Stochastic dualities and Lie algebras.

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The results presented here have been obtained in a series of works in collaboration with:

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Outline

- Lie algebraic approach to duality theory.
- Heisenberg algebra: simple dualities in basic models.
- Construction of Markov processes with algebraic structure and symmetries.
- Classical $\mathfrak{su}(1, 1)$ algebra: duality for Wright-Fisher and Moran models with symmetric mutations.
- Deformed $\mathfrak{su}(1, 1)$ algebra: new processes and new dualities with selection [work in progress!].
1. Lie algebraic approach to duality
Duality

Definition

$(\eta_t)_{t \geq 0}$ Markov process on $\Omega$ with generator $L$,

$(\xi_t)_{t \geq 0}$ Markov process on $\Omega_{dual}$ with generator $L_{dual}$

$\xi_t$ is dual to $\eta_t$ with duality function $D : \Omega \times \Omega_{dual} \rightarrow \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \quad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

$\eta_t$ is self-dual if $L_{dual} = L$. 
Duality

Assume that

\[ D(\cdot, y) \in \mathcal{D}(L) \quad \text{and} \quad S_t D(\cdot, y) \in \mathcal{D}(L) \quad \forall \ y \in \Omega_{\text{dual}} \]
\[ D(x, \cdot) \in \mathcal{D}(L_{\text{dual}}) \quad \text{and} \quad S_t^{\text{dual}} D(x, \cdot) \in \mathcal{D}(L_{\text{dual}}) \quad \forall \ x \in \Omega, \]

with \( \mathcal{D}(L) \) and \( \mathcal{D}(L_{\text{dual}}) \) the generators domain, \( S_t \) and \( S_t^{\text{dual}} \) the processes semigroup.

Then duality is equivalent to

\[ LD(\cdot, \xi)(\eta) = L_{\text{dual}} D(\eta, \cdot)(\xi) \]
A Lie algebra is a vector space $g$ over a field $F$ together with a binary operation $[\cdot, \cdot] : g \times g \to g$ (Lie bracket):

- $\forall a, b \in F$ and $\forall u, v, w \in g$
  \[ [au + bv, w] = a[u, w] + b[v, w], \quad [w, au + bv] = a[w, u] + b[w, v] \]

- $\forall u \in g$: $[u, u] = 0$

- [Jacobi identity]: $\forall u, v, w \in g$
  \[ [u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0 \]

Elements of a Lie algebra $g$ are said to be **generators** of the Lie algebra if the smallest subalgebra of $g$ containing them is $g$ itself.
Algebraic approach

1. Write the Markov generator in abstract form, i.e. as an element of a Lie algebra, using the algebra generators (typically creation and annihilation operators).

2. Duality is related to a change of representation, i.e. new operators that satisfy the same algebra. Duality functions are the intertwiners.

3. Self-duality is associated to symmetries, i.e. conserved quantities.
Duality

Abstract generator

Original generator ↔ Dual generator

D
Self-duality

For Markov chain with countable state space

\[ LD(\cdot, \xi)(\eta) = LD(\eta, \cdot)(\xi) \]

amounts to

\[ LD = DL^T \]

Indeed

\[ \sum_{\eta'} L(\eta, \eta')D(\eta', \xi) = LD(\cdot, \xi)(\eta) = LD(\eta, \cdot)(\xi) = \sum_{\xi'} L(\xi, \xi')D(\eta, \xi') \]
Trivial self-duality functions from reversible measures

From a reversible measure $\mu$, i.e.

$$L(\eta, \xi)\mu(\eta) = L(\xi, \eta)\mu(\xi)$$

a trivial (i.e. diagonal) self-duality function is

$$d(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta, \xi}$$

Indeed

$$\frac{L(\eta, \xi)}{\mu(\xi)} = \sum_{\eta'} L(\eta, \eta')d(\eta', \xi) = \sum_{\xi'} L(\xi, \xi')d(\eta, \xi') = \frac{L(\xi, \eta)}{\mu(\eta)}$$
Symmetries and self-duality

$S$: symmetry of the transposed of the generator, i.e. $[L^T, S] = 0$,

$d$: trivial self-duality function,

$\rightarrow \quad D = dS$ self-duality function.

Indeed

$LD = LdS = dL^T S = dSL^T = DL^T$

Self-duality is related to the action of a symmetry
2. Basic examples
Moran model with two types

Population of \( N \) individuals, each of which can be of types 1 or type 2. A pair of individuals are sampled uniformly at random, one dies with probability 1/2, the other reproduces.

