HIGH ORDER ASYMPTOTIC EXPANSION FOR THE ACOUSTICS IN VISCOUS GASES CLOSE TO RIGID WALLS

KERSTEN SCHMIDT† and ANASTASIA THÖNS-ZUEVA

Institut für Mathematik, Technische Universität Berlin,
Straße des 17. Juni 135, 10623 Berlin, Germany
†*kersten.schmidt@math.tu-berlin.de*

PATRICK JOLY

Laboratoire POems (UMR 7231 CNRS-INRIA-ENSTA),
ENSTA Paristech – UMA, 828, boulevard des Maréchaux, 91762 Palaiseau Cedex, France

We derive a complete asymptotic expansion for the singularly perturbed problem of acoustic wave propagation inside gases with small viscosity. This derivation is for the non-resonant case in smooth bounded domains in two dimensions. Close to rigid walls the tangential velocity exhibits a boundary layer of size $O(\sqrt{\eta})$ where η is the dynamic viscosity. The asymptotic expansion, which is based on the technique of multiscale expansion is expresed in powers of $\sqrt{\eta}$ and takes into account curvature effects. The terms of the velocity and pressure expansion are defined independently by partial differential equations, where the normal component of velocities or the normal derivative of the pressure, respectively, are prescribed on the boundary. The asymptotic expansion is rigorously justified with optimal error estimates.

Keywords: Acoustic wave propagation, Singularly perturbed PDE, Asymptotic Expansions.

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1. Introduction

In this article, we are investigating the acoustic equations in the framework of Landau and Lifschitz [17] as a perturbation of the Navier-Stokes equations around a stagnant ($\mathbf{U}_0 = 0$) uniform fluid with mean density ρ_0, mean pressure p_0 where heat flux is not taken into account. Such conditions are distinguished in literature as a *quiescent* fluid [10,25]. Similar acoustic equations have been derived and studied in Ref. 10, 25, 17 for no mean flow and in Ref. 2, 22, 10, 9, 23 for the case that a mean flow is present. The aim of this study is to take into account viscous effects in the boundary layer near rigid walls.

We consider time-harmonic acoustic velocity $\mathbf{v}$ and acoustic pressure p (the time regime is $\mathrm{e}^{-\mathrm{i}\omega t}$, $\omega \in \mathbb{R}^+$), which are described by the coupled system

$$-\mathrm{i}\omega\rho_0\mathbf{v} + \nabla p - \eta\Delta\mathbf{v} - \eta'\,\nabla\,\mathrm{div}\,\mathbf{v} = \mathbf{f}, \qquad \text{in } \Omega, \tag{1.1a}$$

$$-\mathrm{i}\omega p + \rho_0 c^2\ \mathrm{div}\,\mathbf{v} = 0, \qquad \text{in } \Omega, \tag{1.1b}$$

$$\mathbf{v} = \mathbf{0}, \qquad \text{on } \partial\Omega. \tag{1.1c}$$

In the *momentum equation* (1.1a) with some known source term $\mathbf{f}$ the viscous dissipation in the momentum is not neglected as we consider near wall regions. Here, $\eta > 0$ is the dynamic viscosity and $\eta' = \frac{1}{3}\eta + \zeta$ with $\zeta \geq 0$ the second (volume) viscosity. Since in this article we are mainly interested in the viscous effects, we neglect non-linear convection. The *continuity equation* (1.1b) relates the acoustic pressure linearly to the divergence of the acoustic velocity, where c is the sound velocity. The system is completed by *no-slip* boundary conditions.

For gases the viscosities η and η' are very small and lead to *viscosity boundary layers* close to walls. The comprehension of these boundary layers makes for the subject of many scientific works [2,11,23,27]. In order to resolve the boundary layers with (quasi-)uniform meshes, the mesh size has to be at the same order, which leads to very large linear systems to be solved. This is especially the case for the very small boundary layers of acoustic waves. A common procedure for singularly perturbed problems with small layers close to boundaries is the method of matched asymptotic expansions [30,13], which matches different ansaetze close to the wall and far away. For various model problems with boundary layers, finite schemes or meshes adapted in special ways close to walls have been proposed, so by Il'in [12], Bakhvalov [3] or Shishkin [28,29], which regain the optimal convergence rate of the numerical schemes; see also the review articles Ref. 18, 14.

