On the homogenization of microstructured surfaces

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\begin{abstract}

The direct numerical simulation of microstructured interfaces like multiperforated absorber in acoustics with hundreds or thousands of tiny openings would result in a huge number of basis functions to resolve the microstructure. One is, however, primarily interested in the effective and so homogenized transmission and absorption properties. We introduce the surface homogenization that asymptotically decomposes the solution in a macroscopic part, a boundary layer corrector close to the interface and a near field part close to its ends. The introduction is for a general framework of models of elliptic partial differential equations incorporating the influence of end-points of the microstructured interfaces to the macroscopic part of the solution. The effective transmission and absorption properties are expressed by transmission conditions on an infinitely thin interface and corner conditions at its end-points to ensure the correct singular behaviour, intrinsic to the microstructure. We give details on the computation of the effective parameters and show their dependence on geometrical properties of the microstructure on the example of the wave propagation described by the Helmholtz equation. Numerical experiments indicate with the obtained macroscopic solution representation one can reach very high accuracies with a small number of basis functions.

\textbf{Keywords}

Asymptotic analysis; periodic surface homogenization; singular asymptotic expansions; stress intensity factor; numerical methods.

\textbf{AMS Subject Classification} 32S05, 35C20, 35J05, 35J20, 41A60, 65D15

\section{Introduction}

Microstructured interfaces show effective properties like an absorption of acoustic waves or an impedance for electric fields where much less needs of material or volume of air is needed as if solutions without a microstructure are used. In many engineering applications microstructured surfaces are used to create and tailor such effective
properties. Most prominently are microperforated absorbers and liners for the reduction of acoustic noise of vehicles or aircrafts or for optimal acoustics in conference or lecture halls (see Fig. 1 [9] and [28]). These plates with an array of perforations above a chamber or an array of chambers each of little volume, where the size and distance of the holes are much smaller than the wavelength of the acoustic waves, lead to a damping of waves in a broad or narrow frequency range. Probably equally known is the Faraday cage where a mesh of thin conductors leads to an effective electric shielding. Various examples are shown in Fig. 2 so a channel that is connected to a side chamber by a perforated wall, a channel with a perforated wall in its cross-section and the cross-section of a channel in 3D including a circular wall where a part of it is multiperforated. Direct numerical simulations are exorbitantly expensive for high porosities as for an accurate computation, e.g., with the finite element or finite difference method the size of a large number of mesh cells have to be at the order of the small scale or even smaller. Even so the nature of each of these effects is different due to the different physical phenomena on the microscopic level they all can be modelled in a similar way by a homogenization procedure along the interface. Exactly as the homogenization of volumic microstructures [2, 3, 3] this surface homogenization leads to models with effective parameters representing the microstructure, which can be resolved numerically with a computational effort independently of the ratio of macroscopic and microscopic scales. The procedure of the surface homogenization differs much from the original volumic one, and we expect numerical methods based on the surface homogenization to differ from the numerical methods for the volumic microstructures [24, 32, 1, 31, 6]. The surface homogenization leads to an asymptotic solution representation, which can be used to construct effective boundary or transmission conditions [5, 10, 15, 22, 35], if their end-points are ignored. At the end-points points the asymptotic solution representation has to incorporate the interaction of the microstructure and the singularities correctly (this has been done for the Poisson problem [16, 17] and for the Helmholtz problem [18]). Especially the interaction with the singular behaviour, that is macroscopically measurable, is mathematically involved. It is based on an extension of the singularity theory by Kondrat’ev [27]. This extended theory is due to Nazarov in 1991 in [33] who has introduced the theory for oscillating boundaries ending at a corner (see also [34] and [30, Section 17]). In this article the surface homogenization is presented as a general methodology for an effective description and numerical modelling of microstructured surfaces with emphasis on the interaction with the singular behaviour at its end-points.

The outline of the paper is as follows. Section 2 is dedicated to the major ideas of the surface homogenization in presence of singularities. Based on the solution representation consisting of its macroscopic part, the boundary layer and its near field part effective transmission condition and corner conditions for the macroscopic solution at the limit interface or limit end-points of the microstructured layer, respectively, are introduced. How the nature of the transmission conditions is result of the existence properties of solutions of cell problems for one period of the microstructured layer (see Fig. 5a) and how its parameters are obtained by pre-computations of these solutions is explained in Section 3. Then, in Section 4 the relation of the singular behaviour of the macroscopic part of the solution and the near field part close to the layer end-points is explained. Finally, in Section 5 the accuracy of the surface homogenization is illustrated on numerical experiments.

2 Homogenization for microstructured interfaces at a glance

In this article, the microstructured interfaces considered are constructed of a flat wall from which periodically material is taken away leading to holes or is added leading to an additional roughness, where both are modeled by means of boundary conditions or periodically varying the coefficients in the governing equations.

Figure 2: Illustration of configurations of multiperforated absorbers. The end-points of the multiperforated walls meet the domain boundary at different angles Θ.
Let \( \delta \) be the characteristic distance between two consecutive holes or two consecutive obstacles of the microstructured layer (see e.g. Fig. 3a).

Let \( D(\xi) \) be a complex \( m \times 1 \) vector linearly dependent on the variable \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \). Furthermore, let \( A^\delta, B^\delta \) be two functions with values in the space of complex \( m \times m \) matrices, \((A^\delta, B^\delta) \in (C^\infty(\Omega^\delta))^{m \times m}\) with its limits \( A^0(x) \) and \( B^0(x) \) for \( \delta \to 0 \) and for all \( x \in \Omega \setminus \Gamma \).

In the domain \( \Omega^\delta \), we consider the general problem

\[
\begin{align*}
L^\delta(x, \nabla_x)u &= f, \quad \text{in } \Omega^\delta, \\
N^\delta(x, \nabla_x)u &= g, \quad \text{on } \partial\Omega^\delta,
\end{align*}
\]

where the operators \( L^\delta \) and \( N^\delta \) are defined by

\[
L^\delta(x, \xi) = D(-\xi)A^\delta(x)D(\xi), \quad N^\delta(x, \xi) = D(n)B^\delta(x)D(\xi),
\]

\( n \) being the unit exterior normal vector on \( \partial\Omega^\delta \). We introduce in a similar way the operators \( L^0 \) and \( N^0 \) associated to the matrices \( A^0 \) and \( B^0 \), i.e.,

\[
\lim_{\delta \to 0} L^\delta(x) = L^0(x), \quad \lim_{\delta \to 0} N^\delta(x) = N^0(x), \quad x \in \Omega \setminus \Gamma.
\]

We assume moreover that problem (2.1) is well-posed for any \( \delta \in (0, \delta_0) \) for some \( \delta_0 > 0 \) and admits a solution in the variational space \( \mathcal{V}^\delta(\Omega^\delta) \), and that the limit problem is well posed and admits a solution in the variational space \( \mathcal{V}^0(\Omega) \).

We are going to present the surface homogenization with singularities in a general setting, which we will illustrate with the following example of the scattering on a micro-perforated wall.

**Example 2.1.** We consider for illustration the Helmholtz problem with homogeneous wave-number \( k_0 \) in a wave-guide that is connected to a chamber by a multiperforated wall with holes of distance \( \delta \) and opening width \( \eta(\delta) \).

