Maximum-Likelihood-Estimation of Lévy driven Ornstein-Uhlenbeck Processes

Hilmar Mai
Humboldt-Universität zu Berlin

IRTG Summer School Chorin

3rd Sept 2009

Ornstein-Uhlenbeck (OU) Process

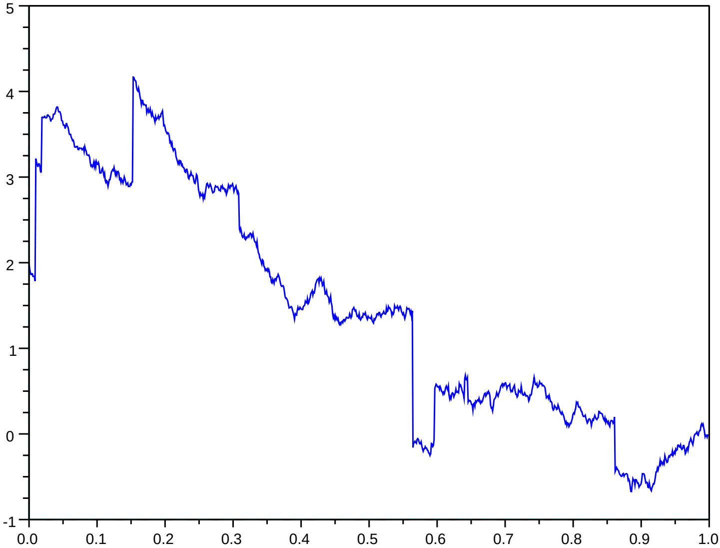
Let $(L_t, t \geq 0)$ be a Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. For every $a \in \mathbb{R}$

$$dX_t = -aX_t\,dt + dL_t, \quad t \in \mathbb{R}_+, \quad X_0 = x, \tag{1}$$

defines an Ornstein Uhlenbeck process driven by the Lévy process L with initial distribution $\pi = \mathcal{L}(X_0)$.
Equivalently,

$$X_t = e^{-at}X_0 + \int_0^t e^{-a(t-s)}dL_s. \tag{2}$$

Sample path from compound Poisson plus Wiener process driver



Setting

Problem: Estimation of a from continuous observations X_t, $0 \leq t \leq T$ and known Lévy-Khintchine triplet of L.

We work throughout in the canonical setting:

- $\Omega = D(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{R};\ f \text{ càdlàg}\}$
- $X(\omega, t) = \omega(t)$ for all $\omega \in \Omega$ coordinate process
- Filtration generated by X

$$\mathcal{F}_t = \bigcap_{s>t} \sigma(X_u : u \leq s) \text{ and } \mathcal{F} = \bigvee_t \mathcal{F}_t$$

For every $a \in \mathbb{R}$ we obtain a solution measure P^a of the OU equation on $D(\mathbb{R}_+)$.

Absolute Continuity/Singularity (ACS) Problem

$P_t^a := P^a_{|\mathcal{F}_t}$ denotes the restriction of P^a to $\mathcal{F}_t$.
Local absolute continuity:

$$P^{a'} \overset{loc}{\ll} P^a \iff P_t^{a'} \ll P_t^a \quad \forall t \in \mathbb{R}_+$$

1. Does
$$P^{a'} \overset{loc}{\ll} P^a \text{ hold for all } a, a' \in \mathbb{R}?$$
2. Can we derive $Z_t = \frac{dP_t^{a'}}{dP_t^a}$ explicitly?

First Problem

Theorem

Let $P^a, P^{a'}$ be two solution measures of the OU equation for the driving Lévy process L with characteristic triplet (b, σ^2, ρ) and initial distributions π and π'. Suppose that $\sigma^2 > 0$ and $\pi' \ll \pi$, then we have

$$P^{a'} \overset{loc}{\ll} P^a.$$

Semimartingale characteristics of X

Let (b, σ^2, μ) denote the Lévy-Khintchine triplet of L.

Then the semimartingale characteristics (B, C, ν) of X are given by

$$\begin{aligned} B(\omega, t) &= bt - a \int_0^t X_{s-}(\omega) ds, \\ C(\omega, t) &= \sigma^2 t, \\ \nu(\omega, dt, dx) &= \mu(dx)\, \lambda(dt), \end{aligned}$$

where λ denotes the Lebesgue measure on $\mathbb{R}$.

The Hellinger Process of X

Proposition

A version of the Hellinger process of two solution measures $P^a, P^{a'}$ is

$$h_t(\alpha; a, a') = \frac{\alpha(1-\alpha)}{2\sigma^2} \int_0^t \left[\int_0^u \left(a' e^{-a'(u-s)} - a e^{-a(u-s)} \right) L(ds) \right]^2 du.$$

for $a, a' \in \mathbb{R}$.

