Predictable projections of Itô SDEs

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Introduction

Motivated by the cubature method on Wiener spaces (developed by Lyons and Victoir in 2003), we introduce an approach to numerical solution of stochastic differential equations based on the predictable projection of stochastic processes. We also obtain preliminary results about complexification of Itô SDEs. In our knowledge, this approach hasn't been studied yet and has some unexplored potential.

Motivation

Assume we have an SDE of the type

$$dX_t = \mu((X_s)_{s\leq t})dt + \sigma((X_s)_{s\leq t})dW_t, \quad X_0 = x,$$

where μ and σ satisfy the usual (Lipschitz) conditions for existence and uniqueness. By considering a time discretization of the type

$$0 = t_0 < t_1 < \cdots < t_n = T,$$

we obtain for $k = 1, \cdots, n$ that

$$X_{t_k} = X_{t_{k-1}} + \int_{t_{k-1}}^{t_k} \mu((X_s)_{s\leq r})dr + \int_{t_{k-1}}^{t_k} \sigma((X_s)_{s\leq r})dW_r.$$

We concentrate our attention on the approximation of $\int_{t_{k-1}}^{t_k} \sigma((X_s)_{s\leq r})dW_r$.

Setting

- $(\Omega, \mathcal{F}, P)$ probability space
- (X, Y) 2-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$
- $Z := X + i.Y$ 1-dimensional complex Brownian motion
- $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$, $\mathbb{F}^Y = (\mathcal{F}_t^Y)_{t \geq 0}$ augmented filtrations of X resp. Y
- $(\mathcal{F}_t)_{t \geq 0}$ augmented filtration of Z

An introductory example

Example

Let $T > 0$ be fixed, and consider the Brownian motion reflected at T given by $\widetilde{Y}_t := Y_{T-t}$. Then, a straightforward application of Itô's formula yields that

$$E\left[\int_0^T (X_t + \widetilde{Y}_t) \circ d(X_t + \widetilde{Y}_t) | \mathcal{F}_T^X\right] = \int_0^T X_t dX_t.$$

We thus obtained an alternative representation of the Itô integral $\int_0^T X_t dX_t$ by considering the random, Brownian perturbation $(X + \widetilde{Y})$, integrating it according to Stratonovich and averaging over the added Brownian noise $\widetilde{Y}$.

Predictable projection

We shortly recall the definition of predictable projection of a stochastic process:

Definition (Predictable Projection)

Let $\mathbb{G}$ be a filtration on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{P}^{\mathbb{G}}$ denote the predictable σ-field with respect to $\mathcal{G}$. Then, for a measurable process X such that X is positive or bounded, there exists a unique process Z such that

1. *Z is predictable with respect to $\mathcal{P}^{\mathbb{G}}$, and*
2. *$E[X_T|\mathcal{F}_{T-}] = Z_T$ P-a.s. on $\{T < \infty\}$ for every predictable stopping time T.*

Z is then called the predictable projection of X on $\mathbb{G}$ and denoted by $X^{p,\mathbb{G}}$.

Predictable projection

For $\varphi : \mathbb{R}^2 \to \mathbb{R}$, we introduce the operator

$$A_s\varphi(x) := E[\varphi(x, Y_s)|\mathcal{F}^X_\infty],$$

where $\mathcal{F}^X_\infty = \bigvee_t \mathcal{F}^X_t$. This operator can be extended to any entire function $f : \mathbb{C} \to \mathbb{C}$ and, by a slight abuse of notation, for such a function we will denote $A_t f(X_t)$ by $A_t f(Z_t)$.

Some properties

$(A_t)_{t\geq 0}$ induces an operator $\mathbf{A}$ on stochastic processes via $\mathbf{A}M := (A_t M_t)_{t\geq 0}$, which corresponds to the predictable projection on $\mathbb{F}^X$ (as long as it exists).
This operator has some interesting properties, and provides alternative representations of well-known stochastic processes:

Examples

- $A_t Z_t^n = H_n(t, X_t)$ *for all* $t \geq 0$, *where* H_n *is the n-th Hermite polynomial.*
- $A_t e^{aZ_t} = \mathcal{E}(a.X)_t$ *for all* $t \geq 0$.

Some properties

Proposition

Let $(g(t,X_t))_{t\geq 0}$ *be a process such that* $g(t,X_t) \in L^2(\sigma(X_t))$*, and* $g(t,X_t) = \sum_{n\in\mathbb{N}} b_n H_n(t,X_t)$*. Then:*

1. $f(z) := \sum_{n\in\mathbb{N}} b_n z^n$ *is an entire function, and* $g(t,X_t) = A_t f(Z_t)$.
2. $\partial_x g(t,X_t) = A_t f'(X_t)$*, i.e.* $\partial_x \circ A_t = A_t \circ \frac{d}{dz}$.

Stochastic integrals

Proposition

Let H be a predictable process with respect to $\mathcal{P}^{\mathbb{F}}$ and such that $H \in L^2(Z)$. Then

$$E\left[\int_0^t H_s dZ_s | \mathcal{F}_\infty^X\right] = \int_0^t E[H_s|\mathcal{F}_\infty^X] dX_s \quad P\text{-a.s. for all } t \geq 0.$$

Corollary

Under above assumptions, we have that

$$\left(\int HdZ\right)^{p,\mathbb{F}^X} = \left(\int HdZ\right)^{p,\mathbb{F}^X} = \int H^{p,\mathbb{F}^X} dX.$$

Complexification of Itô SDEs

This approach seems interesting mainly for two reasons:

1. On the complex plane, under smoothness conditions, both Itô and Stratonovich calculus agree with each other $\Rightarrow$ we can represent Itô SDEs as predictable projections of corresponding complex, Stratonovich ones. Of course, Stratonovich integration is for numerical purposes preferable (fundamental theorem of calculus, Wong-Zakai Theorem).

Complexification of Itô SDEs

Example

Let H be predictable with respect to $\mathcal{P}^{\mathbb{F}^X}$, and consider the 1-dimensional SDE

$$d\widehat{M}_t = \widehat{M}_t H_t dX_t, \quad \widehat{M}_0 = 1.$$

The corresponding complex, Stratonovich version is

$$dM_t = M_t H_t \circ dZ_t, \quad M_0 = 1,$$

and its solution is given by

$$M_t = \exp\left(\int_0^t H_u dZ_u\right).$$

Complexification of Itô SDEs

Example

By the previous Lemma, we get that

$$dM_t^{p,\mathbb{F}^X} = (M_t H_t)^{p,\mathbb{F}^X} dX_t = H_t^{p,\mathbb{F}^X} M_t dX_t.$$

Therefore, $M_t^{p,\mathbb{F}^X} = \mathcal{E}(\int HdX)_t = M_t$ *as expected.*

Cubature on Wiener spaces

2. The approach we presented seems to be compatible with the cubature method on Wiener spaces (Lyons-Victoir, 2003), which has been proven to be very efficient in the weak approximation of SDEs, especially when used together with the reduction algorithm of Lyons and Litterer.

Thank you for your attention!