Duality of Markov processes with respect to a function

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Background

Duality is a useful tool in a variety of areas – e.g., random walks with absorbing / reflecting barriers, queuing theory FELLER, ASMUSSEN, diffusions, interacting particle systems LIGGETT, hydrodynamic limits, population genetics...

Basic idea: in order to understand the long-time behavior of some complicated process, follow a simpler, *dual* process backwards in time.

Related: duality wrt a measure (potential theory HUNT '57-58).
General theory available.

Question: general theory for duality wrt a function?

Outline

1. Duality with respect to a measure
2. Duality of linear operators. Existence and uniqueness
3. Spectrum
4. Convex geometry: cone duality.
5. Quantum mechanics

Definition

- E and F Polish spaces (Borel σ-algebra).
- (X_t) and (Y_t) Markov processes with state spaces E and F
- $H: E \times F \to \mathbb{R}$ bounded, measurable (duality function).

(X_t) and (Y_t) are dual with respect to the function H if for all $t > 0$, $x \in E$ and $y \in F$,

$$\boxed{\mathbb{E}_x H(X_t, y) = \mathbb{E}^y H(x, Y_t)}$$

$\mathbb{E}_x$: (X_t) started in x, $\mathbb{E}^y$: (Y_t) started in y.
In terms of the semi-groups (P_t) and (Q_t):

$$\boxed{\int_E P_t(x, \mathrm{d}x')H(x', y) = \int_F Q_t(y, \mathrm{d}y')H(x, y')}$$

Finite state spaces: matrix equation (A^{T}: transposed)

$$P_t H = H Q_t^{\mathsf{T}}.$$

Special case: $H = \mathrm{diag}(1/\mu(x))$:

$$\mu(x) P_t(x, y) = \mu(y) Q_t(y, x).$$

Duality wrt measure $\approx$ duality wrt diagonal function.

Pathwise duality

Fix a time horizon $T > 0$, starting points $x \in E$, $y \in F$. Dual processes can be defined on common probability space so that $X_0 = x$, $Y_0 = y$ $\mathbb{P}$-a.s. and

$$\forall t \in [0, T]: \quad \mathbb{E}H(x, Y_T) = \mathbb{E}H(X_t, Y_{T-t}) = \mathbb{E}H(X_T, y).$$

= weak pathwise duality. Often, processes can be coupled in such a way that

$$\forall t \in [0, T]: \quad H(x, Y_T) = H(X_t, Y_{T-t}) = H(X_T, y) \quad \mathbb{P}\text{-a.s.}$$

= strong pathwise duality. Examples:

- $H(x, y) = \mathbf{1}(x \le y)$ on totally ordered spaces Clifford, Sudbury '85:

$$\forall t \in [0, T]: \quad x \le Y_T \Leftrightarrow X_t \le Y_{T-t} \Leftrightarrow X_T \le y. \quad \mathbb{P}\text{-a.s.}$$

- Interacting particle systems with "graphical representations".
- Stochastic recursions: common "driving sequence" (W_n), recurrence relations

$$X_{n+1} = f(X_n, W_n), \quad Y_{n+1} = g(Y_n, W_{T-n}).$$

  Ex. random walks, W_n = "up" or "down" Lindley '52, books by Feller, Asmussen.

Duality with respect to a measue

Discrete state spaces: if $\mu(x) > 0$ for all x and (P_t) and (Q_t) are time reversals of each other wrt μ, then $H(x,y) = \frac{1}{\mu(x)}\delta_{x,y}$ is a duality function. What about continuous state spaces?

Definition: μ σ-finite measure on $E = F$. Processes (X_t), (Y_t) with semi-groups (P_t), (Q_t) and resolvents (R_λ), $(\hat{R}_\lambda)$ are in duality wrt μ if

- For all λ, x, y, $R_\lambda(x,\cdot)$ and $\hat{R}_\lambda(x,\cdot)$ are abs. cont. wrt. μ.
- For all λ and $f, g \geq 0$, $\int_E (P_t f) g \mathrm{d}\mu = \int_E f(Q_t g)\mathrm{d}\mu$.