Hellinger Process and Absolute Continuity Problems

Let P, P' be two probability measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

Theorem (Jacod and Mémin (1979))

Let $h(\alpha)$*,* $\alpha \in (0,1)$ *be a version of the Hellinger process* $h(\alpha; P, P')$*. Then for every stopping time* T *there is equivalence between*

1. $P'_T \ll P_T$*;*
2. $\exists \alpha \in (0,1)$ *such that* $P'(h(\alpha)_T < \infty) = 1$ *and* $P'_0 \ll P_0$ *and* $P'(h(0)_T = 0) = 1$*.*

Second Problem

Proposition

There exists a local martingale $N : \Delta \to \mathbb{R}$ *on a random interval* $\Delta \subset \Omega \times \mathbb{R}_+$ *such that the density process is given by*

$$Z_t = \frac{dP_t^{a'}}{dP_t^a} = Z_0 \exp\left(N_t - \frac{(a'-a)^2}{2\sigma^2}\int_0^t X_{s-}^2\,ds\right).$$

Furthermore, for every stopping time S *such that* $[0,S] \subset \Delta$ *the stopped process* N^S *is of the form*

$$N^S = \left(\frac{(a'-a)}{\sigma^2} X_{t-} 1_{[0,S]}\right) \cdot X^c,$$

where X^c *denotes the continuous martingale part of* X *under* P^a*.*

Maximum-Likelihood-Estimator (MLE)

For continuous observations of the Ornstein-Uhlenbeck process X the likelihood function $\mathcal{L}$ for the statistical experiment $(\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a\in\mathbb{R}})$ takes the form

$$\mathcal{L}(a, X^T) = \frac{dP_t^a}{dP_t^0} = \exp\left(-\frac{a}{\sigma^2}\int_0^T X_{s-}\, dX_s^c - \frac{a^2}{2\sigma^2}\int_0^T X_{s-}^2\, ds\right).$$

Hence, the MLE for a is explicitly given by

$$\hat{a}_T = -\frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_{s-}^2 ds}.$$

The continuous martingale part X^c

By the Lévy-Itô decompostition of L we can write X as

$$X_t = X_0 - a \int_0^t X_{s-}\, ds + W_t + J_t \quad , t \geq 0,$$

where W is a Wiener Process and J a quadratic pure jump process given by

$$J_t = \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) + bt + \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}.$$

N is the jump measure of L with compensator μ.

Under P^0 it follows that $X^c = W$, but under P^a

$$\tilde{W}_t = W_t - a \int_0^t X_{s-}\, ds$$

defines a Wiener process such that $X^c = \tilde{W}$ under P^a.
Hence, given observations $(X_t(\omega), t \in [0, T])$

$$X_t^c = X_t - \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) - bt - \sum_{0 \le s \le t} \Delta X_s 1_{\{|\Delta X_s| \ge 1\}}.$$

which can be reconstructed from continuous observations.
Hence, the MLE can be rewritten as

$$\hat{a}_T = -\frac{\int_0^T X_{s-}(dW_s - aX_{s-}ds)}{\int_0^T X_{s-}^2 ds} = a - \frac{\int_0^T X_{s-} dW_s}{\int_0^T X_{s-}^2 ds}.$$

under P^a.

Curved Exponential Families

Let $\{P^\theta, \theta \in \Theta\}$ be a family of measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

Definition (Küchler and Sørensen (1997))

We say that a statistical experiment $\{P^\theta, \theta \in \Theta\}$ forms a ***curved exponential family*** *if the likelihood function exists and is of the form*

$$\frac{dP_t^\theta}{dP_t^{\theta_0}} = \exp\left(\theta' A_t - \kappa(\theta) S_t\right)$$

where $\kappa : \Theta \to \mathbb{R}$, for $\theta_0 \in \Theta$ arbitrary but fixed and $A : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is a càdlàg process. Moreover, $S : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is assumed to be a non-decreasing continuous process with $S_0 = 0$ and $S_t \stackrel{t\to\infty}{\longrightarrow} \infty$.

Strong Consistency

Theorem

Under the condition $\sigma^2 > 0$ the MLE $\hat{a}_T$ for any $a \in \mathbb{R}$ based on continuous observations X_t, $t \in [0, T]$ exists and is given by

$$\hat{a}_T = -\frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_{s-}^2 ds}.$$

Furthermore, under P_a the maximum likelihood estimator is unique for T sufficiently large and

$$\hat{a}_T \to a \quad \textit{almost surely}$$

as $T \to \infty$.

Asymptotic Normality

Theorem

Let X be a stationary OU process and $a > 0$ *such that* $c = E[X^2] < \infty$. *Then under* P^a

$$\sqrt{T}(\hat{a}_T - a) \to N(0, \frac{\sigma^2}{c}) \quad \textit{weakly}$$

as $T \to \infty$.