Define

\[
K^{(N)}(t) = \text{number of individuals of type 1 at time } t \geq 0
\]

\((K^{(N)}(t))_{t \geq 0}\) is a continuous time Markov chain with state space \( \Omega_N = \{0, 1, \ldots, N\} \) and generator

\[
L_N^{Moran} f(k) = \frac{1}{2} k(N - k)(f(k + 1) + f(k - 1) - 2f(k))
\]
Wright-Fisher diffusion with two types

Diffusive scaling limit: the process \( \left( X^{(N)}(t) = \frac{K^{(N)}(N^2 t)}{N} \right) \) with state space \( \Omega'_N = \{0, 1/N, \ldots, 1\} \) has generator

\[
L'_N f(\frac{k}{N}) = N^2 \frac{1}{2} \frac{k}{N} \left( 1 - \frac{k}{N} \right) \left( f\left( \frac{k}{N} + \frac{1}{N} \right) + f\left( \frac{k}{N} - \frac{1}{N} \right) - 2f\left( \frac{k}{N} \right) \right)
\]

In the limit \( N \to \infty \) the process \( (X^{(N)}(t))_{t \geq 0} \) converges to the Wright-Fisher diffusion \( (X(t))_{t \geq 0} \) with state space \([0, 1]\) and generator

\[
L^{WF} f(x) = \frac{1}{2} x(1 - x) \frac{\partial^2 f}{\partial x^2}(x)
\]
Counting blocks of Kingman coalescence

For each $k \in \mathbb{N}$, the $k$-coalescence is a continuous time Markov chain on the space of equivalence relations on $\{1, 2, \ldots, k\}$ with transition rates

$$c(x, y) = \begin{cases} 
1 & \text{if } y \text{ is obtained by coalescing} \\
& \text{two equivalence classes of } x, \\
0 & \text{otherwise.}
\end{cases}$$

By extension the Kingman coalescent on $\mathbb{N}$ is defined by requiring that for each $k$ its restriction to $\{1, \ldots, k\}$ is a $k$-coalescence.

Define

$$N(t) = \text{number of blocks in the } k\text{-coalescence at time } t \geq 0.$$ 

It is a death process on $\{1, \ldots, k\}$ defined by the Markov generator

$$(L^{\text{King}}f)(n) = \frac{n(n-1)}{2}(f(n-1) - f(n)).$$
Heisenberg algebra

The Lie bracket is given by the commutator, i.e. for $u, v$ in the algebra

$$[u, v] = uv - vu$$

The algebra is generated by the elements $(a^+, a^-)$ with commutator

$$[a^-, a^+] = 1$$

Two representations are:

$$\begin{align*}
    a^+ &= x \\
    a^- &= \frac{\partial}{\partial x}
\end{align*}$$

$$\begin{align*}
    a^+ |n\rangle &= |n+1\rangle \\
    a^- |n\rangle &= n|n-1\rangle
\end{align*}$$

where, for $n \in \{0, 1, 2, \ldots\}$, $|n\rangle = e_n$ denote the column vectors

$$(e_n)_i = \begin{cases}
    1 & \text{if } i = n, \\
    0 & \text{if } i \neq n
\end{cases}$$

$$e_n^T \cdot e_m = \langle n|m\rangle = \delta_{n,m}$$
Proposition

The process \( \{ X(t) \}_{t \geq 0} \) with generator \( L^{WF} \) and the process \( \{ N(t) \}_{t \geq 0} \) with generator \( L^{King} \) are dual on \( D(x, n) = x^n \), i.e.

\[
\mathbb{E}_x(X(t)^n) = \mathbb{E}_n(x^{N(t)})
\]

Indeed:

\[
L^{WF} D(\cdot, n)(x) = \frac{1}{2} x(1 - x) \frac{\partial^2}{\partial x^2} x^n
\]

\[
= \frac{n(n - 1)}{2} (x^{n-1} - x^n)
\]

\[
= \frac{n(n - 1)}{2} (D(x, n - 1) - D(x, n))
\]

\[
= L^{King} D(x, \cdot)(n)
\]
Duality Wright-Fisher / Kingman : algebraic approach

The duality is a consequence of the change of representation:

\[
\begin{align*}
& a^+ = x \\
& a^- = \frac{d}{dx}
\end{align*}
\]

\[
\begin{align*}
& a^+|n\rangle = |n + 1\rangle \\
& a^-|n\rangle = n|n - 1\rangle
\end{align*}
\]

The abstract element

\[ L = \frac{1}{2} a^+(1 - a^+)(a^-)^2 \]

\[ L = L^{WF} \quad \text{in the first representation} \]
\[ L^T = L^{King} \quad \text{in the second representation} \]

Duality fct. \( D(x, n) = x^n \) is the intertwiner:

\[
xD(x, n) = D(x, n + 1) \quad \frac{d}{dx} D(x, n) = nD(x, n - 1)
\]
Duality Moran / Kingman

Proposition

The process \( \{ K_N(t) \} \) \( t \geq 0 \) with generator \( L_N^{Moran} \) and the process \( \{ N(t) \} \) \( t \geq 0 \) with generator \( L^{King}_N \) are dual on

\[
D_N(k, n) = \frac{k(k-1) \cdots (k-(n-1))}{N(N-1) \cdots (N-(n-1))}.
\]

with the convention \( D_N(k, 0) = 1 \), \( D_N(k, N+1) = 0 \).

