Maximum-Likelihood-Estimation of Lévy driven Ornstein-Uhlenbeck Processes

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Ornstein-Uhlenbeck (OU) Process

Let \((L_t, t \geq 0)\) be a Lévy process on \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\). For every \(a \in \mathbb{R}\)

\[
dX_t = -aX_t \, dt + dL_t, \quad t \in \mathbb{R}_+, \quad X_0 = x,
\]

defines an Ornstein Uhlenbeck process driven by the Lévy process \(L\) with initial distribution \(\pi = \mathcal{L}(X_0)\).

Equivalently,

\[
X_t = e^{-at}X_0 + \int_0^t e^{-a(t-s)} \, dL_s.
\]
Sample path from compound Poisson plus Wiener process driver
Setting

Problem: Estimation of \( a \) from continuous observations \( X_t, 0 \leq t \leq T \) and known Lévy-Khintchine triplet of \( L \).

We work throughout in the canonical setting:

- \( \Omega = D(\mathbb{R}_+) = \{ f : \mathbb{R}_+ \to \mathbb{R}; \ f \text{ càdlàg} \} \)
- \( X(\omega, t) = \omega(t) \) for all \( \omega \in \Omega \) coordinate process
- Filtration generated by \( X \)

\[
\mathcal{F}_t = \bigcap_{s > t} \sigma(X_u : u \leq s) \text{ and } \mathcal{F} = \bigvee \mathcal{F}_t
\]

For every \( a \in \mathbb{R} \) we obtain a solution measure \( P^a \) of the OU equation on \( D(\mathbb{R}_+) \).
Absolute Continuity/Singularity (ACS)

Problem

\[ P_t^a := P^a \big|_{\mathcal{F}_t} \] denotes the restriction of \( P^a \) to \( \mathcal{F}_t \).

Local absolute continuity:

\[ P^a' \overset{\text{loc}}{\ll} P^a \iff P_t^a' \ll P_t^a \quad \forall t \in \mathbb{R}_+ \]

1. Does \( P^a' \overset{\text{loc}}{\ll} P^a \) hold for all \( a, a' \in \mathbb{R} \)?

2. Can we derive \( Z_t = \frac{dP_t^{a'}}{dP_t^a} \) explicitly?
Theorem

Let $P^a, P^{a'}$ be two solution measures of the OU equation for the driving Lévy process $L$ with characteristic triplet $(b, \sigma^2, \rho)$ and initial distributions $\pi$ and $\pi'$. Suppose that $\sigma^2 > 0$ and $\pi' \ll \pi$, then we have

$$P^{a'} \ll_{\text{loc}} P^a.$$
Semimartingale characteristics of $X$

Let $(b, \sigma^2, \mu)$ denote the Lévy-Khintchine triplet of $L$.

Then the semimartingale characteristics $(B, C, \nu)$ of $X$ are given by

$$B(\omega, t) = bt - a \int_0^t X_{s-}(\omega) ds,$$

$$C(\omega, t) = \sigma^2 t,$$

$$\nu(\omega, dt, dx) = \mu(dx) \lambda(dt),$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$. 
The Hellinger Process of $X$

**Proposition**

A version of the Hellinger process of two solution measures $P^a, P^{a'}$ is

$$h_t(\alpha; a, a') = \frac{\alpha(1 - \alpha)}{2\sigma^2} \int_0^t \left[ \int_0^u \left( a' e^{-a'(u-s)} - ae^{-a(u-s)} \right) L(ds) \right]^2 du.$$  

for $a, a' \in \mathbb{R}$. 
Let $P, P'$ be two probability measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

**Theorem (Jacod and Mémin (1979))**

Let $h(\alpha)$, $\alpha \in (0, 1)$ be a version of the Hellinger process $h(\alpha; P, P')$. Then for every stopping time $T$ there is equivalence between