Theorem Let (P_t), (Q_t) be in duality wrt μ. Write

$$R_\lambda(x, \mathrm{d}y) = r_\lambda(x,y)\mu(\mathrm{d}y), \quad \hat{R}_\lambda(x,\mathrm{d}y) = r_\lambda(y,x)\mu(\mathrm{d}y)$$

Functions $y \mapsto r_\lambda(x,y)$ and $x \mapsto r_\lambda(x,y)$ λ-excessive wrt (R_λ) resp. $(\hat{R}_\lambda)$. Then for all $\lambda > 0$, $r_\lambda(x,y)$ is a duality function for (X_t), (Y_t).

Blumenthal, Getoor: *Markov processes and potential theory*.

Duality of linear operators

Question: Relation with notions of duality in functional analysis?

Natural framework: dual pairs / duality wrt bilinear form.

Def.: V, W vector spaces. Bilinear form $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{R}$ called non-degenerate if right null space is trivial

$$\mathcal{N}_{\mathrm{R}} = \{ w \in W \mid \forall v \in V : \langle v, w \rangle = 0 \} = \{0\}$$

and left null space is trivial. $(V, W, \langle \cdot, \cdot \rangle)$ is a *dual pair*.

Linear operators $S : V \to V$, $T : W \to W$ are dual wrt $\langle \cdot, \cdot \rangle$ if

$$\boxed{\forall v \in V \; \forall w \in W : \quad \langle Sv, w \rangle = \langle v, Tw \rangle.}$$

Notion useful in study of weak topologies, locally convex vector spaces. s

Application to duality of Markov processes

- $\mathcal{M}(E)$ = finite signed Borel measures on E.
- Bilinear form associated with H:

$$B_H : \mathcal{M}(E) \times \mathcal{M}(F) \to \mathbb{R}, \quad B_H(\mu, \nu) := \int_{E\times F} H(x,y)\mu(\mathrm{d}x)\nu(\mathrm{d}y).$$

- $(P_t^* \mu)(A) = (\mu P_t)(A) = \int_E \mu(\mathrm{d}x) P_t(x, A)$.

Observation: Markov semi-groups (P_t) and (Q_t) dual wrt H iff

$$\forall t > 0 \;\forall \mu \in \mathcal{M}(E) \;\forall \nu \in \mathcal{M}(F) : \quad B_H(P_t^* \mu, \nu) = B_H(\mu, Q_t^* \nu).$$

H duality function for (P_t), (Q_t) $\Leftrightarrow$ (P_t^*), (Q_t^*) dual wrt bilinear form B_H.

Question: Given: (P_t) and H, existence of a dual Markov semi-group (Q_t)? Uniqueness?

Uniqueness

Functional analysis fact: The dual of a given operator with respect to a non-degenerate bilinear form is unique.

Consequence: if B_H is non-degenerate then the dual, if it exists, is unique. Applies to a lot of common duality functions:

- Siegmund dual: $H(x,y) = \mathbf{1}(x \leq y)$, $x, y \in \mathbb{R}$.
- Moment dual: $H(x,n) = x^n$, $x \in [0,1]$, $n \in \mathbb{N}_0$.
- Laplace dual: $H(x,\lambda) = \exp(-\lambda x)$, $\lambda, x \in [0,\infty)$.

Uniqueness in moment problems, injectivity of the Laplace transform.

Remark: Connection with kernel identities used in integrable quantum systems (Toda, Calogero-Moser systems) related to Doob h-transforms / non-colliding processes. Non-degeneracy does not come for free.
Ruijsenaars '13 *On positive Hilbet-Schmidt operators*

Existence

Functional analysis fact: Every bounded operator in a Hilbert space has a unique adjoint. Every bounded operator in a Banach space has a unique dual operator. But: for general dual pairs, a given operator need not have a dual (need continuity in suitable weak topology...)

Consequence: semi-group (P_t^*) need not have a dual semi-group of operators $T_t : \mathcal{M}(F) \to \mathcal{M}(F)$.

Additional problem: dual operator semi-group need not come from a Markov semi-group – need to know whether $(T_t \nu)(A) = \int_F \nu(\mathrm{d}y) Q_t(y, B)$ for Markov semi-group (Q_t).

Existence and uniqueness criteria available for Siegmund duality in totally ordered state spaces (stochastic monotonicity). Generalization?

Key: invariance of convex sets Möhle.