Indeed:

\[
L_N^{Moran} D_N(\cdot, n)(k) = L^{King}_N D_N(k, \cdot)(n)
\]
The duality is a consequence of a change of representation. For functions \( f : \{0, \ldots, N\} \rightarrow \mathbb{R} \):

\[
\begin{align*}
ad_N^+ f(k) &= \sum_{r=0}^{k-1} (-1)^{k-1-r} \binom{N}{r} \binom{N}{k} f(r) \\
ad_N^- f(k) &= (N - k) f(k + 1) + (2k - N) f(k) - kf(k - 1)
\end{align*}
\]

with the convention \( f(-1) = f(N + 1) = 0 \), and

\[
\begin{align*}
ad^+ |n\rangle &= |n + 1\rangle \\
ad^- |n\rangle &= n |n - 1\rangle
\end{align*}
\]

The abstract element \( L = \frac{1}{2} a^+(1 - a^+) (a^-)^2 \) is the intertwiner:

\[
\begin{align*}
L &= L^\text{Moran}_N \\
L^T &= L^\text{King}
\end{align*}
\]

in the first representation and in the second representation.
Duality Moran / Kingman: algebraic approach

The intertwiner

\[ D_N(k, n) = \frac{\binom{k}{n}}{\binom{N}{n}} = \frac{k(k - 1) \cdots (k - (n - 1))}{N(N - 1) \cdots (N - (n - 1))}. \]

with the convention \( D_N(k, 0) = 1, \) \( D_N(k, N + 1) = 0 \) is such that on the vector space generated by the functions \( k \mapsto D_N(k, n), 0 \leq n \leq N \)

\[
\begin{align*}
\tilde{a}_N^{-} D_N(\cdot, n)(k) &= nD_N(k, n - 1), \forall 1 \leq n, \forall k \geq n - 1 \\
\tilde{a}_N^{-} D_N(\cdot, 0)(k) &= 0, \forall 0 \leq k \leq N \\
\tilde{a}_N^{+} D_N(\cdot, n)(k) &= D_N(k, n + 1), \forall 0 \leq n \leq N, k \geq n
\end{align*}
\]
Mutation

Consider the Moran model where each individual of type 2 mutates to an individual of type 1 at rate $\theta/N$. Then in the diffusive limit one has

$$L^{WF,\text{mut}} = x(1-x)\frac{d^2}{dx^2} + \theta(1-x)\frac{d}{dx}$$

$$= a^+(1-a^+)(a^-)^2 + \theta(1-a^+)a^-$$

By changing to a discrete representation of the Heisenberg algebra this gives the dual

$$L^{\text{King,mut}} f(n) = n(n-1)(f(n-1) - f(n)) + \theta n(f(n-1) - f(n))$$

which corresponds to Kingman’s coalescent with extra rate $\theta n$ to go down from $n$ to $n-1$, due to mutation.
3. Constructive approach
Algebraic approach

1. Write the Markov generator in **abstract form**, i.e. as an element of a Lie algebra, using the algebra generators (typically creation and annihilation operators).

2. Duality is related to a **change of representation**, i.e. new operators that satisfy the same algebra. Duality functions are the intertwiners.

3. Self-duality is associated to **symmetries**, i.e. conserved quantities.

Conversely, Step 1. can be turned into a constructive step.
Construction of Markov generators with algebraic structure and symmetries

i) *(Lie Algebra)*: Start from a (representation of a) Lie algebra \( g \).

ii) *(Casimir)*: Pick an element in the center of \( g \), e.g. the Casimir \( C \).

iii) *(Co-product)*: Consider a co-product \( \Delta : g \rightarrow g \otimes g \) making the algebra a bialgebra and conserving the commutation relations.

iv) *(Quantum Hamiltonian)*: Compute the co-product \( H = \Delta(C) \).

v) *(Markov generator)*: Apply a ground state transform (often a similarity transformation) to turn \( H \) into a Markov generator \( L \).

vi) *(Symmetries)*: \( S = \Delta(X) \) with \( X \in g \) is a symmetry of \( H \):

\[
[H, S] = [\Delta(C), \Delta(X)] = \Delta([C, X]) = \Delta(0) = 0.
\]
4. Classical $\mathfrak{su}(1, 1)$ algebra.
Classical $\mathfrak{su}(1,1)$ Lie algebra

It turns out that Wright-Fisher diffusion and Moran models have more structure than only the Heisenberg algebra.

