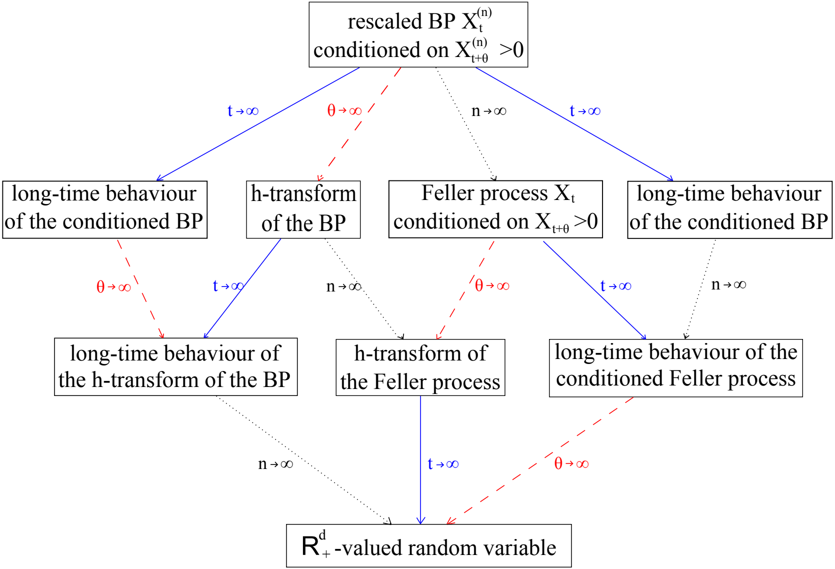
$\mathbb{R}_+^d$-valued random variable



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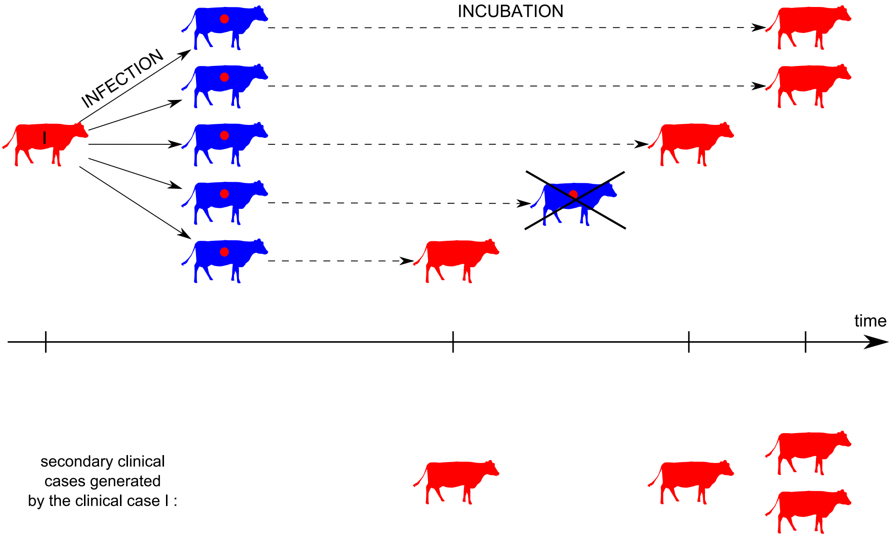
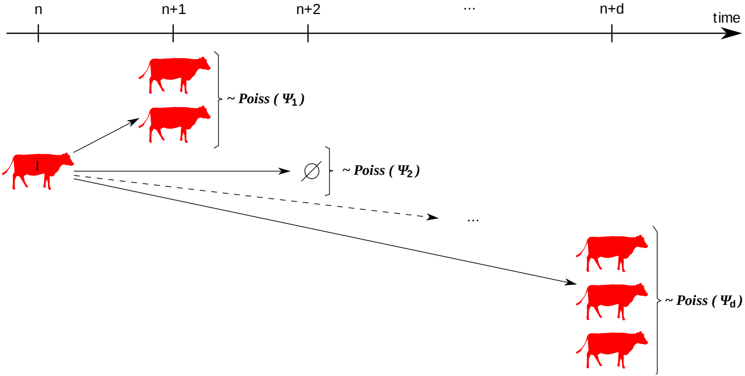
Application to the BSE epidemic