Existence, continued

$\mathcal{M}(F)$ = signed Borel measures, $\mathcal{M}_{1,+}(F)$ = probability measures. Images

$$\mathcal{V} = \Big\{ \int_F H(\cdot, y)\nu(\mathrm{d}y) \mid \nu \in \mathcal{M}(F) \Big\} \subset L^\infty(E)$$
$$\mathcal{V}_{1,+} = \Big\{ \int_F H(\cdot, y)\nu(\mathrm{d}y) \mid \nu \in \mathcal{M}_{1,+}(F) \Big\} \subset \mathcal{V}.$$

Analogous subsets $\mathcal{W}_{1,+} \subset \mathcal{W} \subset L^\infty(F)$.
Finite state spaces: linear / convex combinations of rows / columns of H.

Proposition Let E and F be countable, discrete state spaces and P a $E \times E$ stochastic matrix. Then

1. $PH = HQ^{\mathsf{T}}$ has a solution $Q = (Q(x,y))_{x,y \in F}$ with $\sum_z |Q(y,z)| < \infty$ for all y iff $\mathcal{V}$ is invariant.
2. Q can be chosen as a stochastic matrix iff $\mathcal{V}_{1,+}$ is invariant.

Siegmund duality $H(x,y) = \mathbf{1}(x \leq y)$ in totally ordered state spaces: $\mathcal{V}_{1,+}$ = monotone increasing functions. Recover known relationship Siegmund duality / stochastic monotonicity.

Existence, continued

In continuous time and space, more delicate – Chapman-Kolmogorov and measurability issues, theorems for non-degenerate H given by MÖHLE.

Convex combination of the columns of H:

$$\mathcal{V}_{1,+} = \left\{ \int_F H(\cdot, y)\nu(\mathrm{d}y) \mid \nu \in \mathcal{M}_{1,+}(F) \right\} \subset L^\infty(E).$$

Theorem E, F locally compact, $H : E \times F \to \mathbb{R}$ continuous, non-degenerate. Then

- (P_t) has a H-dual Markov semi-group if and only if $\mathcal{V}_{1,+}$ is invariant.
- The dual (Q_t) is unique.
- The dual is a Feller semi-group if and only if (P_t) is.

Example: moment duality.

Spectrum

Finite state spaces: if $P_t H = H Q_t^{\mathrm{T}}$ and H is invertible, then P_t and Q_t have the same eigenvalues.

In infinite state spaces, non-degenerate duals need not have the same spectrum.

Unless processes are reversible!

Theorem: (P_t), (Q_t) dual with non-degenerate duality function H. Assume that (X_t) and (Y_t) have symmetrizing measures μ and ν. Then (P_t) and (Q_t), as operators in $L^2(E,\mu)$ and $L^2(F,\nu)$, are unitarily equivalent. In particular, same spectrum.

Convex geometry & cone dual: definitions

KLEBANER, RÖSLER, SAGITOV '07, MÖHLE '11:
duality wrt a *convex set* rather than a function.

Simplex in $\mathbb{R}^n$: non-empty convex, compact set such that every point in C is a unique convex combination of its extremal points $\mathrm{ex} C$.

Choquet simplex: generalization to locally convex vector spaces; bijection simplex $\leftrightarrow$ probability measures on $\mathrm{ex} C$, unique integral representation.
Characterization via partial order defined by cone $\{\lambda f \mid f \in C, \lambda \geq 0\}$.

Proposition B_H non-degenerate, $H \in C_0(E \times F)$, F compact $\Rightarrow \mathcal{V}_{1,+} \subset C_0(E)$ compact (topology of uniform convergence), Choquet simplex.

Def. $C \subset L^\infty(E)$ simplex, invariant under (P_t). Cone dual is process (R_t) with state space $\mathrm{ex} C$, duality fct

$$E \times \mathrm{ex} C \to \mathbb{R}, \quad (x, e) \mapsto e(x).$$

Observation: for non-degenerate B_H, bijection $F \to \mathrm{ex} C$, $y \mapsto H(\cdot, y)$, cone dual and usual dual can be identified.

Cone dual vs. usual dual. Lifts and intertwining

More interesting: B_H degenerate but $\mathcal{V}_{1,+}$ simplex. For $y \in F$, let $\Pi(y,\cdot)$ be the unique probability measure on $\mathrm{ex}\mathcal{V}_{1,+}$ such that

$$H(\cdot,y) = \int_{\mathrm{ex}\mathcal{V}_{1,+}} \Pi(y,\mathrm{d}e)e(\cdot).$$

Transition kernel from F to "smaller" space $\mathrm{ex}\mathcal{V}_{1,+}$.

