Quantum spin systems: a dynamical approach

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Classical spin systems

A classical spin system consists of

- $\Lambda \subset\subset \mathbb{Z}^d$, $d \geq 1$;
- a spin configuration $\sigma \in \Omega_\Lambda := \{-1, +1\}^\Lambda$;
- an Hamiltonian $H : \Omega_\Lambda \to \mathbb{R} \rightsquigarrow$ the Gibbs measure

$$\mu(\sigma) := \frac{1}{Z} e^{-\beta H(\sigma)},$$

where $\beta := 1/T$.

How can we view it in quantum terms?

Quantum reformulation of the classical model

- ± 1 are dressed as quantum eigenfunctions of the Pauli matrix $\hat{\sigma}^z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and are indicated as

$$|\,1\,\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\,-1\,\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

- $\Omega_\Lambda \ni \sigma \rightsquigarrow |\,\sigma\,\rangle := \otimes_{i\in\Lambda} |\,\sigma_i\,\rangle \in \mathbb{X}_\Lambda := \bigotimes_{i\in\Lambda} \mathbb{R}^2$;
- $\mathbb{X}_\Lambda$ becomes a Hilbert space with a scalar product: $\langle\,\sigma\,|\,\sigma'\,\rangle := \prod_{i\in\Lambda} \langle\,\sigma_i\,|\,\sigma'_i\,\rangle_2$. $\mathcal{B} := \{|\,\sigma\,\rangle\}_{\sigma\in\Omega_\Lambda}$ is an orthonormal basis of the space.
- For $i \in \Lambda$, $\hat{\sigma}_i^z|\,\sigma\,\rangle := |\,\sigma_1\,\rangle \otimes \ldots \otimes \hat{\sigma}^z|\,\sigma_i\,\rangle \otimes \ldots |\,\sigma_{|\Lambda|}\,\rangle$. So

$$\hat{\sigma}_i^z \hat{\sigma}_j^z |\,\sigma\,\rangle = \sigma_i \sigma_j |\,\sigma\,\rangle$$

Rephrasing of functions

Definition

Given $f : \Omega_\Lambda \to \mathbb{R}$,

$$\mathcal{F} : \mathbb{X}_\Lambda \to \mathbb{X}_\Lambda \quad \text{s. t.} \quad \langle \sigma \,|\, \mathcal{F} \,|\, \sigma \rangle := f(\sigma), \quad \forall \,|\, \sigma \,\rangle \in \mathcal{B} \tag{1}$$

Notation When $f := H$, the associated quantum operator is $\mathcal{H}_1$.

- The partition function $Z := \sum_{\sigma \in \Omega_\Lambda} e^{-\beta H(\sigma)} = \mathrm{Tr}\left(e^{-\beta \mathcal{H}_1}\right)$.
- $< \mathcal{F} >= \frac{\mathrm{Tr}\left(\mathcal{F} e^{-\beta \mathcal{H}_1}\right)}{\mathrm{Tr}\left(e^{-\beta \mathcal{H}_1}\right)}$.
- $< \mathcal{F}; \mathcal{G} >=< \mathcal{F}\mathcal{G} > - < \mathcal{F} >< \mathcal{G} >$.

Definition of a quantum model with transverse field

We now change the Hamiltonian in this way: setting $\hat{\sigma}^x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\hat{\sigma}_i^x$ as before, we introduce a *transverse field* of intensity $\lambda \geq 0$:

$$-\mathcal{H} := -\mathcal{H}_1 + \lambda \sum_{i \in \Lambda} \hat{\sigma}_i^x$$

$\mathcal{H}_1$ and $\sum_{i \in \Lambda} \hat{\sigma}_i^x$ do not commute.
Goal: represent the new model in classical terms.

Path-integral representation

We do this in two steps:

1. for every $i \in \Lambda$ throw points $\sim PPP(\lambda)$ on the loop $\mathbb{S}_\beta$. The arrival configuration is denoted with ξ_i.

2. Color the connected components of each loop with two colors, independently and with probability $1/2$. The coloring is coded by a function $\tau_i(t)$, $t \in [0, \beta)$.

Let $\mathcal{D} := \{\sigma : \mathbb{S}_\beta \to \{-1, +1\} : \textit{piece} - \textit{wise constant}, \ \textit{right continuous}\}$ and $\mathcal{S}$ the set of finite subsets of $\mathbb{S}_\beta$.

Definition (*Spin configuration*)

$$\{(\xi_i, \tau_i(\cdot))\}_{i\in\Lambda} \in (\mathcal{S} \times \mathcal{D})^\Lambda .$$

$P(\mathrm{d}\xi_i)$:= Poisson measure of intensity λ on one site.

$$\pi(\tau, \mathrm{d}\xi) := \frac{1}{Z} \otimes P(\mathrm{d}\xi)\mathbf{1}_{\{\tau\sim\xi\}} \exp\left[\int_0^\beta -H(\tau(t))\mathrm{d}t\right]$$

Theorem (Quantum mean of an observable)

For $\mathcal{F}$ as in (1)

$$< \mathcal{F} >= \int f(\tau(0))\pi(\tau, \mathrm{d}\xi)$$

that is the quantum mean of $\mathcal{F}$ is an average w. r. t. a classical probability measure.

Definition of the dynamics

Definition

If $\sigma \in \mathcal{S}^\Lambda$, $a \in \mathcal{S}$, we denote with $\sigma^i a$ the element of $\mathcal{S}^\Lambda$ obtained from σ by replacing σ_i by a.