In the (multi-type) setting with parent independent mutations their Markov generator can be written using classical $\mathfrak{su}(1,1)$ Lie algebra.

The generators $K^+, K^-, K^o$ of $\mathfrak{su}(1,1)$ algebra satisfy the commutation relations

$$[K^o, K^\pm] = \pm K^\pm$$
$$[K^+, K^-] = -2K^o$$
$\mathfrak{su}(1,1)$ Heisenberg ferromagnet as a population model

The abstract operator

$$L_m = \frac{1}{2} \sum_{1 \leq i < j \leq d} \left( K_i^+ K_j^- + K_i^- K_j^+ - 2 K_i^o K_j^o + \frac{m^2}{8} \right)$$

1. written in terms of a continuous representation, is the generator of the $d$-type Wright-Fisher diffusion with mutation rate $\frac{m}{4}(d - 1)$

2. written in terms of a discrete representation, is the generator of the $d$-type Moran model with mutation rate $\frac{m}{4}(d - 1)$

3. therefore the two processes are dual; in addition - from the symmetries - we will find self-duality.

Let us apply the construction ....
step i): representation in terms of matrices

A discrete representation of $\mathfrak{su}(1,1)$ algebra is

$$
\begin{align*}
K^+ |n\rangle &= (n + \frac{m}{2}) |n + 1\rangle \\
K^- |n\rangle &= n |n - 1\rangle \\
K^o |n\rangle &= (n + \frac{m}{4}) |n\rangle
\end{align*}
$$

In a canonical base

$$
K^+ = \begin{pmatrix}
0 \\
\frac{m}{2} \\
\frac{m}{2} + 1 \\
\ddots \\
\ddots \\
\ddots \\
\end{pmatrix} \\
K^- = \begin{pmatrix}
0 & 1 & \ddots & \\
\ddots & \ddots & \ddots & \\
\ddots & \ddots & \ddots & \\
\end{pmatrix} \\
K^o = \begin{pmatrix}
\frac{m}{4} & 0 \\
\frac{m}{4} + 1 \\
\frac{m}{4} + 2 \\
\ddots \\
\ddots \\
\ddots \\
\end{pmatrix}
$$
step ii): Casimir element

For the $\mathfrak{su}(1, 1)$ algebra the Casimir is

$$C = \frac{1}{2} (K^- K^+ + K^+ K^-) - (K^0)^2$$

$C$ is in the center of the algebra:

$$[C, K^+] = [C, K^-] = [C, K^0] = 0$$

$$C|n\rangle = \frac{1}{2} \left( (n + 1) \left( \frac{m}{2} + n \right) + \left( \frac{m}{2} + n - 1 \right) n \right) |n\rangle - (n + \frac{m}{4})^2 |n\rangle$$

$$= \frac{m}{4} (1 - \frac{m}{4}) |n\rangle$$
step iii): Co-product

The co-product is a morphism that turns the algebra into a bialgebra:

\[ \Delta : \mathfrak{su}(1, 1) \rightarrow \mathfrak{su}(1, 1) \otimes \mathfrak{su}(1, 1) \]

and conserves the commutations relations

\[ [\Delta(K^0), \Delta(K^\pm)] = \pm \Delta(K^\pm) \]

\[ [\Delta(K^-), \Delta(K^+)] = 2\Delta(K^0) \]

For classical Lie-algebras the co-product is just the symmetric tensor product with the identity

\[ \Delta(X) = X \otimes 1 + 1 \otimes X := X_1 + X_2 \]
step iv): Quantum Hamiltonian

\[ \Delta(C) = \frac{1}{2} \left( \Delta(K^-)\Delta(K^+) + \Delta(K^+)\Delta(K^-) \right) - \left( \Delta(K^0) \right)^2 \]

\[ = \frac{1}{2} \left( (K_1^- + K_2^-)(K_1^+ + K_2^+) + (K_1^+ + K_2^+)(K_1^- + K_2^-) \right) \]

\[ - \left( K_1^0 + K_2^0 \right)^2 \]

\[ = K_1^- K_2^+ + K_1^+ K_2^- - 2K_1^0 K_2^0 + C_1 + C_2 \]

\[ = \text{\textit{su}}(1, 1) \text{ Heisenberg ferromagnet} + \text{ diagonal} \]
step v): Markov generator