Theorem $H \in C_0(E \times F)$, F compact, $\mathcal{V}_{1,+}$ simplex, invariant under (P_t).

1. The cone dual (R_t) exists and is unique. It is defined on $\mathrm{ex}\mathcal{V}_{1,+}$ (Borel σ-algebra, uniform topology); $\mathrm{ex}\mathcal{V}_{1,+}$ is Polish.
2. (P_t) has at least one dual Markov semi-group (Q_t).
3. (Q_t) is dual to (P_t) iff for all t, y, B

$$\int_F Q_t(y,\mathrm{d}y')\Pi(y',B) = \int_{\mathrm{ex}\mathcal{V}_{1,+}} \Pi(y,\mathrm{d}e)R_t(e,B).$$

Intertwining relation. Usual duals are lifts of the cone dual.
In terms of processes: cone dual (Z_t), usual dual (Y_t) satisfy

$$\mathbb{P}(Z_t \in B \mid Y_t = y) = \Pi(y,B).$$

Quantum mechanics

Definition: H self-adjoint operator (Hamiltonian) in Hilbert space $\mathcal{H}$. (P_t) Markov semi-group in Polish state space E, symmetrizing σ-finite measure μ, $U : \mathcal{H} \to L^2(E, \mu)$ unitary. Call $((P_t), E, \mu, U)$ a stochastic representation of H if

$$\forall t > 0 : \quad \exp(-tH) = U^* P_t U.$$

Examples: Ornstein-Uhlenbeck process vs. harmonic oscillator, simple symmetric exclusion process vs. Heisenberg spin chain.

Earlier theorem, rephrased If two reversible processes with non-degenerate duality function are associated with Hamiltonians H_1 and H_2, then H_1 and H_2 are unitarily equivalent.

Strategy:

- interpret infinitesimal generator L as quantum Hamiltonian H.
- Interacting particle systems / quantum many-body systems: Break down transitions into birth and death of particles / creation and annihilation operators.
- Look for different representations of the relevant operator algebra.
- Often, this yields unitary equivalences and interesting dualities.

More flexible application: relax requirement of self-adjointness, reversibility. GIARDINA's talk!

Further aspects & conclusion

- Generalized definition allowing for Feynman-Kac corrections.
- Solutions of martingale problems.
- Dualities and scaling limits.
  Swart '06, Alkemper, Hutzenthaler '07,
  J., Kurt '12
- Intertwining of Markov processes
  Diaconis, Fill '90, Carmona, Petit, Yor '98; Huillet, Martinez '11.
- Duality for interacting particle systems as Fourier transforms on groups
  Holley, Stroock '79.

Conclusion: A lot of structure to explore.

Invariant subsets for Siegmund and moment duality

Convex subset $\mathcal{V}_{1,+} \subset L^\infty(E)$:

$H(x,y)$	$x \in E$	$y \in F$	$\mathcal{V}_{1,+} = \{\int_F H(\cdot,y)\nu(\mathrm{d}y) \mid \nu \in \mathcal{M}_{1,+}(F)\}$
$\mathbf{1}(x \le y)$	$x \in \mathbb{R}$	$y \in \mathbb{R}$	monotone increasing, right-continuous functions f, $\lim_{-\infty} f = 0$, $\lim_{+\infty} f = 1$
x^n	$x \in [0,1]$	$n \in \mathbb{N}_0$	absolutely monotone functions f w. $f(1) = 1$

Linear subspace $\mathcal{V} \subset L^\infty(E)$:

$H(x,y)$	$x \in E$	$y \in F$	$\mathcal{V} = \{\int_F H(\cdot,y)\nu(\mathrm{d}y) \mid \nu \in \mathcal{M}(F)\}$
$\chi(x \le y)$	$x \in \mathbb{R}$	$y \in \mathbb{R}$	bounded, right-continuous functions f with bounded variation, $\lim_{-\infty} f = 0$, $\lim_{+\infty} f \in \mathbb{R}$
x^n	$x \in [0,1]$	$n \in \mathbb{N}_0$	functions $f \in C([0,1])$ with analytic extension F to the complex open unit disk, F in positive Wiener algebra ($\subset H^\infty$ Hardy space).