Duality of Markov processes with respect to a function

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Duality is a useful tool in a variety of areas – e.g., random walks with absorbing / reflecting barriers, queuing theory Feller, Asmussen, diffusions, interacting particle systems Liggett, hydrodynamic limits, population genetics...

**Basic idea:** in order to understand the long-time behavior of some complicated process, follow a simpler, dual process backwards in time.

**Related:** duality wrt a measure (potential theory Hunt ’57-58).
General theory available.

**Question:** general theory for duality wrt a function?
Outline

1. Duality with respect to a measure

2. Duality of linear operators. Existence and uniqueness

3. Spectrum


5. Quantum mechanics
Definition

- $E$ and $F$ Polish spaces (Borel $\sigma$-algebra).
- $(X_t)$ and $(Y_t)$ Markov processes with state spaces $E$ and $F$.
- $H : E \times F \to \mathbb{R}$ bounded, measurable (duality function).

$(X_t)$ and $(Y_t)$ are dual with respect to the function $H$ if for all $t > 0$, $x \in E$ and $y \in F$,

$$
\mathbb{E}_x H(X_t, y) = \mathbb{E}^y H(x, Y_t)
$$

$\mathbb{E}_x$: $(X_t)$ started in $x$, $\mathbb{E}^y$: $(Y_t)$ started in $y$.

In terms of the semi-groups $(P_t)$ and $(Q_t)$:

$$
\int_E P_t(x, dx') H(x', y) = \int_F Q_t(y, dy') H(x, y')
$$

Finite state spaces: matrix equation ($A^T$: transposed)

$$
P_t H = HQ_t^T.
$$

Special case: $H = \text{diag}(1/\mu(x))$:

$$
\mu(x) P_t(x, y) = \mu(y) Q_t(y, x).
$$

Duality wrt measure $\approx$ duality wrt diagonal function.
Pathwise duality

Fix a time horizon $T > 0$, starting points $x \in E$, $y \in F$. Dual processes can be defined on common probability space so that $X_0 = x$, $Y_0 = y$ $\mathbb{P}$-a.s. and

$$E H(x, Y_T) = E H(X_t, Y_{T-t}) = E H(X_T, y).$$

= weak pathwise duality. Often, processes can be coupled in such a way that

$$\forall t \in [0, T] : H(x, Y_T) = H(X_t, Y_{T-t}) = H(X_T, y) \quad \mathbb{P}\text{-a.s.}$$

= strong pathwise duality. Examples:

- $H(x, y) = 1(x \leq y)$ on totally ordered spaces Clifford, Sudbury ’85:

  $$\forall t \in [0, T] : x \leq Y_T \iff X_t \leq Y_{T-t} \iff X_T \leq y. \quad \mathbb{P}\text{-a.s.}$$

- Interacting particle systems with “graphical representations”.

- Stochastic recursions: common “driving sequence” $(W_n)$, recurrence relations

  $$X_{n+1} = f(X_n, W_n), \quad Y_{n+1} = g(Y_n, W_{T-n}).$$

  Ex. random walks, $W_n = \text{“up” or “down”}$ Lindley ’52, books by Feller, Asmussen.
Duality with respect to a measure

Discrete state spaces: if \( \mu(x) > 0 \) for all \( x \) and \( (P_t) \) and \( (Q_t) \) are time reversals of each other wrt \( \mu \), then \( H(x, y) = \frac{1}{\mu(x)} \delta_{x,y} \) is a duality function. What about continuous state spaces?

**Definition:** \( \mu \) \( \sigma \)-finite measure on \( E = F \). Processes \( (X_t) \), \( (Y_t) \) with semi-groups \( (P_t) \), \( (Q_t) \) and resolvents \( (R_\lambda) \), \( (\hat{R}_\lambda) \) are in duality wrt \( \mu \) if

- For all \( \lambda, x, y \), \( R_\lambda(x, \cdot) \) and \( \hat{R}_\lambda(x, \cdot) \) are abs. cont. wrt. \( \mu \).
- For all \( \lambda \) and \( f, g \geq 0 \), \( \int_E (P_tf)g \, d\mu = \int_E f(Q_tg) \, d\mu \).

**Theorem** Let \( (P_t), (Q_t) \) be in duality wrt \( \mu \). Write

\[
R_\lambda(x, dy) = r_\lambda(x, y) \mu(dy), \quad \hat{R}_\lambda(x, dy) = r_\lambda(y, x) \mu(dy)
\]

Functions \( y \mapsto r_\lambda(x, y) \) and \( x \mapsto r_\lambda(x, y) \) \( \lambda \)-excessive wrt \( (R_\lambda) \) resp. \( (\hat{R}_\lambda) \). Then for all \( \lambda > 0 \), \( r_\lambda(x, y) \) is a duality function for \( (X_t), (Y_t) \).