The computational domain \( \Omega^\delta = \Omega \setminus \Omega^\delta_{\text{hole}} \) with the periodic array of obstacles \( \Omega^\delta_{\text{hole}} \subset (-L, L) \times (-\delta, \delta) \) and the limit domain \( \Omega \setminus \Gamma \) and limit interface \( \Gamma \) are illustrated in Figure 3. This Helmholtz problem can be stated as

\[
\begin{align*}
\Delta u^\delta + k_0^2 u^\delta &= 0, \quad \text{in } \Omega^\delta, \\
\nabla u^\delta \cdot n &= 0, \quad \text{on } \partial\Omega^\delta \setminus \Gamma^R, \\
\nabla (u^\delta - u_{\text{inc}}) \cdot n - i k_0 (u^\delta - u_{\text{inc}}) &= 0, \quad \text{on } \Gamma^R,
\end{align*}
\]

where \( u_{\text{inc}} \) is an incoming wave (from left or right), which can be assumed to solve the homogeneous Helmholtz equation in an infinite wave-guide with Neumann boundary conditions. The transparent boundary condition on \( \Gamma^R = \{ -L', L' \} \times (0, W) \), \( L' > L \) is a first-order approximation of Robin type. As incoming wave we consider for example the plane wave \( u_{\text{inc}} = \exp(ik_0(x_1 - L')) \) on the left side of \( \Gamma^R \) and \( u_{\text{inc}} = 0 \) on its right side. This example corresponds to the operators

\[
D(\nabla_x) = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ 1 \end{pmatrix}, \quad A^\delta(x) = \begin{pmatrix} 1 & 1 \\ 0 & -k_0^2 \end{pmatrix} = A^0(x),
\]

\[
B^\delta(x) = \begin{pmatrix} 1 & 1 \\ 0 & i k_0 k_{\Gamma^R}(x) \end{pmatrix} = B^0(x).
\]
and the source terms
\[ f = 0, \quad g = 1_{\Gamma_u}(x)(\nabla u_{inc} \cdot n - i k_0 u_{inc}). \]

The natural spaces associated to that problem are \( V^\delta(\Omega^\delta) = H^1(\Omega^\delta) \) and \( V^\delta(\Omega) = H^1(\Omega \setminus \Gamma) \) (see e. g., [11]).

In most cases the solution away from the microstructured interface and its end-points, i. e., the macroscopic part of the solution, is of practical interest. For example, in a wave-guide where part of its boundary is multi-perforated (see Fig. [5a]) the transmission coefficients are of importance, which are macroscopic quantities and, more precisely, functions of the macroscopic solution [37]. However, the macroscopic part is interacting with the solution close to the layer, the boundary layer part, and the solution close to the end-points, known as the near-field part (see Fig. [4]). The macroscopic solution that is defined only in some distance away from the microstructured layer can be smoothly extended to the mid-line of the layer \( \Gamma \) (see Fig. [5b]) including the end-points. On the interface \( \Gamma \) the extensions do not match necessarily as well as their derivatives, but satisfy (effective) transmission conditions. If the macroscopic part of the solution is extended in a smooth way to the end-points, the extension is not necessarily regular, e. g., it may tend to infinity at the end-points of the interface \( \Gamma [17, 18] \). A similar behaviour has been observed for the macroscopic solution for problems with oscillating boundaries with corners [34] or a domain with rounded corners [14].

Solution representation

To obtain an effective description of the macroscopic part up to the interface \( \Gamma \) and its end-points the solution is analyzed asymptotically for \( \delta \to 0 \) based on suitable expansions for the macroscopic part, the boundary layer part and the near field part (see again Fig. [4]). More precisely:

- The macroscopic part of the solution can be written as a modification of its limit term \( u_{0,0} \) by correctors \( u_{n,q}^\delta \) which are weighted with powers of \( \delta \), where the power is a combination of an integer and multiples of \( \frac{\pi}{\Theta} \), with opening angle \( \Theta \) at the macroscopic corner (see Figs. [2] and [3]):
  \[ u^\delta(x) \sim u_{0,0}(x) + \sum_{(n,q) \neq 0} \delta n + q u_{n,q}^\delta(x). \] (2.3)

  The macroscopic terms \( u_{n,q}^\delta \) are defined in the limit domain \( \Omega \setminus \Gamma \) of \( \Omega^\delta \) for \( \delta \to 0 \) (see Fig. [5b]), i. e., up to the corners and the limit interface \( \Gamma \), where they might be two-sided.

- The boundary layer part of the solution corrects the macroscopic part in the neighbourhood on the micro-structured layer. Each macroscopic term \( u_{n,q}^\delta \) is corrected by a boundary layer term \( \Pi_{n,q}^\delta(x) \). It depends on two variables, \( x_G = x_G(x) \) the nearest point of a point \( x \) on the interface \( \Gamma \) and the scaled coordinate \( X = (x - x_G)/\delta \) (see diagonally hatched area in Fig. [4]) and lead to transmission conditions (see Section 3). The boundary layer terms \( \Pi_{n,q}^\delta \) are defined in canonical periodicity cells (see, e. g., Fig. [5a] or [5b]).

- The near field part of the solution corrects its macroscopic part in the neighbourhood on the end-points of the microstructured layer. Each macroscopic term \( u_{n,q}^\delta \) is corrected by a near field term \( U_{n,q}^\delta(X) \) close to the end-point \( x_G^\delta \) depending on the scaled coordinate \( X^\pm = (x - x_G^\pm)/\delta \) (see vertically hatched area in Fig. [4]). The near field terms \( U_{n,q}^\delta \) are defined in canonical domains of the vicinity of one end-point (see, e. g., Fig. [5d] or [5e]) and lead to corner conditions (see Section 4).

![Figure 4: Schematic representation of the overlapping subdomains for the asymptotic expansion. The macroscopic area (dark gray) away from the corners \( x_G^\pm \) and from the limit interface \( \Gamma \), the boundary layer area (light blue vertically hatched) and the near field areas (light green diagonally hatched) are overlapping each other.](image-url)
Numerical computation of effective macroscopic approximations  In this paper, we show how to define and compute the terms of the effective macroscopic approximation numerically after each other. In this way an approximation to the macroscopic part of the solution is obtained that is computable with an effort that is independent of the number of holes or obstacles or its characteristic size $\delta$. For this some pre-computations are performed in domains, which are canonical to the boundary layer part and the near field part of the solution (see Section 3). This is first a domain (see Fig. 5a), which is obtained by taking a zoom to one period (e.g., around one obstacle), where the end-points of the layer and all the other boundaries are relegated towards infinity. The interaction with the other obstacles are taking into account by regarding a periodicity cell of the now infinite array of obstacles. Second, pre-computations are performed on a domain which is obtained by taking a zoom to the end-points of the array obstacles, where the part of boundary that is not touching the end-point, including the other end-point, is relegated to infinity (see Section 4). In this way, a conical domain with a semi-infinite array of obstacles of size and distance of order 1 as shown in Fig. 5d is obtained. This domain has still an infinite number of obstacles and we propose to approximate the near field solution on a truncated sub-domain with well-chosen boundary conditions based on its properties towards infinity.

After this pre-computations, we compute the terms of the macroscopic expansion step-by-step (see Section 5). Each term of the macroscopic expansion depends only on the previous terms. However, each corrector term of the limit solution is singular at the end-points of $\Gamma$. More precisely, they increase towards infinity when approaching the end-point. For this each macroscopic term to be computed is decomposed into a regular part and a singular part. The unbounded singular part is given analytically as a function of the previous terms, whereas the regular part lives in the energy space $V^0(\Omega)$, e.g. the Sobolev space like $H^1(\Omega \setminus \Gamma)$ and can be computed with classical adaptive finite element methods [38], where the data depends on the terms of lower order.

Justification  The asymptotic solution representation can be verified theoretically with rigorous estimates on the modelling error of the macroscopic approximation, which has been done in [16] for the Poisson problem in a waveguide with Dirichlet boundary conditions connected to a side chamber by a multi-perforated wall and in [18] for the Helmholtz problem with all-over Neumann boundary conditions. The error estimates are based on the above mentioned theory of the solutions of the near field problems in the conical domain with the semi-infinite array of obstacles (see Fig. 5d) in special weighted Sobolev spaces and a matching procedure of the different expansions. In general, one expects an optimal macroscopic modelling error in a subdomain $\Omega_\alpha$ of $\Omega^0$ of fixed distance $\alpha > 0$ away from the microstructured layer that is of the order of the first neglected term, i.e., for any $s > 0$ it holds in the energy norm of the problem

$$
\|u^\delta - \sum_{\hat{\gamma} \in \mathbb{N}^+ \times \mathbb{N}} \delta^{\hat{\gamma} \cdot \theta \eta} u_{\gamma,\eta}^\theta \|_{\Omega_\alpha} = O(\delta^s \ln \delta)^{\kappa(s)},
$$

(2.4)
where the exponent $\kappa(s) \in \mathbb{N}$ of the logarithm $\ln \delta$ depends on $s$. An optimal error also in the vicinity of the microstructured layer cannot be expected with the macroscopic part alone, but if combinations of the macroscopic terms and near field terms multiplied with well-suited cut-off functions and respective boundary layer terms are added.