There is no need of a “ground state transformation”. In the discrete representation we find

$$\Delta(C) = (L_{1,2}^{SIP(m)})^* + \left(\frac{m}{2} \left(1 - \frac{m}{2}\right)\right) 1 \otimes 1$$

where

$$L_{1,2}^{SIP(m)} f(\eta_1, \eta_2) = \eta_1 \left(\eta_2 + \frac{m}{2}\right) \left[f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2)\right] + \eta_2 \left(\eta_1 + \frac{m}{2}\right) \left[f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2)\right]$$

is the generator of the Symmetric Inclusion Process SIP(m). If \(\eta_1 + \eta_2 = N\) it is the Moran model with \(N\) individuals, two types and symmetric mutation rate \(m/2\).
step vi): symmetries

As a consequence of the construction, \( \Delta(K^\alpha) \) with \( \alpha \in \{+, -, o\} \) are symmetries of the process:

\[
\begin{align*}
[(L_{1,2}^{\text{SIP}(m)})^*, K_1^o + K_2^o] &= 0 \\
[(L_{1,2}^{\text{SIP}(m)})^*, K_1^+ + K_2^+] &= 0 \\
[(L_{1,2}^{\text{SIP}(m)})^*, K_1^- + K_2^-] &= 0
\end{align*}
\]
Self-duality of the multi-type Moran model
Theorem [Carinci, G., Giberti, Redig (2013), to appear on SPA]

On the simplex $\sum_{i=1}^{d} \eta_i = N$, the $d$-types Moran model with $N$ individuals and parent-independent mutation at rate $\frac{m}{4}(d-1)$ coincides with the $\text{SIP}(m)$ on the complete graph with $d$ sites

$$\mathcal{L}_{N,d,\frac{m}{4}(d-1)}^{\text{Moran}} = \mathcal{L}_{d}^{\text{SIP}(m)}$$

$$\mathcal{L}_{d}^{\text{SIP}(m)} f(\eta) = \frac{1}{2} \sum_{1 \leq i < j \leq d} \eta_i \left( \eta_j + \frac{m}{2} \right) \left( f(\eta + e_i - e_j) - f(\eta) \right)$$

$$\frac{1}{2} \sum_{1 \leq i < j \leq d} \eta_j \left( \eta_i + \frac{m}{2} \right) \left( f(\eta - e_i + e_j) - f(\eta) \right)$$

The process is self-dual with self-duality function

$$D(\eta_1, \ldots, \eta_d; \xi_1, \ldots, \xi_d) = \prod_{i=1}^{d} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma \left( \frac{m}{2} \right)}{\Gamma \left( \frac{m}{2} + \xi_i \right)}$$
Trivial self-duality function $d$

Reversible product measure are product of Negative Binomial $(p, \frac{m}{2})$

$$
\mu_{rev}(\eta) = \prod_{i=1}^{d} p_{\xi_i} (1 - p)^{\frac{m}{2}} \frac{\Gamma(\frac{m}{2} + \eta_i)}{\eta_i!} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2})}
$$

and a trivial (i.e. diagonal) self-duality function was obtained as

$$
d(\eta, \xi) = \frac{1}{\mu_{rev}(\eta)} \delta_{\eta, \xi}
$$

Since the total number of particles is constant, one can take

$$
d(\eta, \xi) = \prod_{i=1}^{d} \frac{\eta_i! \Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + \eta_i)} \delta_{\eta_i, \xi_i}
$$
The symmetry $S = \exp(\sum_{i=1}^{d} K_i^+)$

$$S(\eta, \xi) = \prod_{i=1}^{d} \langle \eta_i | \exp(K_i^+) | \xi_i \rangle$$

$$= \prod_{i=1}^{d} \left\langle \eta_i \right| \sum_{s_i \geq 0} \frac{(K_i^+)^{s_i}}{s_i!} | \xi_i \rangle$$

$$= \prod_{i=1}^{d} \left\langle \eta_i \right| \sum_{s_i \geq 0} \frac{(m/2 + \xi_i + s_i - 1)!}{(m/2 + \xi_i - 1)! s_i!} | \xi_i + s_i \rangle$$

$$= \prod_{i=1}^{d} \frac{(m/2 + \eta_i - 1)!}{(m/2 + \xi_i - 1)! (\eta_i - \xi_i)!}$$

$$= \prod_{i=1}^{d} \frac{\Gamma(m/2 + \eta_i)}{\Gamma(m/2 + \xi_i) (\eta_i - \xi_i)!} \frac{1}{\Gamma(m/2 + \xi_i)}$$
The self-duality function $D$