**Blumenthal, Getoor:** *Markov processes and potential theory.*
Duality of linear operators

**Question**: Relation with notions of duality in functional analysis?

**Natural framework**: dual pairs / duality wrt bilinear form.

**Def.**: \( V, W \) vector spaces. Bilinear form \( \langle \cdot, \cdot \rangle : V \times W \to \mathbb{R} \) called non-degenerate if right null space is trivial

\[
\mathcal{N}_R = \{ w \in W | \forall v \in V : \langle v, w \rangle = 0 \} = \{0\}
\]

and left null space is trivial. \((V, W, \langle \cdot, \cdot \rangle)\) is a dual pair.

Linear operators \( S : V \to V, T : W \to W \) are dual wrt \( \langle \cdot, \cdot \rangle \) if

\[
\forall v \in V \forall w \in W : \langle Sv, w \rangle = \langle v, Tw \rangle.
\]

Notion useful in study of weak topologies, locally convex vector spaces. s
Application to duality of Markov processes

- $\mathcal{M}(E)$ = finite signed Borel measures on $E$.
- Bilinear form associated with $H$:
  \[ B_H : \mathcal{M}(E) \times \mathcal{M}(F) \rightarrow \mathbb{R}, \quad B_H(\mu, \nu) := \int_{E \times F} H(x, y) \mu(dx)\nu(dy). \]
- \[(P_t^* \mu)(A) = (\mu P_t)(A) = \int_E \mu(dx)P_t(x, A).\]

**Observation:** Markov semi-groups $(P_t)$ and $(Q_t)$ dual wrt $H$ iff
\[
\forall t > 0 \ \forall \mu \in \mathcal{M}(E) \ \forall \nu \in \mathcal{M}(F) : \quad B_H(P_t^* \mu, \nu) = B_H(\mu, Q_t^* \nu).
\]

$H$ duality function for $(P_t), (Q_t) \Leftrightarrow (P_t^*), (Q_t^*)$ dual wrt bilinear form $B_H$.

**Question:** Given: $(P_t)$ and $H$, existence of a dual Markov semi-group $(Q_t)$? Uniqueness?
**Uniqueness**

**Functional analysis fact:** The dual of a given operator with respect to a non-degenerate bilinear form is unique.

**Consequence:** if $B_H$ is non-degenerate then the dual, if it exists, is unique. Applies to a lot of common duality functions:

- Siegmund dual: $H(x, y) = \mathbf{1}(x \leq y), x, y \in \mathbb{R}$.
- Moment dual: $H(x, n) = x^n, x \in [0, 1], n \in \mathbb{N}_0$.
- Laplace dual: $H(x, \lambda) = \exp(-\lambda x), \lambda, x \in [0, \infty)$.

Uniqueness in moment problems, injectivity of the Laplace transform.

**Remark:** Connection with kernel identities used in integrable quantum systems (Toda, Calogero-Moser systems) related to Doob $h$-transforms / non-colliding processes. Non-degeneracy does not come for free. **Ruijsenaars ’13** *On positive Hilbet-Schmidt operators*
Existence

**Functional analysis fact:** Every bounded operator in a Hilbert space has a unique adjoint. Every bounded operator in a Banach space has a unique dual operator. But: for general dual pairs, a given operator need not have a dual (need continuity in suitable weak topology...)

**Consequence:** semi-group \((P^*_t)\) need not have a dual semi-group of operators \(T_t : M(F) \rightarrow M(F)\).

**Additional problem:** dual operator semi-group need not come from a Markov semi-group – need to know whether \((T_t \nu)(A) = \int_F \nu(dy)Q_t(y, B)\) for Markov semi-group \((Q_t)\).

Existence and uniqueness criteria available for Siegmund duality in totally ordered state spaces (stochastic monotonicity). Generalization?

**Key:** invariance of convex sets Möhle.
Existence, continued

\[ \mathcal{M}(F) = \text{signed Borel measures}, \quad \mathcal{M}_{1,+}(F) = \text{probability measures}. \]

Images

\[ \mathcal{V} = \left\{ \int_F H(\cdot, y) \nu(dy) \mid \nu \in \mathcal{M}(F) \right\} \subset L^\infty(E) \]

\[ \mathcal{V}_{1,+} = \left\{ \int_F H(\cdot, y) \nu(dy) \mid \nu \in \mathcal{M}_{1,+}(F) \right\} \subset \mathcal{V}. \]

Analogous subsets \( \mathcal{W}_{1,+} \subset \mathcal{W} \subset L^\infty(F) \).