**Example 2.2 (Wave-guide connected to a chamber by a multiperforated wall).** We consider for the numerical illustration and verification of Example 2.1 the domain $\Omega_{\text{hom}}^\delta$ as a thin plate of length 1 (i.e. $L = 0.5$) and width $0.075\delta$, containing $1/\delta \in \mathbb{N}$ holes that are periodically spaced. For this domain the periodicity cell is shown in Fig. 5a and the near-field domain close to the end-point $x^-_0$ in Fig. 5d. We denote by $\rho$ the porosity of the thin plate, i.e. the characteristic size of a hole is $\eta(\delta) = \rho \delta$. The value of the angle at the end-points is $\Theta = 3\pi/2$. The width of the chamber and of the wave-guide are both equal to $W = 0.5$. The length of the wave-guide is $L' = 2.5$ and the wave number in (P) is $k_0 = 5\pi$.

### 3 The periodic layer and transmission conditions

As we said already in the introduction, the main interest is an effective description of the macroscopic part taking into account the interaction with the periodic layer and the corner singularities. This section focuses on the interaction with the periodic layer. For the effective description the macroscopic solution is extended to the mid-line $\Gamma$ of the layer close to the end-points, and the macroscopic solution as well as its normal derivative can become discontinuous and fulfill transmission conditions which compromise the periodic layer and its impedance in an effective way.

To expose this effective behaviour of a macroscopic solution, it is expanded in powers of $\delta$, the distance between the size of the holes. This becomes

$$u^\delta(x) = u_0^\delta(x) + \delta u_1^\delta(x) + \delta^2 u_2^\delta(x) + \ldots,$$

where the dependence of the terms $u_\ell^\delta$ on $\delta$ is due to the end-points, which we will suspend at this moment, and to a possibly smaller scale of the geometry (e.g. $\eta(\delta) = o(\delta)$ in Example 2.1).

It has been already widely spread in the literature (see e.g. [10] and the references within) that the transmission conditions for the limit solution $u_0^\delta$ take then the general form

$$(B_{\Gamma} u_0^\delta)(x) = 0 \quad \text{on } \Gamma,$$

where $B_{\Gamma}$ is an operator taking the two limits of $u_0^\delta$ and its normal derivative on the limit interface $\Gamma$. Introducing the jump $[v]_\Gamma$ of a function $v$ over $\Gamma$ and the jump $[\nabla v \cdot n]_\Gamma$ of its normal derivative by

$$[v]_\Gamma(x^\Gamma) := \lim_{h \to 0} v(x^\Gamma + nh) - v(x^\Gamma + nh),$$

$$[\nabla v \cdot n]_\Gamma(x^\Gamma) := \lim_{h \to 0} (\nabla v(x^\Gamma + nh) - \nabla v(x^\Gamma - nh)) \cdot n,$$

where the averages $\langle v \rangle_\Gamma$ and $\langle \nabla v \cdot n \rangle_\Gamma$ are introduced in a similar way, for Example 2.1 and $\eta(\delta) \sim \delta$, this operator becomes

$$B_{\Gamma} v = \left[ \frac{[v]_\Gamma}{[\nabla v \cdot n]_\Gamma} \right]$$

It corresponds to the acoustics of a in the limit vanishing layer, and $B_{\Gamma} v = ([v]_\Gamma - \langle \nabla v \cdot n \rangle_\Gamma, [\nabla v \cdot n]_\Gamma)^\top$ for $\delta \log \eta(\delta) \sim 1$ (i.e. $\eta(\delta) = \beta^1/\delta$ for some $\beta \in (0, 1)$) corresponding to an impedance boundary condition in the limit (see also [35]). If Dirichlet conditions on the boundary of the obstacles of (P) are taken, then for $\eta(\delta) \sim \delta$ one obtains $B_{\Gamma} v = ([v]_\Gamma, \langle v \rangle_\Gamma)^\top$ corresponding to a closed wall (see also [23] for the electromagnetic scattering on a cylindrical Faraday cage).

The first corrector satisfies similar transmission conditions with a source term depending on the limit solution

$$(B_{\Gamma} u_1^\delta)(x) = (B_{\Gamma}^1 u_0^\delta)(x) \quad \text{on } \Gamma$$

as well as the higher order correctors with a source term depending on all previous terms

$$(B_{\Gamma} u_p^\delta)(x) = \sum_{\ell=0}^{q-1} (B_{\Gamma}^{\ell+p} u_p^\delta)(x) \quad \text{on } \Gamma.$$
For Example 2.1 and \( \eta(\delta) \sim \delta \) the first order corrector is

\[
B^1_\Gamma v = \left( 2D_\infty (\nabla v \cdot n)_{\Gamma} \right)
\]

with two parameters \( D_\infty, N_0 \in \mathbb{R} \).

The nature of the transmission conditions and their parameters depend on existence and uniqueness of the so called cell problems defined in the periodicity domain \( \Omega \) (see, e.g., Fig. 5a) with suitable boundary conditions. To define such a domain, one has to scale around one hole with respect to \( \delta \), take an appropriate ansatz and plug this ansatz in the rescaled problem. If a smaller scale is involved (e.g. \( \eta(\delta) = o(\delta) \)), then these periodicity cell problems will contain a point contribution (see Fig. 5b), coming from resolution of another problem in a geometry scaled with \( \eta(\delta) \) around one hole (see Fig. 5c).

To obtain the parameters, in general, the solution of a cell problem has to be computed, but sometimes they appear just as a function of geometrical parameters or are even simple constants. For example, the impedance parameter \( \frac{\epsilon_B}{\rho} \) contains a point contribution (see Fig. 5b), coming from resolution of another problem in a geometry scaled with \( \eta(\delta) \) around one hole (see Fig. 5c).

More specifically, for this example, the condition \( \nabla u^0_1 \cdot n \rightarrow \hat{\Omega} \) originates from the existence of the blockage function \( D \)

\[
\left\{ \begin{array}{l}
-\Delta \Pi = 0, \quad \text{in } \hat{\Omega}, \\
\nabla \Pi \cdot n = 0, \quad \text{on } \partial \hat{\Omega}_{\text{hole}} \cap \hat{\Omega},
\end{array} \right.
\]

where \( \Pi \) and its derivative are 1-periodic and \( \Pi \) is bounded. The boundary \( \partial \hat{\Omega}_{\text{hole}} \cap \hat{\Omega} \) is illustrated by the blue line in Fig. 5a. The condition \( [\nabla u^0_1]_\Gamma = 0 \) comes from the solution of

\[
\left\{ \begin{array}{l}
-\Delta D = 0, \quad \text{in } \hat{\Omega}, \\
\nabla D \cdot n = 0, \quad \text{on } \partial \hat{\Omega}_{\text{hole}} \cap \hat{\Omega},
\end{array} \right.
\]

where \( D \) and its derivative are 1-periodic and \( D - X_2 \) is bounded. This problem defines \( D \) up to an additive constant. This constant is chosen such that the two limits of \( \pm(D - X_2) \) for \( \pm X_2 \rightarrow \infty \) are the same, and \( D_\infty \) denote the value of this limit, i.e.,

\[
\lim_{X_2 \rightarrow \pm \infty} (D(X_1, X_2) - X_2 \mp \Delta \infty) = 0, \quad X_1 \in (0, 1).
\]

The problem (3.8) can be solved numerically on a truncated periodicity cell \( \hat{\Omega}_B \) for given \( B \geq 2 \) using Dirichlet-to-Neumann (DtN) boundary operators \( \Lambda_B \) based on a Fourier expansion in \( X_1 \) in the spirit of [25, 26], using a spectral decomposition of \( D \) with the theory of self-adjoint compact operators [7 Theorem VI.11], for \( X_1 \in (0, 1) \):

\[
\Lambda_B^\pm D(X_1, \pm B) := -\sum_{m \neq 0} \frac{2\pi |m|}{\rho} \left( \int_0^1 D(\hat{X}_1, \pm B) e^{-2\pi i m \hat{X}_1} d\hat{X}_1 \right) e^{2\pi i m X_1}.
\]

With this DtN boundary operator, problem (3.8) can be truncated on \( \hat{\Omega}_B \), adding the condition

\[
\nabla D \cdot n + \Lambda_B^\pm D = \pm 1, \quad \text{on } \Gamma_B^\pm = (0, 1) \times \{ \pm B \}.
\]

and we look for a periodic solution \( D \in H^1(\hat{\Omega}_B) \). Then, the condition (3.9) and the spectral decomposition of \( D \) lead to

\[
(i) \quad \int_{\Gamma_B^+} D + \int_{\Gamma_B^-} D = 0, \quad (ii) \quad \int_{\Gamma_B^+} D - \int_{\Gamma_B^-} D = 2B = 2D_\infty.
\]

We illustrate the function \( D \) and the constant \( D_\infty \) for Example 2.2 as functions of the porosity \( \rho \) in Figs. 6 and 7 using a finite element discretization and truncate DtN operators with 8 modes.