Mainly based on experiments, the physics community has introduced slip boundary conditions for tangential velocity components, also known as wall laws, see for example Ref. 11, 23, 24. The boundary layers of incompressible and compressible fluid flows have been addressed by many authors [7,8,16,20]. For acoustic boundary layers on straight (and rough) walls in presence of a shear flow Aurégan et. al. [2] derived effective impedance boundary conditions of first order accuracy with the multiscale expansion (the authors call the method composite expansion). However, there is no mathematical justification of wall laws for acoustic boundary layers in literature. In this work we are going to derive a complete asymptotic expansion using the method of multiscale expansion which we will rigorously justify.

The present paper consists of three parts. In Section 2 we state the problem and present the main results of our work. In Section 3 the complete asymptotic expansion under curvilinear coordinates will be derived, and the far field equations including boundary conditions, as well as the near field equations will be explicitly given up to order 1. The last section (Section 4) is devoted to the justification of the results for the asymptotic analysis. It comprises the proofs for stability and regularity of the solution as well as the error analysis.

2. Formulation of the problem and main results

2.1. *The geometrical setting*

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega$. The boundary shall be described by a mapping $\mathbf{x}_{\partial\Omega}(t)$ from an interval $\Gamma \subset \mathbb{R}$. We assume the boundary to be Lipschitz, which is enough to define a weak solution of (1.1) (or its version for asymptotically small viscosity), and we will indicate whenever we will rely on a C^∞ boundary, which will particularly be needed for the definition of an asymptotic expansion. In the latter case the points close to $\partial\Omega$ can be uniquely written as

$$\mathbf{x}(t,s) = \mathbf{x}_{\partial\Omega}(t) - s\mathbf{n}(t) \tag{2.1}$$

where $\mathbf{n}(t)$ is the outer normalised normal vector and s the distance from the boundary (see Fig. 1). Without loss of generality we can assume $|\mathbf{x}'_{\partial\Omega}(t)| = 1$ for all $t \in \Gamma$. The orthogonal unit vectors in these tangential and normal coordinate directions are $\mathbf{e}_t(t) = -\mathbf{n}^\perp(t)$ and $\mathbf{e}_s(t) = -\mathbf{n}(t)$, where we use the notation $\mathbf{u}^\perp = (u_2, -u_1)^\top$ for a vector rotated clockwise by 90°. Furthermore, let $s_0 \in \mathbb{R}$ such that all points with distance smaller than s_0 to $\partial\Omega$ have a unique closest point on the boundary.

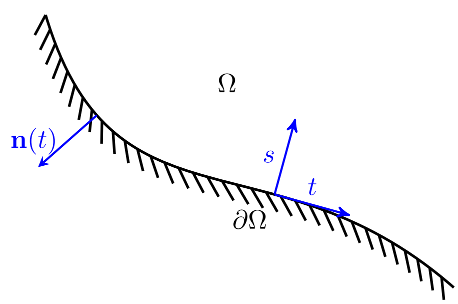

Fig. 1. Definition of a local coordinate system (t, s) close to the wall.

2.2. *The time-harmonic Navier-Stokes equation for viscous gases*

Obviously, the acoustic velocity and pressure can be decoupled

$$\left(1 - \frac{\mathrm{i}(\eta+\eta')\omega}{\rho_0 c^2}\right)\nabla \operatorname{div}\mathbf{v} + \frac{\omega^2}{c^2}\mathbf{v} + \frac{\mathrm{i}\eta\omega}{\rho_0 c^2}\,\mathbf{curl}_{2D}\operatorname{curl}_{2D}\mathbf{v} = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}, \quad \text{in } \Omega \tag{2.2a}$$

$$\mathbf{v} = \mathbf{0}, \quad \text{on } \partial\Omega, \tag{2.2b}$$

$$p = -\frac{\mathrm{i}\rho_0 c^2}{\omega}\operatorname{div}\mathbf{v}, \quad \text{in } \Omega \tag{2.2c}$$

Here, we have used the 2D rotation operators

$$\operatorname{curl}_{2D} \mathbf{u} := \partial_1 u_2 - \partial_2 u_1, \qquad \mathbf{curl}_{2D}\, u := \begin{pmatrix} \partial_2 u \\ -\partial_1 u \end{pmatrix}.$$

