

On ballisticity criteria for RWRE

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(joint work with Alejandro Ramírez)



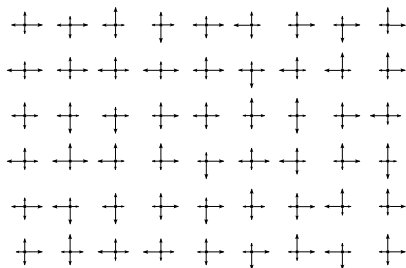
Contents

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- 2 Problem and results
- 3 Outline & ideas of the proof
- 4 Open questions

Model

Heuristics:

- *RWRE*: \mathbb{Z}^d -valued nearest-neighbour “random walk” with transition probabilities randomly chosen for each site;
- *environment*: family of transition probabilities (indexed by \mathbb{Z}^d);



Model

Formally:

- *environment* $\omega \in \Omega := (\mathcal{M})^{\mathbb{Z}^d}$, \mathcal{M} the set of probability measures on $\{\mathbf{e} \in \mathbb{Z}^d : |\mathbf{e}| = 1\}$;

given environment $(\omega(z, \cdot))_{z \in \mathbb{Z}^d}$, *quenched RWRE* is the Markov chain (X_n) with state space \mathbb{Z}^d starting in x with law $P_{x,\omega}$ s.t.

$$P_{x,\omega}(X_{n+1} = z + \mathbf{e} | X_n = z) := \omega(z, \mathbf{e}),$$

$$z \in \mathbb{Z}^d, |\mathbf{e}| = 1.$$

- *environment measure* $\mathbb{P} = \mu^{\mathbb{Z}^d}$, μ measure on \mathcal{M} with $\mu(\inf_{\mathbf{e}} \omega(0, \mathbf{e}) \geq \kappa) = 1$ some $\kappa > 0$.
Annealed (or averaged) measures

$$P_x = \int_{\Omega} P_{x,\omega} \mathbb{P}(d\omega).$$

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Theorem (Sznitman, Zerner, [SZ99], [Zer02])

Assume

$$P_0(\lim_n X_n \cdot e = \infty) \in \{0, 1\} \quad \forall e \in \mathbb{Z}^d \text{ with } |e| = 1.$$

Then there exists $v \in \mathbb{R}^d$ deterministic s.t. P_0 -a.s.

$$\lim_n \frac{X_n}{n} = v.$$

Question: For $d \geq 2$, can we find characterisations of ballisticity, i.e. of the case $v \neq 0$?

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Ballisticity criteria

Definition (Sznitman)

For $\gamma \in (0, 1)$, $l \in \mathbb{S}^{d-1}$ condition $(T)_\gamma|l$ is satisfied if for each l' in a neighbourhood of l and for all $b > 0$ one has

$$\limsup_{L \rightarrow \infty} L^{-\gamma} \log P_0(X_{T_{U_{l',b,L}}} \cdot l' < 0) < 0,$$

where $U_{l',b,L} := \{x \in \mathbb{Z}^d : -bL < x \cdot l' < L\}$.

Furthermore: $(T)'|l : \iff (T)_\gamma|l$ holds for all $\gamma \in (0, 1)$.

Ballisticity criteria

[Szn02]:

- 1 $(T')|I$ implies ballisticity.
- 2 For $\gamma \in (\frac{1}{2}, 1)$: $(T)_\gamma|I \iff (T')|I$.

Theorem (D. Ramírez, [DR09])

Let $d \geq 2$ and

$$\gamma_d := \frac{\sqrt{3d^2 - d} - d}{2d - 1}.$$

Then, for each $\gamma \in (\gamma_d, 1)$ and $I \in \mathbb{S}^{d-1}$, $(T)_\gamma|I$ is equivalent to $(T')|I$.

Remark

Depending on $d \in \mathbb{N}$, γ_d takes values in the interval $(0.366, 0.388)$

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Main ideas of the proof

- *Recall:* In $d = 1$, Solomon [Sol75] showed that (X_n) is ballistic (to the right) iff

$$\mathbb{E} \frac{\omega(\mathbf{0}, -1)}{\omega(\mathbf{0}, 1)} < 1.$$

- *Higher-dimensional "analogue":*

Theorem (Sznitman, [Szn02])

Effective criterion $\iff (T^1) \uparrow \Psi$.