1. $P'_T \ll P_T$;
2. $\exists \alpha \in (0, 1)$ such that $P'(h(\alpha)_T < \infty) = 1$ and $P'_0 \ll P_0$ and $P'(h(0)_T = 0) = 1$. 
Second Problem

Proposition

There exists a local martingale $N : \Delta \to \mathbb{R}$ on a random interval $\Delta \subset \Omega \times \mathbb{R}_+$ such that the density process is given by

$$Z_t = \frac{dP_{t}^{a'}}{dP_{t}^{a}} = Z_0 \exp \left( N_t - \frac{(a' - a)^2}{2\sigma^2} \int_0^t X_s^2 ds \right).$$

Furthermore, for every stopping time $S$ such that $[0, S] \subset \Delta$ the stopped process $N^S$ is of the form

$$N^S = \left( \frac{(a' - a)}{\sigma^2} X_{t-1[0,S]} \right) \cdot X^c,$$

where $X^c$ denotes the continuous martingale part of $X$ under $P^a$. 
For continuous observations of the Ornstein-Uhlenbeck process $X$ the likelihood function $\mathcal{L}$ for the statistical experiment $(\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a \in \mathbb{R}})$ takes the form

$$\mathcal{L}(a, X^T) = \frac{dP^a_t}{dP^0_t} = \exp \left( -\frac{a}{\sigma^2} \int_0^T X_{s-} \, dX^c_s - \frac{a^2}{2\sigma^2} \int_0^T X^2_{s-} \, ds \right).$$

Hence, the MLE for $a$ is explicitly given by

$$\hat{a}_T = -\frac{\int_0^T X_{s-} \, dX^c_s}{\int_0^T X^2_{s-} \, ds}.$$
The continuous martingale part $X^c$

By the Lévy-Itô decompostition of $L$ we can write $X$ as

$$X_t = X_0 - a \int_0^t X_s^- \, ds + W_t + J_t, \quad t \geq 0,$$

where $W$ is a Wiener Process and $J$ a quadratic pure jump process given by

$$J_t = \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) + bt + \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}.$$

$N$ is the jump measure of $L$ with compensator $\mu$. 
Under $P^0$ it follows that $X^c = W$, but under $P^a$

$$
\tilde{W}_t = W_t - a \int_0^t X_{s-} \, ds
$$

defines a Wiener process such that $X^c = \tilde{W}$ under $P^a$. Hence, given observations $(X_t(\omega), t \in [0, T])$

$$
X^c_t = X_t - \int \mathbb{1}_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) - bt - \sum_{0 \leq s \leq t} \Delta X_s \mathbb{1}_{\{\Delta X_s \geq 1\}}.
$$

which can be reconstructed from continuous observations. Hence, the MLE can be rewritten as

$$
\hat{a}_T = -\frac{\int_0^T X_{s-} (dW_s - aX_{s-} \, ds)}{\int_0^T X_{s-}^2 \, ds} = a - \frac{\int_0^T X_{s-} \, dW_s}{\int_0^T X_{s-}^2 \, ds}.
$$

under $P^a$. 
Let \( \{ P^\theta, \theta \in \Theta \} \) be a family of measures on \((\Omega, \mathcal{F}, (\mathcal{F}_t))\).

**Definition (Küchler and Sørensen (1997))**

We say that a statistical experiment \( \{ P^\theta, \theta \in \Theta \} \) forms a **curved exponential family** if the likelihood function exists and is of the form

\[
\frac{dP^\theta_t}{dP^\theta_0} = \exp \left( \theta' A_t - \kappa(\theta) S_t \right)
\]

where \( \kappa : \Theta \to \mathbb{R} \), for \( \theta_0 \in \Theta \) arbitrary but fixed and \( A : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d \) is a càdlàg process. Moreover, \( S : \Omega \times \mathbb{R}_+ \to \mathbb{R} \) is assumed to be a non-decreasing continuous process with \( S_0 = 0 \) and \( S_t \xrightarrow{t \to \infty} \infty \).
Strong Consistency