The first equation (2.2a) is a ∇ div-Helmholtz equation with absorption terms in the two highest derivatives. The ∇ div-Helmholtz equation, naturally stated in $H(\operatorname{div},\Omega)$, needs only a prescribed normal component of the velocity. The prescribed tangential component is the essential boundary condition for the $\mathbf{curl}_{2D} \operatorname{curl}_{2D}$ operator. As there is a small factor – η is assumed to be small – in front of the $\mathbf{curl}_{2D} \operatorname{curl}_{2D}$ operator, the system is singularly perturbed, *i. e.*, first, its formal limit $\eta \to 0$ does not provide a meaningful solution, and secondly, a boundary layer close to the wall $\partial\Omega$ appears. Since only the $\mathbf{curl}_{2D} \operatorname{curl}_{2D}$ operator has a small factor, only the tangential velocity component exhibits a boundary layer, whose size is of the order

$$\delta = \sqrt{\frac{\eta}{\rho_0 \omega}}. \tag{2.3}$$

This observation will be justified later on in this article by asymptotic expansion. In case of non-smooth $\operatorname{curl}_{2D} \mathbf{f}$ there appear also (internal) boundary layers at discontinuities of $\operatorname{curl}_{2D} \mathbf{f}$ or its higher derivatives. To exclude those, we assume $\operatorname{curl}_{2D} \mathbf{f} \in H^m(\Omega)$ for any $m \in \mathbb{N}_0$.

2.3. *The equations for asymptotically small viscosity*

To investigate the solution of (1.1) for small viscosities, we introduce a small parameter $\varepsilon \in \mathbb{R}^+$ and replace η, η' by $\varepsilon^2\eta_0$, $\varepsilon^2\eta_0'$ with $\eta_0, \eta_0' \in \mathbb{R}^+$. In this way the boundary layer width will become proportional to ε. We will label the solution of (1.1) or (2.2), respectively, $\mathbf{v}^\varepsilon$ and p^ε due to its dependence on ε, which satisfy

$$-\mathrm{i}\omega\rho_0\mathbf{v}^\varepsilon + \nabla p^\varepsilon - \varepsilon^2\eta_0\Delta\mathbf{v}^\varepsilon - \varepsilon^2\eta_0'\nabla \operatorname{div} \mathbf{v}^\varepsilon = \mathbf{f}, \qquad \text{in } \Omega, \tag{2.4a}$$

$$-\mathrm{i}\omega p^\varepsilon + \rho_0 c^2 \operatorname{div} \mathbf{v}^\varepsilon = 0, \qquad \text{in } \Omega, \tag{2.4b}$$

$$\mathbf{v}^\varepsilon = \mathbf{0}, \qquad \text{on } \partial\Omega. \tag{2.4c}$$

In this study we consider the non-resonant case, *i. e.*, for vanishing viscosity, and so absorption the kernel of the system is empty – there is no eigensolution. The eigenvalues of the limit problem coincide with the Neumann eigenvalues of $-\Delta$, a fact that we will address in the proof – which will be given later in Sec. 4.3 – of the following lemma.

Lemma 2.1 (Stability for the non-resonant case). *For any* $\mathbf{f} \in (H_0(\operatorname{div},\Omega) \cap H(\operatorname{curl}_{2D},\Omega))'$ *the system* (2.4) *has a unique solution* $(\mathbf{v}^\varepsilon, p^\varepsilon) \in H_0(\operatorname{div},\Omega) \cap H(\operatorname{curl}_{2D},\Omega) \times L^2(\Omega)$. *If* $\frac{\omega^2}{c^2}$ *is not a Neumann eigenvalue of* $-\Delta$, *then there exists* *a constant* $C > 0$ *independent of* ε *such that*

$$\|\mathbf{v}^\varepsilon\|_{H(\mathrm{div},\Omega)} + \varepsilon\,\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon\|_{L^2(\Omega)} + \|p^\varepsilon\|_{L^2(\Omega)} \le C\,\|\mathbf{f}\|_{(H_0(\mathrm{div},\Omega)\cap H(\mathrm{curl}_{2D},\Omega))'}, \tag{2.5a}$$

$$\|\nabla p^\varepsilon\|_{L^2(\Omega)} \le C\,\|\mathbf{f}\|_{L^2(\Omega)}. \tag{2.5b}$$

For any $\omega > 0$ *and for* C^∞ *boundary* $\partial\Omega$ *it holds*

$$\varepsilon\|\mathbf{v}^\varepsilon\|_{(H^1(\Omega))^2} \le C\,\|\mathbf{f}\|_{(H_0(\mathrm{div},\Omega)\cap H(\mathrm{curl}_{2D},\Omega))'}. \tag{2.5c}$$

The proof will be given later in Sec. 4.3. See Sec. 4.1 for the definition of the used Sobolev spaces.