Finite state spaces: linear / convex combinations of rows / columns of \( H \).

**Proposition** Let \( E \) and \( F \) be countable, discrete state spaces and \( P \) a \( E \times E \) stochastic matrix. Then

1. \( PH = HQ^T \) has a solution \( Q = (Q(x, y))_{x, y \in F} \) with \( \sum_z |Q(y, z)| < \infty \) for all \( y \) iff \( \mathcal{V} \) is invariant.

2. \( Q \) can be chosen as a stochastic matrix iff \( \mathcal{V}_{1,+} \) is invariant.

Siegmund duality \( H(x, y) = 1(x \leq y) \) in totally ordered state spaces: \( \mathcal{V}_{1,+} \) = monotone increasing functions. Recover known relationship Siegmund duality / stochastic monotonicity.
Existence, continued

In continuous time and space, more delicate – Chapman-Kolmogorov and measurability issues, theorems for non-degenerate $H$ given by Möhle.

Convex combination of the columns of $H$:

$$V_{1,+} = \left\{ \int_F H(\cdot, y) \nu(dy) \mid \nu \in \mathcal{M}_{1,+}(F) \right\} \subset L^\infty(E).$$

**Theorem** $E, F$ locally compact, $H : E \times F \to \mathbb{R}$ continuous, non-degenerate. Then

- $(P_t)$ has a $H$-dual Markov semi-group if and only if $V_{1,+}$ is invariant.
- The dual $(Q_t)$ is unique.
- The dual is a Feller semi-group if and only if $(P_t)$ is.

**Example:** moment duality.
Finite state spaces: if $P_t H = HQ_t^T$ and $H$ is invertible, then $P_t$ and $Q_t$ have the same eigenvalues.

In infinite state spaces, non-degenerate duals need not have the same spectrum.

Unless processes are reversible!

**Theorem**: $(P_t)$, $(Q_t)$ dual with non-degenerate duality function $H$. Assume that $(X_t)$ and $(Y_t)$ have symmetrizing measures $\mu$ and $\nu$. Then $(P_t)$ and $(Q_t)$, as operators in $L^2(E, \mu)$ and $L^2(F, \nu)$, are unitarily equivalent. In particular, same spectrum.
Convex geometry & cone dual: definitions

Klebaner, Rösler, Sagitov ’07, Möhle ’11: duality wrt a **convex set** rather than a function.

**Simplex** in $\mathbb{R}^n$: non-empty convex, compact set such that every point in $C$ is a unique convex combination of its extremal points $\text{ex}C$.

**Choquet simplex**: generalization to locally convex vector spaces; bijection simplex $\leftrightarrow$ probability measures on $\text{ex}C$, **unique integral representation**. Characterization via partial order defined by cone $\{\lambda f \mid f \in C, \lambda \geq 0\}$.

**Proposition** $B_H$ non-degenerate, $H \in C_0(E \times F)$, $F$ compact $\Rightarrow \mathcal{V}_{1,+} \subset C_0(E)$ compact (topology of uniform convergence), Choquet simplex.

**Def.** $C \subset L^\infty(E)$ simplex, invariant under $(P_t)$. **Cone dual** is process $(R_t)$ with state space $\text{ex}C$, duality fct

$$E \times \text{ex}C \to \mathbb{R}, \quad (x, e) \mapsto e(x).$$

**Observation**: for non-degenerate $B_H$, bijection $F \to \text{ex}C$, $y \mapsto H(\cdot, y)$, cone dual and usual dual can be identified.
More interesting: $B_H$ degenerate but $\mathcal{V}_{1,+}$ simplex. For $y \in F$, let $\Pi(y, \cdot)$ be the unique probability measure on $ex\mathcal{V}_{1,+}$ such that
\[
H(\cdot, y) = \int_{ex\mathcal{V}_{1,+}} \Pi(y, de)e(\cdot).
\]

Transition kernel from $F$ to “smaller” space $ex\mathcal{V}_{1,+}$.

Theorem $H \in C_0(E \times F)$, $F$ compact, $\mathcal{V}_{1,+}$ simplex, invariant under $(P_t)$.