In Fig. 6 it is clearly observed that the blockage coefficient \( D_\infty \) depends mainly on the relative size of the holes and less on the relative thickness of the wall. It is visible that the function \( D - X_2 \) vary mostly inside and in the neighborhood of the holes and that the limit values \( \pm D_\infty \) in (3.9) are rapidly attained. Fig. 7 shows then that the blockage coefficient \( D_\infty \) is in fact a decreasing function of the porosity \( \rho \). Moreover, this coefficient behaves asymptotically like \( 1/\rho \) as \( \rho \rightarrow 0 \). In particular, it diverges to infinity and the jump corrector (3.4) becomes less and less negligible, which corresponds, however, to a non-physical behaviour.
\[ H = 0.3, \rho = 0.9, \quad D_\infty = 0.038775. \]
\[ H = 0.3, \rho = 0.5, \quad D_\infty = 0.37172. \]
\[ H = 0.3, \rho = 0.1, \quad D_\infty = 3.2658. \]
\[ H = 0.6, \rho = 0.9, \quad D_\infty = 0.022052. \]
\[ H = 0.6, \rho = 0.5, \quad D_\infty = 0.273737. \]
\[ H = 0.6, \rho = 0.1, \quad D_\infty = 1.91781. \]

Figure 6: Plot of \( D - X_2 \) for different values of the porosity \( \rho \) and relative wall thickness \( H = 0.3 \) or \( H = 0.6 \) for Example 2.2. The periodicity domain \( \hat{\Omega} \) is obtained by identification with \( \delta = 1/4 \) for \( H = 0.3 \) and \( \delta = 1/8 \) for \( H = 0.6 \).

Figure 7: Plot of \( D_\infty \) with respect to the porosity \( \rho \) of the obstacle. Close to \( \rho = 0 \), the quantity \( D_\infty \rho \) remains constant.

### 4 The end-point of the periodic layer and corner conditions

In the previous section, we derived an effective description for the macroscopic part extended to the mid-line \( \Gamma \) of the microstructured layer through the description of transmission conditions (3.5). In these transmission conditions the termination of the microstructured layer are not considered. Therefore one can ask if these transmission conditions are still valid when approaching these end-points \( x_{\hat{\Omega}} \) of the interface \( \Gamma \). Equivalently, the question of the correct singular behaviour of \( u_{\delta q} \) close to the end-points arises.

To expose this effective behaviour of a macroscopic solution, it is expanded in powers of \( \delta \hat{\pi} \), where \( \Theta \) is the
opening angle of the end-points. This becomes
\[ u^0_i(x) = u^0_{0,q}(x) + \delta \hat{\pi} u^0_{1,q}(x) + \delta \hat{\pi}^2 u^0_{2,q}(x) + \ldots, \]  
(4.1)
where the dependence of the terms \( u^0_{n,q} \) on \( \delta \) may be logarithmic (i.e. \( \ln \delta \)) caused by the end-point singularities, and possibly smaller scales (e.g. \( \eta(\delta) = \exp(-C/\delta) \)) in Example 2.1.

For each macroscopic term \( u^0_{n,q} \), one defines the stress intensity factor \( s^0_{n,q} \) that are not in \( \mathcal{V}^0(\Omega \setminus \Gamma) \) in general, to have the difference \( u^0_{n,q} - s^0_{n,q} \in \mathcal{V}^0(\Omega \setminus \Gamma) \). Here, the set \( \mathcal{V}^0(\Omega \setminus \Gamma) \) is the set of functions such that their restriction to any connected domain \( K \subset \Omega \) belongs to \( \mathcal{V}^0(K) \). The notion of stress intensity factors have been introduced by the mechanical engineering community for corners and cracks, see e.g. [8, 13, 39]. The reason for the existence of the stress intensity factors for microstructured interfaces are twofold:

(i) Due to the transmission conditions (3.4) with a source term depending on the limit solution, we obtain a singular behaviour for the first corrector close to the end-points, which is automatically consistent when the matching with the near field. Numerical pre-computations in a neighborhood of the end-points of the periodic layer are not necessary. This point will be more deeply studied in Sec. 4.1.

(ii) In addition higher-order correctors exhibit a singular behavior that is not caused by the source term in the transmission condition only, since this singular behavior is in the kernels of \( \mathcal{L}^0 \) and \( \mathcal{B}_\Gamma \) close to the end-points, but can be explained only with the matching to the near field. For this, we need pre-computations of singular enhancement functions \( \mathcal{S}^\pm \) and singularity enhancement factors \( \mathcal{L}(\mathcal{S}^\pm) \) in a neighborhood of the end-points of the periodic layer. This point will be more deeply studied in Sec. 4.2.

In general, a part of the singularity is correctly obtained studying the behaviour due to the source terms in the transmission conditions and a part is not correctly obtained and one needs to study the matching with the near field and pre-compute singularity enhancement functions and factors. The stress intensity factors and their nature depend on the shape of singularity enhancement functions in the two conical domains containing an infinite periodic layer \( \Omega^\pm \) (see Fig. 5d). To define the problem for the singular enhancement functions, one has to scale around one corner with respect to \( \delta \) to obtain the domain \( \Omega^\pm \), take an appropriate ansatz and plug this ansatz in the rescaled problem. If a smaller scale is involved (e.g. \( \eta(\delta) = o(\delta) \)), then these near field problems will contain an infinite periodic point contribution (see Fig. 5e), originating from resolution of another problem in a geometry scaled with \( \eta(\delta) \) around one hole (see Fig. 5c).

To obtain these functions, in general, the solution of a near field problem has to be computed (Sec. 4.2), but sometimes they can be computed analytically, using the impedance parameters that were computed in the previous section (Sec. 5).

### 4.1 Singular behaviour due to source terms in the transmission conditions

In this section, we are interested in the singular behaviour of the macroscopic solution close to the end-points of the interface \( \Gamma \). For this we consider the the boundary value problem
\[
\begin{align*}
\mathcal{L}^0 v &= 0, & \text{ in } \mathcal{K}^\pm \setminus \Gamma^\pm, \\
\mathcal{N}^0 v &= 0, & \text{ on } \partial \mathcal{K}^\pm, \\
\mathcal{B}_\Gamma^\pm v &= f^\pm, & \text{ on } \Gamma^\pm,
\end{align*}
\]
(4.2)
in the infinite cones (see Fig. 8)
\[ \mathcal{K}^\pm = \{(r^\pm \cos \theta^\pm, r^\pm \sin \theta^\pm), \quad r^\pm > 0, \quad \theta^- \in (\pi - \Theta, \pi), \quad \theta^+ \in (0, \Theta)\}. \]
The semi-infinite straight interfaces \( \Gamma^\pm \) are tangent to \( \Gamma \) at its end-points \( x^\pm_0 \). We consider \( f^\pm = \mathcal{B}_\Gamma^\pm u \) with \( u \in \mathcal{V}^0(\mathcal{K}^\pm) \) being an homogeneous solution of (4.2) where \( \mathcal{V}^0(\mathcal{K}^\pm) \) is the set of functions \( \phi \) such that the function \( \phi \chi^\pm \in \mathcal{V}^0(\Omega) \), where \( \chi^\pm \) are \( C^\infty \) truncation functions compactly supported in a vicinity of \( x^\pm_0 \). The operators \( \mathcal{B}_\Gamma^\pm \) and \( \mathcal{B}_1^\pm \) are formally the operators \( \mathcal{B}_\Gamma \) and \( \mathcal{B}_1 \) written on \( \Gamma^\pm \) instead of \( \Gamma \).