Combining trivial self-duality and symmetry leads to

\[
D(\eta, \xi) = dS(\eta, \xi)
\]

\[
= \prod_{i=1}^{d} \frac{\eta_i! \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} + \eta_i\right)} \cdot \frac{\Gamma\left(\frac{m}{2} + \eta_i\right)}{\Gamma\left(\frac{m}{2} + \xi_i\right)(\eta_i - \xi_i)!} \cdot 1
\]

\[
= \prod_{i=1}^{d} \frac{\eta_i! \Gamma\left(\frac{m}{2}\right)}{(\eta_i - \xi_i)! \Gamma\left(\frac{m}{2} + \xi_i\right)}
\]
Duality between multi-type Wright-Fisher diffusion and Moran model
Theorem [Carinci, G., Giberti, Redig (2013), to appear on SPA]

On the simplex $\sum_{i=1}^{d} x_i = 1$, the $d$-types Wright-Fisher diffusion with parent-independent mutation at rate $\frac{m}{4} (d - 1)$ coincides with the $BEP(m)$ on the complete graph with $d$ sites

$$\mathcal{L}_{d, \frac{m}{4} (d-1)}^{WF} = \mathcal{L}_{d}^{BEP(m)}$$

$$\mathcal{L}_{d, \theta}^{WF} = \sum_{i=1}^{d-1} \frac{1}{2} x_i (1 - x_i) \frac{\partial^2}{\partial x_i^2} - \sum_{1 \leq i < j \leq d-1} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\theta}{d - 1} \sum_{i=1}^{d-1} (1 - dx_i) \frac{\partial}{\partial x_i}$$

$$\mathcal{L}_d^{BEP(m)} = \frac{1}{2} \sum_{1 \leq i < j \leq d} x_i x_j \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)^2 - \frac{m}{4} \sum_{1 \leq i < j \leq d} (x_i - x_j) \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)$$

The process is dual to Moran model with $N = \sum_{i=1}^{d} \xi_i$ individuals on duality function

$$D_N(x_1, \ldots, x_d; \xi_1, \ldots, \xi_d) = \prod_{i=1}^{d} x_i^{\xi_i} \frac{\Gamma \left( \frac{m}{2} \right)}{\Gamma \left( \frac{m}{2} + \xi_i \right)}$$
Duality explained

The abstract operator (quantum Heisenberg ferromagnet)

\[ \mathcal{L} = \sum_{(i,j) \in E} \left( K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^0 K_j^0 + \frac{m^2}{8} \right) \]

with \( \{ K_i^+, K_i^-, K_i^0 \}_{i \in V} \) satisfying \( \mathfrak{su}(1,1) \) commutation relations:

\[ [K_i^0, K_j^\pm] = \pm \delta_{i,j} K_i^\pm \quad [K_i^+, K_j^-] = -2 \delta_{i,j} K_i^0 \]

can be looked at in different representations.

Duality between \( L^{WF} \) e \( L^{Moran} \) corresponds to two different representations of the operator \( \mathcal{L} \).
Duality fct is the intertwiner.
Representation of $\mathfrak{su}(1,1)$ algebra in terms of differential operators

Continuous representation

$$\mathcal{K}_i^+ = x_i \quad \mathcal{K}_i^- = x_i \frac{\partial^2}{\partial x_i^2} + \frac{m}{2} \frac{\partial}{\partial x_i} \quad \mathcal{K}_i^0 = x_i \frac{\partial}{\partial x_i} + \frac{m}{4}$$

satisfy commutation relations

$$[\mathcal{K}_i^0, \mathcal{K}_j^\pm] = \pm \delta_{i,j} \mathcal{K}_i^\pm \quad [\mathcal{K}_i^-, \mathcal{K}_j^+] = 2 \delta_{i,j} \mathcal{K}_i^0$$

In this representation

$$\mathcal{L} = \mathcal{L}^{BEP}(m)$$
Duality function as intertwiner

Intertwiner

\[ \mathcal{K}_i^+ D_i(\cdot, \xi_i)(x_i) = K_i^+ D_i(x_i, \cdot)(\xi_i) \]
\[ \mathcal{K}_i^- D_i(\cdot, \xi_i)(x_i) = K_i^- D_i(x_i, \cdot)(\xi_i) \]
\[ \mathcal{K}_i^0 D_i(\cdot, \xi_i)(x_i) = K_i^0 D_i(x_i, \cdot)(\xi_i) \]