1. The cone dual $(R_t)$ exists and is unique. It is defined on $ex\mathcal{V}_{1,+}$ (Borel $\sigma$-algebra, uniform topology); $ex\mathcal{V}_{1,+}$ is Polish.
2. $(P_t)$ has at least one dual Markov semi-group $(Q_t)$.
3. $(Q_t)$ is dual to $(P_t)$ iff for all $t, y, B$
\[
\int_F Q_t(y, dy')\Pi(y', B) = \int_{ex\mathcal{V}_{1,+}} \Pi(y, de)R_t(e, B).
\]

Intertwining relation. Usual duals are lifts of the cone dual.
In terms of processes: cone dual $(Z_t)$, usual dual $(Y_t)$ satisfy
\[
P(Z_t \in B \mid Y_t = y) = \Pi(y, B).
\]
Quantum mechanics

**Definition:** $H$ self-adjoint operator (Hamiltonian) in Hilbert space $\mathcal{H}$. $(P_t)$ Markov semi-group in Polish state space $E$, symmetrizing $\sigma$-finite measure $\mu$, $U : \mathcal{H} \to L^2(E, \mu)$ unitary. Call $((P_t), E, \mu, U)$ a stochastic representation of $H$ if

$$\forall t > 0 : \exp(-tH) = U^* P_t U.$$ 

**Examples:** Ornstein-Uhlenbeck process vs. harmonic oscillator, simple symmetric exclusion process vs. Heisenberg spin chain.

**Earlier theorem, rephrased** If two reversible processes with non-degenerate duality function are associated with Hamiltonians $H_1$ and $H_2$, then $H_1$ and $H_2$ are unitarily equivalent.

**Strategy:**

- interpret infinitesimal generator $L$ as quantum Hamiltonian $H$.
- **Interacting particle systems / quantum many-body systems**: Break down transitions into birth and death of particles / creation and annihilation operators.
- Look for different representations of the relevant operator algebra.
- Often, this yields unitary equivalences and interesting dualities.

More flexible application: relax requirement of self-adjointness, reversibility. Giardina’s talk!
Further aspects & conclusion

- Generalized definition allowing for **Feynman-Kac corrections**.
- Solutions of martingale problems.
- Dualities and **scaling limits**.
  Swart ’06, Alkemper, Hutzenthaler ’07, J., Kurt ’12
- **Intertwining** of Markov processes
  Diaconis, Fill ’90, Carmona, Petit, Yor ’98; Huillet, Martinez ’11.
- Duality for interacting particle systems as **Fourier transforms** on groups
  Holley, Stroock ’79.

**Conclusion:** A lot of structure to explore.
Invariant subsets for Siegmund and moment duality

Convex subset $\mathcal{V}_{1,+} \subset L^\infty(E)$:

<table>
<thead>
<tr>
<th>$H(x, y)$</th>
<th>$x \in E$</th>
<th>$y \in F$</th>
<th>$\mathcal{V}<em>{1,+} = { \int_F H(\cdot, y) \nu(dy) \mid \nu \in \mathcal{M}</em>{1,+}(F) }$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1(x \leq y)$</td>
<td>$x \in \mathbb{R}$</td>
<td>$y \in \mathbb{R}$</td>
<td>monotone increasing, right-continuous functions $f$, $\lim_{-\infty} f = 0$, $\lim_{+\infty} f = 1$</td>
</tr>
<tr>
<td>$x^n$</td>
<td>$x \in [0, 1]$</td>
<td>$n \in \mathbb{N}_0$</td>
<td>absolutely monotone functions $f$ w. $f(1) = 1$</td>
</tr>
</tbody>
</table>

Linear subspace $\mathcal{V} \subset L^\infty(E)$:

<table>
<thead>
<tr>
<th>$H(x, y)$</th>
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<th>$\mathcal{V} = { \int_F H(\cdot, y) \nu(dy) \mid \nu \in \mathcal{M}(F) }$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi(x \leq y)$</td>
<td>$x \in \mathbb{R}$</td>
<td>$y \in \mathbb{R}$</td>
<td>bounded, right-continuous functions $f$ with bounded variation, $\lim_{-\infty} f = 0$, $\lim_{+\infty} f \in \mathbb{R}$</td>
</tr>
<tr>
<td>$x^n$</td>
<td>$x \in [0, 1]$</td>
<td>$n \in \mathbb{N}_0$</td>
<td>functions $f \in C([0, 1])$ with analytic extension $F$ to the complex open unit disk, $F$ in positive Wiener algebra ($\subset H^\infty$ Hardy space).</td>
</tr>
</tbody>
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