More specifically, in Example 2.1 the limit term \( u_{0,0} \in H^1_{\text{loc}}(\mathcal{K}^\pm) \) meaning that \( u_{0,0} \chi^\pm \in H^1(\Omega) \). Using again the self-adjoint operators for the 1D Laplace-Beltrami operator \( \partial_\theta^2 \) with Neumann boundary conditions at \( \theta^- = \{\pi - \Theta, \pi\} \) (resp. \( \theta^+ = \{0, \Theta\} \), we can express \( u_{0,0} \) close to the end-point \( x^\pm_0 \) as a linear combination of radial Bessel functions of first kind \( J_m(k_0 r^\pm) \) times cosine functions in \( \theta^\pm \). This leads in view of the transmission conditions (3.4) for \( u^0_{0,1} \) to problem (4.2) with the right-hand side \( f^\pm_m := \mathcal{B}_\Gamma^\pm w^\pm_m, w^\pm_m(r^\pm, \theta^\pm) := \)
solve the system

These functionals are obtained by projecting the macroscopic terms of lower order on their regular part. For

stress intensity factors

4.2 Singular behaviour coming from the matching with the near field

To obtain the singular enhancement factor, we extract from $u_{0,1}^\pm$ the contribution corresponding to the Bessel function $J_{-\ell}(kr^\pm)$

Then the singular behaviour of $u_{0,1}^\pm$ is given in terms of the stress intensity factor $K(4.5)$

The function $\phi_1^\pm(r^\pm, \theta^\pm)$ that we specify later in Appendix A.

Note that $u_{0,1}^\pm$ is given by $(4.4)$

4.2 Singular behaviour coming from the matching with the near field

As it was already stated in the introduction of this section, there exists two singularity enhancement factors $\mathcal{L}(S^\pm)$ such that the singular behaviour of $u_{2,0}^\pm$ is given by

where $I^- = (\pi - \Theta, \pi), I^+ = (0, \Theta)$. (4.5)

Then the singular behaviour of $u_{0,1}^\pm$ is given in terms of the stress intensity factor $K_{o}^\pm(4.6)$

Note that the function $\phi_2^\pm(r^\pm, \theta^\pm) := (r^\pm, \theta^\pm) \mapsto Y_{\pm}(kr^\pm) \cos \frac{\pi}{\Theta}(\theta^\pm - \Theta_0^\pm) \chi_{\pm} \in H^1(\Omega \setminus \Gamma)$. (4.7)

More generally, a higher order macroscopic term has a singular behaviour as a linear combination of canonical stress intensity factors $y_{k,\pm}(r^\pm)\phi_{k,\pm}(\theta^\pm)$ that are solution of (4.2) with source terms that are products of a functional of lower order macroscopic terms and a related singularity enhancement factor. These functionals are obtained by projecting the macroscopic terms of lower order on their regular part. For example, $\ell^\pm(u_{0,0})$ is given by relation (4.3).

To obtain the singular enhancement factor $\mathcal{L}(S^\pm)$, one has to compute a particular near field function (also called singular enhancement function) $S^\pm$ in a stretched multi-perforated domain around one end-point (see Fig. 5d) that is solution of a Laplace equation with a prescribed behavior at infinity away from the perforations, i.e. for radial coordinate $R \to \infty$ and for $\theta$ different from the angle of the interface. More precisely, we are looking for $S^\pm$ that solve the system

$$
\Delta S^\pm = 0, \quad \text{in} \ \tilde{\Omega}^\pm,
$$

$$
\nabla S^\pm \cdot n = 0, \quad \text{on} \ \partial \tilde{\Omega}^\pm,
$$

$$
S^\pm = R^{2/\Theta} \cos \frac{\pi}{\Theta}(\theta^\pm - \Theta_0^\pm) + o(1), \quad R \to \infty, \theta \neq \pi - \Theta_0^\pm.
$$

Figure 8: Semi-infinite conical domain $\mathcal{K}^-$ with the semi-infinite interface $\Gamma^-$. 
We can see that the equation and the boundary conditions that we have to consider are just the principal symbol of the Helmholtz equation and the Neumann boundary conditions of (4.1). In the general case of linear operators $L^0$, $N^0$, given e.g. by (4.2), and denoting respectively $L^0$, $N^0$ their principal part, we have to solve

$$L^0V = F, \quad \text{in } \widehat{\Omega}^\pm,$$

$$N^0V = G, \quad \text{on } \partial\widehat{\Omega}^\pm,$$  \hspace{1cm} (4.9)

with a prescribed behavior towards infinity, which originates from the expansion of the homogeneous solutions of (4.2). Such a problem has been studied by Sergei Nazarov in the case of a periodic boundary with Dirichlet boundary conditions[33] and with Neumann boundary conditions[35] for a general linear differential operator and has been studied by the authors [16, 18] and relies on the use of Mellin transform, as well as on the extension of the Kondrat’ev theory [27]. The possible non-zero right-hand sides $F$ and $G$ in (4.9) are obtained from the study of the high-order near field terms.

The standard variational space to solve problem (4.9) in the case of the Laplace equation is

$$\mathcal{V}^\pm(\widehat{\Omega}^\pm) = \left\{ V \in H^1_{\text{loc}}(\widehat{\Omega}^\pm), \nabla V \in L^2(\widehat{\Omega}^\pm), \frac{V}{(1 + R) \ln(2 + R)} \in L^2(\widehat{\Omega}^\pm) \right\},$$

which, equipped with the norm

$$\| V \|_{\mathcal{V}^\pm(\widehat{\Omega}^\pm)}^2 = \left\| \frac{V}{(1 + R) \ln(2 + R)} \right\|_{L^2(\widehat{\Omega}^\pm)}^2 + \| \nabla V \|_{L^2(\widehat{\Omega}^\pm)}^2,$$  \hspace{1cm} (4.10)

is a Hilbert space. However, it is clear that with the requested condition towards infinity (4.8), the singular enhancement function $S^\pm$ cannot belong to $\mathcal{V}^\pm(\widehat{\Omega}^\pm)$. Therefore, we shall decompose it into a particular function (also called asymptotic block) $S^\pm$ that has this prescribed behavior towards infinity, and its remainder $R^\pm = S^\pm - S^\pm$ that belongs to $\mathcal{V}^\pm(\widehat{\Omega}^\pm)$.

To write the asymptotic block $S^\pm$, one starts from the limit behavior $R^\pi e^{\pm \cos \Theta (\theta^\pm - \Theta_0^\pm)}$ of (4.4). In the particular case $\Theta = \pi$ (see e.g. Fig. 2a), we need to take into account one additional term in that expansion. Neglecting the $O(R^{\pi/2 - 2})$ part and multiplying by $\chi(R)$ to have a regular behavior towards $R = 0$, the remainder $\mathcal{R}^\pm$ satisfies problem (4.9) with $F = -L^0 S^\pm = -\Delta S^\pm$ and $G = -N^0 S^\pm = -\nabla S^\pm \cdot \mathbf{n}$. This problem is well-posed and admits a unique solution in $\mathcal{V}^\pm(\widehat{\Omega}^\pm)$. It can then be shown that the leading part of this remainder towards infinity is the same as the leading part of problem (4.9) written in the conical domain $\mathcal{K}^\pm$ instead of the domain $\widehat{\Omega}^\pm$, i.e. there exists a constant $\mathcal{L}(S^\pm)$ independent of the choice of the truncating function such that

$$\mathcal{R}^\pm \sim \mathcal{L}(S^\pm)(R^\pm)^{-\frac{\pi}{\Theta}} \cos \frac{\pi}{\Theta} (\theta^\pm - \Theta_0^\pm), \quad R^\pm \to \infty, \theta^\pm \neq \pi - \Theta_0^\pm.$$  \hspace{1cm} (4.11)