Theorem
Under the condition $\sigma^2 > 0$ the MLE $\hat{a}_T$ for any $a \in \mathbb{R}$ based on continuous observations $X_t$, $t \in [0, T]$ exists and is given by

$$\hat{a}_T = -\frac{\int_0^T X_s dX_s^c}{\int_0^T X_s^2 ds}.$$ 

Furthermore, under $P_a$ the maximum likelihood estimator is unique for $T$ sufficiently large and

$$\hat{a}_T \to a \text{ almost surely}$$

as $T \to \infty$. 
Asymptotic Normality

Theorem
Let $X$ be a stationary OU process and $a > 0$ such that $c = E[X^2] < \infty$. Then under $P^a$

$$\sqrt{T}(\hat{a}_T - a) \to N(0, \frac{\sigma^2}{c}) \quad \text{weakly}$$

as $T \to \infty$. 
Discrete Observations

Given discrete observations $X_{\Delta}, X_{2\Delta}, \ldots, X_{n\Delta}$ with step size $\Delta$ a discretized version of $\hat{a}$ is

$$\hat{a}_n = -\frac{\sum_{m=0}^{n-1} X_{m\Delta} \delta X_m^c}{\Delta \sum_{m=0}^{n-1} X_{m\Delta}}$$

with increments $\delta X_m^c = X_{(m+1)\Delta}^c - X_m^c$.

**Theorem**

Under the assumption that $X$ is stationary and $K_a(t) = E_a(X_tX_0)$ is continuously differentiable in $t$ the MLE satisfies

$$\hat{a}_n \xrightarrow{n \to \infty} a + \Delta a \frac{\dot{K}_a(0)}{E_a(X_0^2)} + O(\Delta^2) \quad P_a\text{-a.s.}$$
Proof

We rewrite $\hat{a}_n = \frac{A_n}{l_n} = -\frac{\sum_{m=0}^{n-1} X_{m\Delta} \Delta X_m}{\Delta \sum_{m=0}^{n-1} X_{m\Delta}}$ as

$$A_n = \sum_{m=0}^{n-1} X_{m\Delta} \delta W_m - a \sum_{m=0}^{n-1} X_{m\Delta} \int_{m\Delta}^{(m+1)\Delta} X_s \ ds$$

$$= \sum_{m=0}^{n-1} X_{m\Delta} \delta W_m - a \sum_{m=0}^{n-1} X_{m\Delta} \int_{m\Delta}^{(m+1)\Delta} (X_s - X_{m\Delta}) \ ds - a \Delta \sum_{m=0}^{n-1} X_{m\Delta}$$

$$=: Z_n - aR_n - aI_n$$
By ergodicity of $X$ it follows that $P_a$-a.s.

$$\frac{I_n}{n} \xrightarrow{n \to \infty} \Delta n, \quad \frac{Z_n}{n} \xrightarrow{n \to \infty} 0,$$

and

$$\frac{R_n}{n} \xrightarrow{n \to \infty} \int_0^\Delta K_a(t) \, dt - \Delta E(X_0^2),$$

such that finally

$$\hat{a}_n = \frac{Z_n}{I_n} - a \frac{R_n}{I_n} - a \xrightarrow{n \to \infty} a \frac{\int_0^\Delta K_a(t) \, dt}{\Delta E_a(X_0^2)} \quad P_a\text{-a.s.}$$
Simulations

Boxplot for $\hat{a}_n$ from a Wiener process plus compound Poisson ($\lambda = 4$, $N(0,1)$-jumps) driver and true parameter $a = 2$. 

T=50, n=2000  T=200, n=4000  T=500, n=6000
Summary and Outlook

• Under $\sigma^2 > 0$ the solution measures $\{P^a, a \in \mathbb{R}\}$ are locally equivalent.
• The MLE takes an explicit form and is efficient and asymptotically normal.
• Computation from discrete observations is straightforward.
• Asymptotics for $a < 0$ and without second moments of $X$?
• Delay estimation