The model



secondary clinical
cases generated
by the clinical case I :

The model



$$\Psi_i = \frac{P_{inc}(i)}{\sum_{k=1}^{d+1} P_{surv}(k)} \left(\theta_{env} \sum_{k=i+1}^{d+1} P_{surv}(k) + p_{mat} P_{surv}(i+1) \right), \quad i = 1 \ldots d.$$

θ_{env}= environmental infection parameter (mean number of newly infected animals via the environment, per infective and per year)

The model

X_n = number of clinical cases at time n

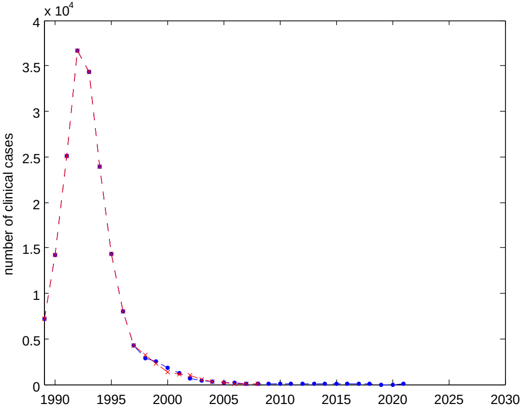
Markovian process of order d, with transition law

$$X_n|\mathcal{F}_{n-1} \sim Poiss\left(\sum_{i=1}^{d} X_{n-i}\Psi_i\right)$$

or Markovian d-types branching process (types ↔ memory), with mean matrix

$$\mathbf{M} = \begin{pmatrix} \Psi_1 & 1 & 0 & \dots & 0 \\ \Psi_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \Psi_{d-1} & 0 & \dots & \dots & 1 \\ \Psi_d & 0 & \dots & \dots & 0 \end{pmatrix}$$

Extinction of the epidemic



Bayesian estimation:

Extinction of the epidemic $\sim$ 2022

Extinction after 2030 with probability 2.5%

Conditioning on very late extinction

Aim = study of the trajectories with late extinction:

$$\mathbb{P}\left(\mathbf{X}_n = . \,|\, \mathbf{X}_{n+k} \neq \mathbf{0}\right), \quad k \text{ very large.}$$

Approximation by

$$\mathbb{P}\left(\mathbf{X}_n^* = .\right) := \lim_{k\to\infty} \mathbb{P}\left(\mathbf{X}_n = . \,|\, \mathbf{X}_{n+k} \neq \mathbf{0}\right).$$

Proposition (S. P.)

$$X_n^* | \mathcal{F}_{n-1}^* \sim Poiss\left(\sum_{k=1}^{d} X_{n-i}^* \Psi_i\right) + \mathcal{B}\left(\frac{\sum_{i=1}^{d} X_{n-i}^* \Psi_i \xi_1}{\sum_{k=1}^{d} X_{n-i}^* \Psi_i \xi_1 + X_{n-1}^* \xi_2 + \ldots + X_{n-(d-1)}^* \xi_d}\right)$$

Estimation

Estimation of the environmental infection parameter θ_{env}

$$\Psi_i = a_i\theta_{env} + b_i.$$

Bayesian estimation: $\widetilde{\theta} = 2.43$

Conditional Least Squares Estimation:

"Classical" CLSE estimation

$$\widehat{\theta}_{|\mathbf{x}_0|,n} = \arg\min_{\theta} \sum_{k=1}^{n} \frac{\left[x_k - \mathbb{E}_\theta\left(X_k|\mathbf{X}_{k-1} = \mathbf{x}_{k-1}\right)\right]^2}{\sum_{i=1}^{d} a_i x_{k-i}}$$