The problem (4.9) can be solved numerically on a truncated near field domain $\widehat{\Omega}_{R_e}^\pm$ for given $R_e \geq 2$ using an approximate Robin boundary condition that takes into account the behavior of $\mathcal{R}^\pm$ given by (4.14). With this approximate Robin boundary condition, problem (4.9) can be truncated on $\widehat{\Omega}_{R_e}^\pm$, adding the condition

$$\nabla \mathcal{R}^\pm \cdot \mathbf{n} + \frac{\pi}{\Theta R_e} \mathcal{R}^\pm = 0, \quad \text{on } \Gamma_{R_e}^\pm,$$  \hspace{1cm} (4.15)

where the artificial boundary $\Gamma_{R_e}^\pm$ is given by $\Gamma_{R_e}^\pm = \{ (R_e \cos \theta^-, R_e \sin \theta^-, \theta^- \in (\pi - \Theta, \pi) \}$ and the artificial boundary $\Gamma_{R_e}^+$ is given similarly. Additionally, for these artificial boundaries we choose $R_e$ such that $\Gamma_{R_e}^\pm \subset \Omega^\pm$, i.e. they do not intersect any hole. Using again the behavior of $\mathcal{R}^\pm$ (4.14), we have

$$\mathcal{L}(S^\pm) \sim \frac{2}{\Theta} R_e^{\pi/\Theta} \int_{\Gamma_{R_e}^\pm} \mathcal{R}^\pm, \quad R_e \to \infty.$$  \hspace{1cm} (4.16)
Figure 9: Plot of $\mathcal{L}(S^\pm)$ with respect to the porosity $\rho$ of the obstacle, for the truncation radius $R_e = 30.5$. The periodicity domain is obtained by identification with $\delta = 1/4$ for $H = 0.3$ and $\delta = 1/8$ for $H = 0.6$. Close to $\rho = 0$, the quantity $\mathcal{L}(S^\pm)\rho$ remains constant.

Computations of $\mathcal{L}(S^\pm)$ are illustrated for Example 2.2. In Fig. 9, the singular enhancement coefficient $\mathcal{L}(S^\pm)$ is plotted with respect to the characteristic size $\rho$ of the obstacle for the truncation radius $R_e = 30.5$. In Fig. 10a, the singular enhancement coefficient $\mathcal{L}(S^\pm)$ is plotted with respect to the characteristic truncating radius $R_e$ of the near-field domain for the porosity $\rho = 0.3$. Contrarily to the computation of the blockage coefficient $D_\infty$ which exponentially converges with respect to the characteristic domain size $B$ (see e.g. [21]), the convergence rate of the singular enhancement coefficient $\mathcal{L}(S^\pm)$ is only polynomial (see Fig. 10b). Using at least two different computations (e.g. for $R_e = 30.5$ and $R_e = 35.5$) and an extrapolation one can improve the accuracy of the coefficient.

5 Computation of the macroscopic solution

This section is dedicated to the computation of each macroscopic term $u_{n,q}^\delta$ of the expansion (2.3). These terms solve the problem

$$
\begin{align*}
\mathcal{L}^0(x, \nabla x)u_{n,q}^\delta &= f_{n,q}, & \text{in } \Omega \setminus \Gamma, \\
\mathcal{N}^0(x, \nabla x)u_{n,q}^\delta &= g_{n,q}, & \text{on } \partial\Omega,
\end{align*}
$$

(5.1)

Figure 10: Plot of singular enhancement factor (a) $\mathcal{L}(S^\pm)$ and (b) $\mathcal{L}(S^\pm) - 0.167$ with respect to the truncation radius $R_e$, for the porosity $\rho = 0.3$. The periodicity cell is obtained by identification with $\delta = 1/8$ for $H = 0.6$. The convergence of the singular enhancement factor to its limit value is like $R_e^{-2/3}$. 

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Computation of the corrector $u$.

Moreover, $u$ is defined by (3.5) and with possibly corner singularities that have been studied in Section 3. One important point to notice is that, as it was already explained in Section 2, the computational effort of each macroscopic term is independent of the parameter $\delta$. Obviously this is the case since the linear differential operators involved in these equations as well as the computational domain are independent of $\delta$.

In the following the different macroscopic terms of the expansion are computed for Example 2.1 (Section 5.1) and a finite sum of the expansion is compared with a reference solution computed by resolving all the of the many holes of the microstructured interface (Section 5.2).

### 5.1 Computation of the macroscopic term of the expansion

For the example problem (P), the macroscopic term $u_{1,0}^\delta$ corresponding to the weight $\delta \hat{\pi}$ is solution of (5.1)-(5.2) with right-hand side $f_{1,0}^\delta = g_{1,0}^\delta = 0$ and contains no stress intensity factor (i.e. $u_{1,0}^\delta \in H^1(\Omega \setminus \Gamma)$). Therefore it vanished to 0 and $u_{1,0}^\delta = 0$. In a similar way, and using the transmission conditions (5.2), the macroscopic term $u_{1,1}^\delta$ corresponding to the weight $\delta \pi / \Theta^{+1}$ is equal to 0. Therefore the first non-negligible macroscopic terms are $u_{0,0}^\delta$, $u_{0,1}^\delta$ and $u_{2,0}^\delta$.

For the presented numerical simulations the C++ Finite Element Library Concepts [12, 19] has been used. The macroscopic terms are computed on a quadrilateral mesh generated using GMSH [20]. This mesh is refined close to the corners and solves the limit interface $\Gamma$ (see Fig. 11). In particular, the interface $\Gamma$ is refined close to the end-points.

**Computation of the limit solution** In this paragraph, the problem (5.1)-(5.2) for $n = q = 0$, with $f_{0,0} = 0$ and $g_{0,0} = 1_{\Gamma_n}(\nabla u_{\text{inc}} \cdot n - i k_{\text{inc}} u_{\text{inc}})$ is studied. The transmission condition (5.2) degenerates to the no jump conditions (Dirichlet and Neumann jumps vanish), so that the limit interface $\Gamma$ is transparent for $u_{0,0}^\delta$. This problem admits then a unique solution $u_{0,0} \in H^1(\Omega)$ independent of $\delta$ and is resolved numerically using an $h$-$p$-refinement strategy towards the end-points.

From the resolution of this limit problem, we compute the trace operator $B_{\Gamma}^1 u_{0,0}$ on $\Gamma$ using (3.6) and the values $\ell^\pm(u_{0,0})$ that will be used for the determination of the stress intensity factors of the upcoming terms.

**Computation of the corrector $u_{0,1}^\delta$** The corrector $u_{0,1}^\delta$ is defined by (5.1) and (5.2) for $n = 0$, $q = 1$, with $f_{0,1} = g_{0,1} = 0$ and with the transmission operator $B_{\Gamma}^1$ given by (3.6)

$$B_{\Gamma}^1 v = \left( 2D_{\infty}(\nabla v \cdot n)_\Gamma \right) \left( N_0(\partial_k^2 + k_0^2)(v)_\Gamma \right).$$

Moreover, $u_{0,1}^\delta$ admits a prescribed stress intensity factor given by the relation (4.6)

$$u_{0,1}^\delta = \sum_{\pm} \frac{k_0 \Theta}{2\pi} \ell^\pm(u_{0,0}) \phi^\pm x \in H^1(\Omega \setminus \Gamma).$$