Set of transitions G:

$$\gamma_{i,+}^{x,a,b}(\sigma,\xi) := \begin{cases} (\sigma^i b, \xi \overset{i}{\cup} \{x\}) & \text{if } \sigma_i = a,\ b \sim \xi_i \cup \{x\} \\ (\sigma,\xi) & \text{otherwise} \end{cases}$$

$$\gamma_{i,-}^{x,a,b}(\sigma,\xi) := \begin{cases} (\sigma^i b, \xi \overset{i}{\setminus} \{x\}) & \text{if } x \in \sigma_i,\ \sigma_i = a,\ b \sim \xi_i \setminus \{x\} \\ (\sigma,\xi) & \text{otherwise} \end{cases}$$

$$\nabla_\gamma f(\sigma,\xi) := f(\gamma(\sigma,\xi)) - f(\sigma,\xi).$$

We construct a dynamics reversible w. r. t. π with generator

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\begin{aligned}
&\mathcal{L}f(\sigma,\xi) := \\
&\lambda \sum_{i\in\Lambda} \int_0^\beta \mathrm{d}x \frac{1}{2^{\#\xi_i+1}} \sum_{a\sim\xi_i\cup\{x\}} \exp\left[\int_0^\beta \left(\nabla_{\gamma_{i,+}^{x,\sigma_i,a}} H(\sigma(t))\right) \mathrm{d}t\right] \times \\
&\times \left[\nabla_{\gamma_{i,+}^{x,\sigma_i,a}} f\left((\sigma,\xi)\right)\right] + \\
&+ \sum_{i\in\Lambda} \frac{1}{2^{\#\xi_i}} \sum_{y\in\xi_i} \sum_{\eta\sim\xi_i\setminus\{y\}} \left[\nabla_{\gamma_{i,-}^{y,\sigma_i,\eta}} f\left((\sigma,\xi)\right)\right].
\end{aligned}
$$

Properties of the Hamiltonian

We require that $\forall\, \gamma \in G$ the following hold:

- Bounded gradient

$$\|\nabla_\gamma H\|_{L^\infty(\pi)} \leq C \tag{P1}$$

- Local function of range R: if γ acts on the i-th site,

$$\nabla_\gamma H(\sigma) = \nabla_\gamma H(\sigma') \tag{P2}$$

if $\sigma_k = {\sigma'}_k,\ \forall\, k :\ d(i,k) \leq R$.

Spectral gap

Using the Bochner-Bakry-Emery method we obtain the following

Theorem

If $(P1)$-$(P2)$ *are satisfied, for every* β *sufficiently small there exists* $k > 0$ *such that*

$$gap(\mathcal{L}) > k$$

uniformly in the volume $|\Lambda|$.

Local functions and decay of correlations

Let $\Lambda_f \subset\subset \Lambda$, $\Lambda_g \subset\subset \Lambda$, and $\Lambda_f \cap \Lambda_g = \emptyset$; let f (resp. g) be a local function on Λ_f (resp. Λ_g); $\mathcal{F}$, $\mathcal{G}$ as in (1).

Definition

A quantum spin model has *decay of covariances* if

$$< \mathcal{F}; \mathcal{G} >= o\left(d(\Lambda_f, \Lambda_g)\right)$$

Definition

A quantum model has an *exponential decay of correlations* if there exist C, C' constants s. t.

$$|< \mathcal{F}; \mathcal{G} >| \leq Ce^{-C'\, d(\Lambda_f, \Lambda_g)}$$

Positivity of the spectral gap $\Rightarrow$ decay of correlations

Theorem (Exponential decay of correlations)

Let H be an Hamiltonian with (P1)-(P2). If $gap(\mathcal{L}) > 0$ then the following holds: for every $\Lambda_f \subset\subset \Lambda$, $\Lambda_g \subset\subset \Lambda$ and $\Lambda_f \cap \Lambda_g = \emptyset$, f (resp. g) local on Λ_f (resp. Λ_g), there exist $C < +\infty$, $C' < +\infty$ constants not depending on $|\Lambda|$ s. t.

$$|< \mathcal{F}; \mathcal{G} >| = |\pi[f; g]| \leq C e^{-C' d(\Lambda_f, \Lambda_g)}$$

Sketch of the proof.

Lemma There are constants $A < +\infty$, $B < +\infty$, $\delta > 0$ that depend on $|\Lambda_f|$, $|\Lambda_g|$, $||f||_{L^2(\pi)}||$, $||g||_{L^2(\pi)}$, $|||f|||$ and $|||g|||$ s. t. for all t

$$|\pi[S_t(fg) - S_t f\, S_t g]| \leq A e^{Bt - \delta d(\Lambda_f, \Lambda_g)}$$

■

$$\begin{aligned} |\pi[f;g]| &= |\pi[S_t(fg)]| = \\ &= |\pi[S_t f\, S_t g] + \pi[S_t(fg) - S_t f\, S_t g]| \leq \\ &\leq ||S_t f||_{L^2(\pi)}\, ||S_t g||_{L^2(\pi)} + |\pi[S_t(fg) - S_t f\, S_t g]| \end{aligned}$$

by Cauchy-Schwarz inequality.

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- By Poincaré inequality ($G := 1/k$)

$$|\pi[f;g]| \leq e^{-\frac{2t}{G}}||f||_{L^2(\pi)}||g||_{L^2(\pi)} + Ae^{Bt-\delta d(\Lambda_f,\Lambda_g)}$$

- If A and B are chosen as in the Lemma, set $t := \frac{\delta}{2B}d(\Lambda_f,\Lambda_g)$ and the proof is concluded.

$\square$