"Paranoiac" CLSE estimation

$$\widehat{\theta}^*_{|\mathbf{x}_0|,n} = \arg\min_{\theta} \sum_{k=1}^{n} \frac{\left[x_k - \mathbb{E}_\theta\left(X^*_k|\mathbf{X}^*_{k-1} = \mathbf{x}_{k-1}\right)\right]^2}{\sum_{i=1}^{d} a_i x_{k-i}}$$

Classical estimation

$$\widehat{\theta}_{|\mathbf{x}_0|,n} = \arg\min_{\theta} \sum_{k=1}^{n} \frac{\left[x_k - \mathbb{E}_\theta\left(X_k | \mathbf{X}_{k-1} = \mathbf{x}_{k-1}\right)\right]^2}{\sum_{i=1}^{d} a_i x_{k-i}}$$

- explicit form of $\widehat{\theta}_{|\mathbf{x}_0|,n}$
- no asymptotic property as $n \to \infty$
- consistency and asymptotic behaviour as $|\mathbf{x}_0| \to \infty$

$$\lim_{|\mathbf{x}_0|\to\infty} \sqrt{\sum_{k=1}^{n}\sum_{i=1}^{d} a_i x_{k-i}} \left(\widehat{\theta}_{|\mathbf{x}_0|,n} - \theta_0\right) \overset{\mathcal{D}}{=} \mathcal{N}\left(0, \sigma^2\left(\theta_0\right)\right)$$

$$\widehat{\theta}_{|\mathbf{x}_0|,n} = 2.4486$$

Paranoiac estimation

$$\widehat{\theta}^*_{|\mathbf{x}_0|,n} = \arg\min_{\theta} \sum_{k=1}^{n} \frac{\left[x_k - \mathbb{E}_\theta\left(X_k^* | \mathbf{X}_{k-1}^* = \mathbf{x}_{k-1}\right)\right]^2}{\sum_{i=1}^{d} a_i x_{k-i}}$$

- no explicit form of $\widehat{\theta}^*_{|\mathbf{x}_0|,n}$
- consistency as $|\mathbf{x}_0| \to \infty$
- consistence et comportement asymptotique lorsque $n \to \infty$

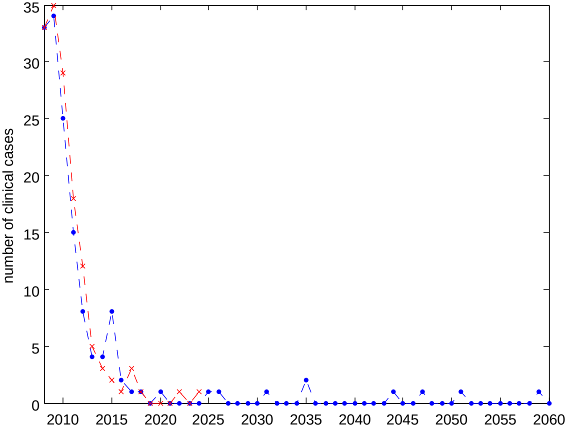
$$\lim_{n\to\infty} \sqrt{n}\left(\widehat{\theta}^*_{|\mathbf{x}_0|,n} - \theta_0\right) \overset{\mathcal{D}}{=} \mathcal{N}\left(0, \sigma^*\left(\theta_0\right)^2\right)$$

$$\widehat{\theta}^*_{|\mathbf{x}_0|,n} = 2.3977$$

$$\widehat{\theta}^*_{|\mathbf{x}_0|,n} \leqslant \widehat{\theta}_{|\mathbf{x}_0|,n}$$

Simulation

Simulation of the *unconditioned* and *conditioned* process with infection parameter $\theta_{env} = 2.3977$



Prediction

Simulation of 1000 trajectories of the *conditioned* process with infection parameter $\theta_{env} = 2.3977$