Figure 11: Mesh used for the computation of the macroscopic solution. The blue line is the limit interface $\Gamma$. The red arcs are the domains of integration to obtain the functionals $\ell^\pm(u_{0,0})$. These lines are exactly resolved by the curved edges of the mesh.
To solve this problem, one introduces the regular part \( \tilde{u}_{0,1} \) by subtracting its stress intensity factor. Therefore, the function \( \tilde{u}_{0,1} \) has to satisfy the following problem

\[
\mathcal{L}^0(\mathbf{x}, \nabla \mathbf{x}) \tilde{u}_{0,1}^\delta = -\sum_{\pm} \frac{k_0 \Theta}{2\pi} \ell^\pm (u_{0,0}) \mathcal{L}^0(\mathbf{x}, \nabla \mathbf{x}) (\phi^\pm_1), \quad \text{in } \Omega \setminus \Gamma,
\]

\[
\mathcal{N}^0(\mathbf{x}, \nabla \mathbf{x}) \tilde{u}_{0,1}^\delta = -\sum_{\pm} \frac{k_0 \Theta}{2\pi} \ell^\pm (u_{0,0}) \mathcal{N}^0(\mathbf{x}, \nabla \mathbf{x}) (\phi^\pm_1 \chi_{\pm}), \quad \text{on } \partial \Omega, \quad (5.3)
\]

\[
(B_{\Gamma} \tilde{u}_{0,1}^\delta)(\mathbf{x}) = (B_{\Gamma}^L \tilde{u}_{0,0}) = -\sum_{\pm} \frac{k_0 \Theta}{2\pi} \ell^\pm (u_{0,0}) (B_{\Gamma} \phi^\pm_1 \chi_{\pm})(\mathbf{x}), \quad \text{on } \Gamma.
\]

Problem (5.3) seems a priori as complicated to solve as the problem satisfied by \( u_{0,1}^\delta \), since the right-hand side of the first line, for example, may not belong to \( L^2(\Omega) \). However, this is the case as the singular enhancement function \( \phi^\pm_1 \) is solution of (4.2). We introduce a suitable choice for the cut-off functions \( \chi_{\pm} \) that are identically equal to 1 in a vicinity of the end-points \( x_{\pm}^{\delta} \). To do so, with two suitable numbers \( r_i < r_{e} \), the functions \( \chi_{\pm} \) depend only on the radius \( r^\pm = |x - x_{\pm}^{\delta}| \) and is chosen such that \( \chi_{\pm} = 1 \) for \( r^\pm < r_i \) and \( \chi_{\pm} = 0 \) for \( r^\pm > r_e \). It will ensure then that the boundary operator \( \mathcal{N}^0(\mathbf{x}, \nabla \mathbf{x}) \) can commute with the truncating function, i.e. \( \mathcal{N}^0(\mathbf{x}, \nabla \mathbf{x}) (\phi^\pm_1 \chi_{\pm}) = \chi_{\pm} \mathcal{N}^0(\mathbf{x}, \nabla \mathbf{x}) (\phi^\pm_1) = 0 \). Introducing for a linear operator \( A \) the commutator operator \( [A, \chi_{\pm}] = A \chi_{\pm} - \chi_{\pm} A \) which will be compactly supported in the support of \( \nabla \chi_{\pm} \), and using that \( \phi^\pm_1 \) is solution of (4.2), problem (5.3) can be simplified to

\[
\mathcal{L}^0(\mathbf{x}, \nabla \mathbf{x}) \tilde{u}_{0,1}^\delta = -\sum_{\pm} \frac{k_0 \Theta}{2\pi} \ell^\pm (u_{0,0}) [\mathcal{L}^0(\mathbf{x}, \nabla \mathbf{x}), \chi_{\pm}] \phi^\pm_1, \quad \text{in } \Omega \setminus \Gamma,
\]

\[
\mathcal{N}^0(\mathbf{x}, \nabla \mathbf{x}) \tilde{u}_{0,1}^\delta = 0, \quad \text{on } \partial \Omega, \quad (5.4)
\]

\[
(B_{\Gamma} \tilde{u}_{0,1}^\delta)(\mathbf{x}) = (B_{\Gamma}^L \tilde{u}_{0,0}) = -\sum_{\pm} \frac{k_0 \Theta}{2\pi} \ell^\pm (u_{0,0}) (B_{\Gamma} \phi^\pm_1 \chi_{\pm})(\mathbf{x}), \quad \text{on } \Gamma.
\]

Numerically, the function \( \tilde{u}_{0,1}^\delta \) is computed using finite elements that are discontinuous over the interface \( \Gamma \), since the jump of \( \tilde{u}_{0,1}^\delta \) across the interface \( \Gamma \) is zero. The Neumann jump of \( \tilde{u}_{0,1}^\delta \) appears naturally when writing the variational formulation associated to the problem (5.4), whereas the Dirichlet jump has to be taken into account, e.g., using a mixed formulation.

**Computation of the corrector \( u_{2,0}^\delta \)** The corrector \( u_{2,0}^\delta \) is defined by (5.1) and (5.2) for \( n = 2, q = 0 \) with \( f_{2,0} = g_{2,0} = 0 \). The transmission conditions (5.2) become again the no jump conditions, so that the limit interface \( \Gamma \) is transparent for \( u_{2,0}^\delta \). But, contrarily to the resolution of the limit solution, the function \( u_{2,0}^\delta \) is not in \( H^1 \) close to the corners. Hence, one has to introduce the regular part \( \tilde{u}_{2,0}^\delta \) of \( u_{2,0}^\delta \) by subtracting its stress intensity factor given in (4.7). Similarly as for \( \tilde{u}_{0,1}^\delta \), one sees that the function \( \tilde{u}_{2,0}^\delta \) satisfies the problem

\[
\mathcal{L}^0(\mathbf{x}, \nabla \mathbf{x}) \tilde{u}_{2,0}^\delta = -\sum_{\pm} \frac{\pi \mathcal{L}(S^\pm) \ell^\pm (u_{0,0})(k_0/2)^{2\pi/\Theta}}{\Gamma(\pi/\Theta) \Gamma(\pi/\Theta + 1)} \mathcal{L}^0(\mathbf{x}, \nabla \mathbf{x}), \chi_{\pm}^{\delta} \phi^\pm_2, \quad \text{in } \Omega \setminus \Gamma,
\]

\[
\mathcal{N}^0(\mathbf{x}, \nabla \mathbf{x}) \tilde{u}_{2,0}^\delta = 0, \quad \text{on } \partial \Omega, \quad (5.5)
\]

\[
(B_{\Gamma} \tilde{u}_{2,0}^\delta)(\mathbf{x}) = 0, \quad \text{on } \Gamma.
\]

Note that the numerical effort to compute the corrector \( \tilde{u}_{2,0}^\delta \) is the same as the numerical effort to compute the limit solution \( u_{0,0}^\delta \).

**5.2 Computation of a reference solution and comparison**

We compare the numerically obtained macroscopic solution representation with the exact solution for Example 2.2. For this we compute a reference solution with a mesh that resolves the microstructured interfaces (see Fig. 12a for a thin periodic perforated plate with 4 holes and Fig. 12b for a thin periodic perforated plate with 8 holes).


(a) $\delta = 1/4$

(b) $\delta = 1/8$

Figure 12: Mesh used for the computation of the reference solution, for the porosity $\rho = 0.3$.

(a) $\delta = 1/4$

(b) $\delta = 1/8$

Figure 13: Reference solution, for the porosity $\rho = 0.3$.

**Study of the robustness of the modelling error**

It was already studied in [18] that the error estimate (2.4) holds for $(s, \kappa(s)) = (1, 0)$, $(s, \kappa(s)) = (\frac{2}{3}, 0)$ and $(s, \kappa(s)) = (2, 1)$, with constants $C_s$ depending on the canonical hole domain $\hat{\Omega}_{\text{hole}}$. In particular, $C_s$ depend on the porosity $\rho$ of the thin plate. As it was already shown on Figures 7 and 9, this constant could possibly degenerate as $\rho \to 0$. To study the robustness of the modelling error, let $\delta = 1/8$ and let the height $H$ of the canonical obstacle be obtained by parameter identification (i.e. $H = 0.6$).