Example

\[ \mathcal{K}_i^+ D_i(\cdot, \xi_i)(x_i) = x_i D_i(x_i, \xi_i) \]
\[ = x_i^{\xi_i+1} \frac{\Gamma(m/2)}{\Gamma(m/2 + \xi_i)} \]
\[ = (\frac{m}{2} + \xi_i)x_i^{\xi_i+1} \frac{\Gamma(m/2)}{\Gamma(m/2 + \xi_i + 1)} \]
\[ = (\frac{m}{2} + \xi_i)D_i(x_i, \xi_i + 1) = K_i^+ D_i(x_i, \cdot)(\xi_i) \]
5. Deformed $\mathfrak{su}(1, 1)$ algebra
$q$-numbers

For $q \in (0, 1)$ and $n \in \mathbb{N}_0$ introduce the $q$-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Remark: $\lim_{q \to 1} [n]_q = n$.

The first $q$-number’s are:

$[0]_q = 0$, $[1]_q = 1$, $[2]_q = q + q^{-1}$, $[3]_q = q^2 + 1 + q^{-2}$, ...
The deformed Lie algebra $\mathfrak{su}_q(1, 1)$

For $q \in (0, 1)$ consider the algebra with generators $K^+, K^-, K^0$ with commutation relations

$$[K^+, K^-] = -[2K^0]_q, \quad [K^0, K^\pm] = \pm K^\pm$$

where

$$[2K^0]_q := \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

The Casimir element is

$$C = [K^0]_q[K^0 - 1]_q - K^+K^-$$

A standard discrete representation is given by

$$\begin{align*}
K^+|n\rangle &= \sqrt{[n + \frac{m}{2}]_q[n + 1]_q} |n + 1\rangle \\
K^-|n\rangle &= \sqrt{[n]_q[n + \frac{m}{2} - 1]_q} |n - 1\rangle \\
K^0|n\rangle &= (n + \frac{m}{4}) |n\rangle
\end{align*}$$
Co-product

A co-product $\Delta : su_q(1, 1) \to su_q(1, 1) \otimes su_q(1, 1)$ is defined as

$$\Delta(K^\pm) = K^\pm \otimes q^{-K^o} + q^{K^o} \otimes K^\pm$$
$$\Delta(K^o) = K^o \otimes 1 + 1 \otimes K^o$$

The co-product is an isomorphism such that

$$[\Delta(K^+), \Delta(K^-)] = -[2\Delta(K^o)]_q \quad [\Delta(K^o), \Delta(K^\pm)] = \pm \Delta(K^\pm)$$

The co-product applied to the Casimir in the discrete representation gives, after a suitable ground state transformation, the generator of a new asymmetric process which we called Asymmetric Inclusion Process ASIP$(q, m)$. 
ASIP(q,m) process

For $0 < q \leq 1$ the generator is given by

$$(\mathcal{L}_{1,2}^{\text{ASIP}} f)(\eta_1, \eta_2)$$

$$= q^{\eta_1-\eta_2+(\frac{m}{2}-1)}[\eta_1]_q[\eta_2 + \frac{m}{2}]_q(f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2))$$

$$+ q^{\eta_1-\eta_2-(\frac{m}{2}-1)}[\eta_2]_q[\eta_1 + \frac{m}{2}]_q(f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2))$$

By construction, the process has natural symmetries:

$$\Delta(K^\pm) = K^\pm \otimes q^{-K^o} + q^{K^o} \otimes K^\pm$$

$$\Delta(K^o) = K^o \otimes 1 + 1 \otimes K^o$$
Self-duality of ASIP\((q, m)\)

Theorem [Carinci, G., Redig, Sasamoto (2014 + in progress)]

The ASIP\((q, m)\) on \([1, L] \cap \mathbb{Z}\) with generator \(\sum_{i=0}^{L-1} L_{i,i+1}^{ASIP(q,m)}\) is self-dual on

\[
D_q(\eta, \xi) = \prod_{i=1}^{L} \frac{[\eta_i]_q!}{[\eta_i - \xi_i]_q!} \frac{\Gamma_q\left(\frac{m}{2}\right)}{\Gamma_q\left(\frac{m}{2} + \xi_i\right)} \cdot q^{(\eta_i - \xi_i)[2 \sum_{k=1}^{i-1} \xi_k + \xi_i] - mi \xi_i}
\]

The proof is an immediate consequence of the constructive method.
Applications in non-equilibrium statistical physics

Let \( \xi^{(i)} \) be the configuration with only one dual particle

\[
\xi^{(i)}_m = \begin{cases} 
    1 & \text{if } m = i \\
    0 & \text{otherwise}
\end{cases}
\]

and \( N_i(\eta) := \sum_{k \geq i} \eta_k \)

then

\[
D_q(\eta, \xi^{(i)}) = \frac{q^{-4mi+1}}{q^2 - q^{-2}} \cdot (q^{2N_i(\eta)} - q^{2N_{i+1}(\eta)})
\]