2.4. *Asymptotic expansion*

The solution $\mathbf{v}^\varepsilon$, p^ε should be approximated by a two-scale asymptotic expansion in the framework of Vishik and Lyusternik [31] in the form

$$\mathbf{v}^\varepsilon \approx \mathbf{v}^{\varepsilon,N}(\mathbf{x}) := \sum_{j=0}^{N} \varepsilon^j \left(\mathbf{v}^j(\mathbf{x}) + \mathbf{v}^j_{\mathrm{BL},\varepsilon}(\mathbf{x})\right), \tag{2.6a}$$

$$p^\varepsilon \approx p^{\varepsilon,N}(\mathbf{x}) := \sum_{j=0}^{N} \varepsilon^j \left(p^j(\mathbf{x}) + p^j_{\mathrm{BL},\varepsilon}(\mathbf{x})\right), \tag{2.6b}$$

where $\mathbf{v}^j$ and p^j are the *far field* and $\mathbf{v}^j_{\mathrm{BL},\varepsilon}$ and $p^j_{\mathrm{BL},\varepsilon}$ are the *near field* velocity and pressure. The subscript $\cdot_{\mathrm{BL},\varepsilon}$ stands for "boundary layer" expressing the nature of the near field terms, which in fact depend on ε.

Lemma 2.2 (Asymptotic exactness of the two-scale asymptotics). *If* $\frac{\omega^2}{c^2}$ *is not a Neumann eigenvalue of* $-\Delta$ *and* $\partial\Omega$ *is* C^∞ *then there exist functions* $\mathbf{v}^j \in (H^1_0(\Omega))^2$, $p^j \in L^2(\Omega)$, $\mathbf{v}^j_{\mathrm{BL},\varepsilon} \in (H^1_0(\Omega))^2$ *such that for any* $N \in \mathbb{N}_0$ *the approximate solution* $\mathbf{v}^{\varepsilon,N}$, $p^{\varepsilon,N}$ *defined by* (2.6a) *with* $p^j_{\mathrm{BL},\varepsilon} \equiv 0$ *for any* $j \in \mathbb{N}_0$ *satisfies*

$$\|\mathbf{v}^\varepsilon - \mathbf{v}^{\varepsilon,N}\|_{H(\mathrm{div},\Omega)} + \sqrt{\varepsilon}\|\operatorname{curl}_{2D}(\mathbf{v}^\varepsilon - \mathbf{v}^{\varepsilon,N})\|_{L^2(\Omega)} + \|p^\varepsilon - p^{\varepsilon,N}\|_{H^1(\Omega)} \le C\,\varepsilon^{N+1}, \tag{2.7}$$

where the constant $C > 0$ *does not depend on* ε. *Furthermore, for any* $\delta > 0$ *there is a constant* $C_\delta > 0$ *independent of* ε *such that outside a* δ*-neighbourhood* Ω_δ *of* $\partial\Omega$ *holds*

$$\|\mathbf{v}^\varepsilon - \sum\nolimits_{j=0}^{N} \varepsilon^j \mathbf{v}^j\|_{(H^1(\Omega\setminus\overline{\Omega}_\delta))^2} \le C_\delta\,\varepsilon^{N+1}. \tag{2.8}$$

The far field velocity and pressure terms $\mathbf{v}^j$ and p^j in Lemma 2.2 can not be defined uniquely unless we do not assume them to be independent of ε. We will give a unique definition of the ε-independent terms in Sec. 2.4.1 or Sec. 2.4.2, respectively. The near field velocity terms $\mathbf{v}^j_{\mathrm{BL},\varepsilon}$ depend on ε and are in general not unique. We will give a (unique) choice in Sec. 3.2. The proof of Lemma 2.2 will be given in Sec. 4.5.