On Figure 14, several plots of the $L^2$ macroscopic error (2.4) computed on the domain $\Omega_{0.25}$ are shown for different values of $s$: $(s, \kappa(s)) = (1, 0)$ corresponds to the macroscopic error between the reference solution $u_\delta$ and the limit macroscopic term $u_{0,0}$ (blue solid plot), $(s, \kappa(s)) = (\frac{2}{3}, 0)$ corresponds to the macroscopic error in which we take into account the first order corrector (red dashed plot), and $(s, \kappa(s)) = (2, 1)$ corresponds to the macroscopic error in which we take into account the second order corrector (brown dotted plot). For comparison, the plot of the $L^2$ macroscopic error (2.4) computed with $s$ small and with a limit term obtained from a three scale strategy (i.e. $B_\Gamma v = ([v]_\Gamma - Z \langle \nabla v \cdot n \rangle_\Gamma, \langle \nabla v \cdot n \rangle_\Gamma$) is shown using a green plot with squares. These different error curves show that, if the porosity $\rho$ of the material is not too small, the application of this method with a two scale strategy gives a more and more accurate solution, when increasing the number of considered macroscopic terms in the expansion. *A contrario*, when the $\rho$ is too small, the correctors degrade the obtained error, and it would be more appropriate to consider a three scale strategy.

This manuscript has no data.

Both authors contributed equally in formulating, carrying out and writing up the results of this research. The final version has been approved by both authors for publication.

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Figure 14: Computation of the relative macroscopic error for the problem (P), taking into account more and more terms of the expansion, with respect to the porosity, and comparison with the macroscopic error obtained with a three scale strategy.
A Appendix: definition of the profile function $\psi_1^{\pm}$

In this appendix, we show the construction of the angular function $\psi_1^{\pm}$, that is used in the transmission conditions \([4,3]\) and in the stress intensity factor \([4,0]\) for the example of the Helmholtz problem \([5]\). This function which is derived from the function $f := J_{\psi} (k_{0r}^{\pm}) \cos \frac{m \pi}{\Theta} (\theta^\pm - \Theta_0^\pm)$ can be seen as the first element of a family of functions that can be derived by a general behavior of the form

$$J_{\psi} (k_{0r}^{\pm}) \cos \frac{m \pi}{\Theta} (\theta^\pm - \Theta_0^\pm), \quad m \in \mathbb{N}.$$

The function $\psi_1^{\pm}$ is decomposed as

$$\psi_1^{\pm} (\ln r^{\pm}, \theta^{\pm}) = \sum_{q=0}^{2} \psi_{1,q}^{\pm} (\theta^{\pm}) (\ln r^{\pm})^q, \quad w_{1,1,q,\pm} \in C^\infty (I_1^{\pm}) \cap C^\infty (I_2^{\pm}), \quad (A.1)$$

where $I_1^{\pm} = (a^{\pm}, \gamma^{\pm})$, $I_2^{\pm} = (\gamma^{\pm}, b^{\pm})$ with $a^{\pm} = 0$, $\gamma^{\pm} = \pi$, $b^{\pm} = \Theta$, and $a^{-} = \pi - \Theta$, $\gamma^{-} = 0$, $b^{-} = \pi$. The functions $\psi_{1,q}^{\pm}$, $q = 0, 1, 2$ in \((A.1)\), depend only on the angular variable. Then, $\psi_1^{\pm}$ is constructed such that the function

$$\psi_1^{\pm} (r^{\pm}, \theta^{\pm}) := J_{\psi} (k_{0r}^{\pm}) \psi_1^{\pm} (\ln r^{\pm}, \theta^{\pm})$$

satisfies

$$\begin{aligned}
\Delta \psi_1^{\pm} + k_0^2 \psi_1^{\pm} &= 0 \quad \text{in } K_1^{\pm} \cap K_2^{\pm}, \\
\frac{\partial \psi_1^{\pm}}{\partial r} (r^{\pm}, a^{\pm}) &= 0, \quad r^{\pm} > 0, \\
\frac{\partial \psi_1^{\pm}}{\partial r} (r^{\pm}, b^{\pm}) &= 0, \quad r^{\pm} > 0, \\
[\psi_1^{\pm} (r^{\pm}, \gamma^{\pm})]_{\partial K_1^{\pm} \cap \partial K_2^{\pm}} &= J_{\psi} (k_{0r}^{\pm}) a_{1,1,\pm}, \quad r^{\pm} > 0, \\
[\psi_1^{\pm} (r^{\pm}, \gamma^{\pm})]_{\partial K_1^{\pm} \cap \partial K_2^{\pm}} &= J_{\psi} (k_{0r}^{\pm}) b_{1,1,\pm}, \quad r^{\pm} > 0,
\end{aligned} \quad (A.2)$$

where

$$K_j = \{(r^{\pm} \cos \theta^{\pm}, r^{\pm} \sin \theta^{\pm}) \in K_j^{\pm}, \quad r^{\pm} \in \mathbb{R}^+_j, \theta^{\pm} \in I_j^{\pm}\}, \quad j = \{1, 2\}, \quad (A.3)$$

and,

$$a_{1,1,\pm} = \mp D_0^\pm \frac{\pi}{\Theta} \sin \frac{\pi^2}{\Theta}, \quad (A.4)$$

$$b_{1,1,\pm} = \mp N_2^\pm \frac{\pi}{\Theta} \left( \cos \frac{\pi^2}{\Theta} \mp N_2^\pm \frac{\pi}{\Theta} \right) \cos \frac{\pi^2}{\Theta}. \quad (A.5)$$

In view of \([18]\) Lemma A.1, since $\lambda \Theta = -\Theta$ is not a multiple of $\pi$, $\sin (\pi - \Theta) \neq 0$ the functions $\psi_{1,1}^{\pm}$ and $\psi_{1,2}^{\pm}$ are identically equal to 0. Therefore, $\psi_1^{\pm}$ does not depend on $\ln r^{\pm}$, and there exists two constants $w_{1,1}^{\pm, +}$ and $w_{1,1}^{\pm, -}$ such that

$$\psi_1^{\pm} (\theta^{\pm}) = \begin{cases} w_{1,1}^{\pm, +} \cos \left( \frac{\pi}{\Theta} \right) - 1 \right) (\theta^{\pm} - \Theta_0^{\pm}), & \sin \theta^{\pm} > 0, \\
\psi_1^{\pm, -} \cos \left( \frac{\pi}{\Theta} \right) - 1 \right) (\theta^{\pm} - \Theta_0^{\pm} \mp \Theta), & \sin \theta^{\pm} < 0. \end{cases} \quad (A.6)$$

We insert expression \((A.6)\) in the Dirichlet and Neumann jump conditions, that are the fourth and fifth lines of \((A.2)\), giving, that the jump is the limit value for $\theta^{\pm} > \gamma^{\pm}$ minus the limit value for $\theta^{\pm} < \gamma^{\pm}$ and using that $\gamma^{\pm} - \Theta_0^{\pm} = \pm \pi$:

$$w_{1,1}^{\pm, +} \cos \left( \frac{\pi}{\Theta} \right) - 1 \right) \pi - w_{1,1}^{\pm, -} \cos \left( \frac{\pi}{\Theta} \right) - 1 \right) (\pi - \Theta) = \mp a_{1,1,\pm},$$

$$w_{1,1}^{\pm, +} \sin \left( \frac{\pi}{\Theta} - 1 \right) \pi - w_{1,1}^{\pm, -} \sin \left( \frac{\pi}{\Theta} - 1 \right) (\pi - \Theta) = \frac{b_{1,1,\pm}}{\Theta - 1}. \quad (A.7)$$

Then, the determinant of system \((A.7)\) is just $\sin (\pi - \Theta)$ which is non-zero by the assumption on $\Theta$. Therefore, this system is invertible, and we get

$$w_{1,1}^{\pm, +} = \frac{1}{\sin (\Theta - \pi)} \left( \mp a_{1,1,\pm} \sin \left( \frac{\pi}{\Theta} - 1 \right) \pi \mp \frac{\Theta b_{1,1,\pm}}{\pi - \Theta} \cos \left( \frac{\pi}{\Theta} - 1 \right) (\pi - \Theta) \right),$$

$$w_{1,1}^{\pm, -} = \frac{1}{\sin (\Theta - \pi)} \left( \mp a_{1,1,\pm} \sin \left( \frac{\pi}{\Theta} - 1 \right) \pi \mp \frac{\Theta b_{1,1,\pm}}{\pi - \Theta} \cos \left( \frac{\pi}{\Theta} - 1 \right) (\pi - \Theta) \right). \quad (A.8)$$
References