Sufficient for establishing effective criterion: show that for some $\alpha > 0$,

$$\mathbb{E} \rho^{L^{-\alpha}} \rightarrow 0$$

fast enough as $L \rightarrow \infty$, where

$$\rho(\omega) := \frac{P_{0,\omega}(X_{T_B} \notin \partial_+ B)}{P_{0,\omega}(X_{T_B} \in \partial_+ B)}.$$

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Main ideas of the proof

Decomposition

$$\mathbb{E} \rho^{L^{-\alpha}} = (I) + (II) + (III),$$

where

$$(I) := \mathbb{E} (\rho^{L^{-\alpha}}, P_{0,\omega}(X_{T_B} \in \partial_+ B) > e^{-k_0 L^\gamma})$$

as well as

$$(II) := \mathbb{E} (\rho^{L^{-\alpha}}, e^{-k_1 L^{\beta_1}} < P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_0 L^\gamma})$$

and

$$(III) := \sum_{j=1}^n \mathbb{E} (\rho^{L^{-\alpha}}, e^{-k_{j+1} L^{\beta_{j+1}}} < P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_j L^{\beta_j}}),$$

with constants β_j s.t.

$$1 = \beta_{n+1} > \beta_n > \cdots > \beta_1 > (1 + \gamma)^{-1}.$$

Estimate first summand

$$(I) := \mathbb{E} \left(\rho^{L^{-\alpha}}, P_{0,\omega}(X_{T_B} \in \partial_+ B) > e^{-k_0 L^\gamma} \right)$$

Lemma

For all $L > 0$,

$$(I) \leq e^{k_0 L^{\gamma-\alpha} - \delta_1 L^{\gamma-\alpha} + o(L^{\gamma-\alpha})},$$

where

$$\delta_1 := - \limsup_{L \rightarrow \infty} L^{-\gamma} \log P_0(X_{T_B} \notin \partial_+ B) > 0.$$

Proof.

Jensen implies $(I) \leq e^{k_0 L^{\gamma-\alpha}} P_0(X_{T_B} \notin \partial_+ B)^{L^{-\alpha}}$.

$(T)_\gamma$ entails $P_0(X_{T_B} \notin \partial_+ B)^{L^{-\alpha}} \leq e^{-\delta_1 L^{\gamma-\alpha} + o(L^{\gamma-\alpha})}$. □

\implies Choose $\alpha < \gamma$.

Estimate second summand

$$(II) := \mathbb{E}(\rho^{L^{-\alpha}}, e^{-k_1 L^{\beta_1}} < P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_0 L^\gamma})$$

Lemma

For all $L > 0$,

$$(II) \leq e^{k_1 L^{\beta_1 - \alpha} - \delta_1 L^\gamma + o(L^\gamma)}.$$

Proof.

$$\begin{aligned} (II) &\leq e^{k_1 L^{\beta_1 - \alpha}} \mathbb{E}(P_{0,\omega}(X_{T_B} \notin \partial_+ B)^{L^{-\alpha}}, P_{0,\omega}(X_{T_B} \notin \partial_+ B) \geq 1 - e^{-k_0 L^\gamma}) \\ &\leq e^{k_1 L^{\beta_1 - \alpha}} P_0(X_{T_B} \notin \partial_+ B) (1 - e^{-k_0 L^\gamma})^{-1}. \end{aligned}$$



\implies Choose $\beta_1 < 2\gamma$, $\alpha \in (\beta_1 - \gamma, \gamma)$.

Estimate third summand

Lemma

Let the β_j 's be chosen as above. Then, for all $j \in \{1, \dots, n\}$ and $\zeta \in (0, f(\beta_j))$,

$$\mathbb{E}(\rho^{L^{-\alpha}}, e^{-k_{j+1}L^{\beta_{j+1}}} \leq P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_j L^{\beta_j}}) \leq e^{k_{j+1}L^{\beta_{j+1}-\alpha - \delta_2 L^\zeta + o(L^\zeta)}$$

where f is defined by

$$f : ((1 + \gamma)^{-1}, 1) \ni \beta \mapsto d\left(\beta - \frac{1}{1 + \gamma}\right) \frac{1 + \gamma}{\gamma},$$

and

$$\delta_2 := - \limsup_{L \rightarrow \infty} L^{-\zeta} \log \mathbb{P}(P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-cL^{\beta_j}}) > 0.$$

(Proof omitted, via renormalisation estimate from [Szn01])

\implies Need $\beta_{j+1} - \alpha < f(\beta_j)$

Fixed-point iteration

- 1 Conditions from estimating (I) to (III) :

$$\beta_1 < 2\gamma, \quad \alpha \in (\beta_1 - \gamma, \gamma), \quad \beta_{j+1} - \alpha < f(\beta_j).$$

- 2 Leads to

$$\beta_{j+1} < f(\beta_j) + \gamma.$$

- 3 Unstable fixed point of the iteration of $x \mapsto f(x) + \gamma$ is given by

$$x^* := \frac{d - \gamma^2}{(1 + \gamma)d - \gamma}.$$

- 4 Finally, the starting condition $\beta_1 > x^*$ yields

$$\gamma > \gamma_d = \frac{-2d + \sqrt{12d^2 - 4d}}{2(2d - 1)}.$$

Open questions

- 1 Can one lower γ_d to 0 (for all $d \geq 2$)?
- 2 Does ballisticity imply $(T)'|\hat{v}$?

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