Discrete Observations

Given discrete observations $X_\Delta, X_{2\Delta}, \ldots, X_{n\Delta}$ with step size Δ a dicretized version of $\hat{a}$ is

$$\hat{a}_n = -\frac{\sum_{m=0}^{n-1} X_{m\Delta}\ \delta X_m^c}{\Delta \sum_{m=0}^{n-1} X_{m\Delta}}$$

with increments $\delta X_m^c = X_{(m+1)\Delta}^c - X_{m\Delta}^c$.

Theorem

Under the assumption that X is stationary and $K_a(t) = E_a(X_t X_0)$ is continuously differentiable in t the MLE satisfies

$$\hat{a}_n \stackrel{n\to\infty}{\longrightarrow} a + \Delta a \frac{\dot{K}_a(0)}{E_a(X_0^2)} + O(\Delta^2) \qquad P_a\text{-}a.s.$$

Proof

We rewrite $\hat{a}_n = \frac{A_n}{I_n} = -\frac{\sum_{m=0}^{n-1} X_{m\Delta}\ \delta X_m}{\Delta \sum_{m=0}^{n-1} X_{m\Delta}}$ as

$$
\begin{aligned}
A_n &= \sum_{m=0}^{n-1} X_{m\Delta}\ \delta W_m - a \sum_{m=0}^{n-1} X_{m\Delta} \int_{m\Delta}^{(m+1)\Delta} X_{s-}\ ds \\
&= \sum_{m=0}^{n-1} X_{m\Delta}\ \delta W_m - a \sum_{m=0}^{n-1} X_{m\Delta} \int_{m\Delta}^{(m+1)\Delta} (X_{s-} - X_{m\Delta})\ ds - a\Delta \sum_{m=0}^{n-1} X \\
&=: Z_n - aR_n - aI_n
\end{aligned}
$$

By ergodicity of X it follows that P_a-a.s.

$$\frac{I_n}{n} \stackrel{n\to\infty}{\longrightarrow} \Delta n, \quad \frac{Z_n}{n} \stackrel{n\to\infty}{\longrightarrow} 0,$$

and

$$\frac{R_n}{n} \stackrel{n\to\infty}{\longrightarrow} \int_0^\Delta K_a(t)\, dt - \Delta E(X_0^2),$$

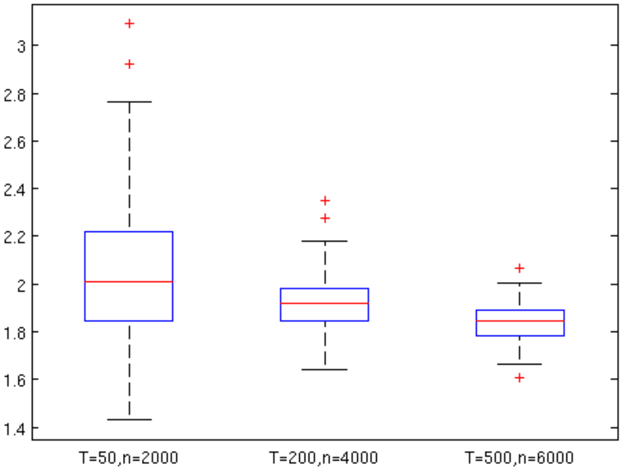
such that finally

$$\hat{a}_n = \frac{Z_n}{I_n} - a\frac{R_n}{I_n} - a \stackrel{n\to\infty}{\longrightarrow} a\frac{\int_0^\Delta K_a(t)\, dt}{\Delta E_a(X_0^2)} \quad P_a\text{-a.s.}$$

□

Simulations

Boxplot for $\hat{a}_n$ from a Wiener process plus compound Poisson ($\lambda = 4$, N(0,1)-jumps) driver and true parameter $a = 2$.



Summary and Outlook

- Under $\sigma^2 > 0$ the solution measures $\{P^a, a \in \mathbb{R}\}$ are locally equivalent.
- The MLE takes an explicit form and is efficient and asymptotically normal.
- Computation from discrete observations is straight forward.
- Asymptotics for $a < 0$ and without second moments of X?
- Delay estimation

Bibliography

Ole E. Barndorff-Nielsen and Neil Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc.*, 63(2), 2001.

Yaozhong Hu and Hongwei Long. Least squares estimator for ornstein-uhlenbeck processes driven by [alpha]-stable motions. *Stochastic Processes and their Applications*, 2009.

G. Jongbloed, F.H. van der Meulen, and A.W. van der Vaart. Nonparametric inference for Lévy-driven Ornstein-Uhlenbeck processes. *Bernoulli*, 11(5), 2005.

L. Valdivieso, W. Schoutens, and F. Tuerlinckx. Maximum Likelihood Estimation in Processes of Ornstein-Uhlenbeck type. *Stat. Infer. Stoch Process.*, 12:1–19, 2009.