\( N_i(\eta(t)) \) is related to the total current \( J_i(t) \) in the time interval \([0, t]\)

across bond \((i - 1, i)\) in the right direction: \( J_i(t) = N_i(\eta(t)) - N_i(\eta(0)) \)

The duality relation gives

\[
\mathbb{E}_\eta D_q(\eta(t), \xi^{(i)}) = \mathbb{E}_i \frac{q^{-4mY(t)+1}}{q^2 - q^{-2}} \cdot (q^{2N_{Y(t)}(\eta)} - q^{2N_{Y(t)+1}(\eta)})
\]

where \( Y(t) \) is a continuous time random walk jumping to the right at rate \( q^\frac{m}{2} \lfloor \frac{m}{2} \rfloor q \) and to the left at rate \( q^{-\frac{m}{2}} \lfloor \frac{m}{2} \rfloor q \).
Applications in population dynamics

Let \((\eta(t))_{t \geq 0}\) be the \textit{ASIP}(1 - \frac{\sigma}{N}, m) process with \(N\) particle and \textbf{weak} asymmetry.

Scaling limit: let \(X_i(t) = \lim_{N \to \infty} \frac{\eta_i(t)}{N}\), this process is a diffusion with generator

\[
\mathcal{L}^{\text{ABEP}}_{i,i+1}(\sigma,m) = \frac{1}{4\sigma^2} (1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)^2 \\
+ \frac{1}{2\sigma} \left\{ (1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) + \frac{m}{2} \left( 2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}} \right) \right\} \left( \frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i} \right)
\]

If \(\sigma \to 0\) we recover the \textit{BEP}(m), i.e. the Wright-Fisher model.
Applications in population dynamics

\[
L_{i,i+1}^{\text{ABEP}}(\sigma, m) = \frac{1}{4\sigma^2} \left( 1 - e^{-2\sigma x_i} \right) \left( e^{2\sigma x_{i+1}} - 1 \right) \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)^2 \\
+ \frac{1}{2\sigma} \left\{ (1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) + \frac{m}{2} \left( 2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}} \right) \right\} \left( \frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i} \right)
\]

Expanding to first order in \( m \) and \( \sigma \) one has

\[
L_{i,i+1}^{\text{ABEP}}(\sigma, m) = x_i x_{i+1} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)^2 + \left\{ 2\sigma x_i x_{i+1} + \frac{m}{2} (x_i - x_{i+1}) \right\} \left( \frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i} \right) + \ldots
\]

To first order, we recover the Wright-Fisher model with mutation and selection
Duality between \textit{ABEP}(\(\sigma, m\)) and \textit{SIP}(m)

Theorem [Carinci, G., Redig, Sasamoto (2014 + in progress)]

The \textit{ABEP}(\(\sigma, m\)) on \([1, L] \cap \mathbb{Z}\) is dual to the \textit{SIP}(m) on \([1, L] \cap \mathbb{Z}\) with self-duality function

\[
D^{\sigma}(x, \xi) = \prod_{i=1}^{L} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} + \xi_i\right)} \cdot (\sinh(\sigma x_i))\xi_i \cdot e^{-\sigma x_i} \left[2 \sum_{k=1}^{i-1} \xi_k + \xi_i\right]
\]

The proof follows from the self-duality of \textit{ASIP} and the scaling limit:

\[
\tilde{D}_q(\eta, \xi) := (1 - q)|\xi| D_q(\eta, \xi) \quad \lim_{N \to \infty} \tilde{D}_{1-\sigma/N}(N x, \xi) = D^{\sigma}(x, \xi)
\]

\[
\left[\left(\mathcal{L}^{\text{ABEP}(\sigma, m)} D^{\sigma}\right)(\cdot, \xi)\right](x) = \lim_{N \to \infty} \left[\left(\mathcal{L}^{\text{ASIP}\left(1-\sigma/N, m\right)} \tilde{D}_{1-\sigma/N}\right)(\cdot, \xi)\right](N x)
\]

\[
= \lim_{N \to \infty} \left[\left(\mathcal{L}^{\text{ASIP}\left(1-\sigma/N, m\right)} \tilde{D}_{1-\sigma/N}\right)(N x, \cdot)\right](\xi)
\]

\[
= \left[\left(\mathcal{L}^{\text{SIP}(m)} D^{\sigma}\right)(x, \cdot)\right](\xi)
\]
Thank you for your attention.