Remark 2.1. The estimate (2.8) shows that the far field velocity taken alone, *i. e.*, with correction by the near field velocity, is an optimal approximation for any $N \in \mathbb{N}_0$ when measuring the error only in some distance from the boundary. The pressure approximation does not include a correcting near field close to the wall ($p^j_{\mathrm{BL},\varepsilon} \equiv 0$), hence the far field pressure $p^{\varepsilon,N} = \sum_{j=0}^{N} \varepsilon^j p^j$ is according to (2.7) an optimal approximation, even up to the boundary. Note, that $\mathrm{curl}_{2D}\, \mathbf{v}^0$ is even accurate up to $O(\varepsilon^2)$ in some distance from the wall as $\mathrm{curl}_{2D}\, \mathbf{v}^1 \equiv 0$, which we observe after applying curl_{2D} to (2.9a) below.

The far field terms of velocity and pressure can equivalently be defined by solving a PDE for the far field velocity, where the far field pressure follows by an explicit equation, or by solving an Helmholtz equation for the far field pressure, where an explicit equation fixes the far field velocity. We prefer the first characterisation, which will be given in Sec. 2.4.1 for the analysis of the modelling error. The second characterisation is easier for the numerical computation of the far field pressure and associated impedance boundary conditions, which we will address in a forthcoming article.

2.4.1. *First characterisation of the far field terms*

Lemma 2.2 holds with a particular family of far field terms which is intrinsic to the problem. They will be derived later on in this article as solutions to the partial differential equations

$$\nabla \operatorname{div} \mathbf{v}^j + \frac{\omega^2}{c^2}\mathbf{v}^j = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}\cdot\delta_{j=0} + \frac{\mathrm{i}\eta_0\omega}{\rho_0 c^2}\Delta\mathbf{v}^{j-2} + \frac{\mathrm{i}\eta_0'\omega}{\rho_0 c^2}\nabla \operatorname{div} \mathbf{v}^{j-2}, \quad \text{in } \Omega, \tag{2.9a}$$

$$\mathbf{v}^j \cdot \mathbf{n} = \sum_{\ell=1}^{j} G_\ell \operatorname{div} \mathbf{v}^{j-\ell} + H_j(\mathbf{f}), \qquad \text{on } \partial\Omega, \tag{2.9b}$$

where $\mathbf{v}^{-1} = \mathbf{v}^{-2} = \mathbf{0}$, G_ℓ and H_ℓ are tangential differential operators acting on terms of lower orders or the trace of $\mathbf{f}$ on $\partial\Omega$, respectively. Furthermore, $\delta_{j=0}$ stands for the Kronecker symbol which is 1 if $j = 0$ and 0 otherwise. The reader shall be easily convinced, that in contrast to the PDEs (2.9a) the boundary conditions (2.9b) will be derived in several steps, where the main ingredient is the solution of ordinary differential equation in normalised coordinates. This will be detailed in Sec. 3 and in the Appendix the operators G_ℓ and H_ℓ for general orders will be iteratively derived, see Cor. A.2.

Let us state now the boundary conditions (2.9b) up to $j = 2$, which are given by

$$
\begin{aligned}
\mathbf{v}^0\cdot\mathbf{n} &= \quad 0,\\
\mathbf{v}^1\cdot\mathbf{n} &= (1+\mathrm{i})\sqrt{\frac{\eta_0}{2\omega\rho_0}}\Big(\frac{c^2}{\omega^2}\partial_t^2\operatorname{div}\mathbf{v}^0+\frac{\mathrm{i}}{\omega\rho_0}\partial_t(\mathbf{f}\cdot\mathbf{n}^\perp)\Big),\\
\mathbf{v}^2\cdot\mathbf{n} &= \frac{c^2}{\omega^2}\Big((1+\mathrm{i})\sqrt{\frac{\eta_0}{2\omega\rho_0}}\partial_t^2\operatorname{div}\mathbf{v}^1+\frac{\mathrm{i}\eta_0}{2\omega\rho_0}\partial_t(\kappa\partial_t\operatorname{div}\mathbf{v}^0)\Big)-\frac{\eta_0}{2\omega^2\rho_0^2}\partial_t(\kappa\,\mathbf{f}\cdot\mathbf{n}^\perp).
\end{aligned}
\tag{2.10}
$$

Applying recursively curl_{2D} to (2.9a) for $j=0,1,\dots$ we get expressions for $\operatorname{curl}_{2D}\mathbf{v}^j$ by the source term only

$$
\operatorname{curl}_{2D}\mathbf{v}^j=\begin{cases}0 & j \text{ is odd},\\ \frac{\mathrm{i}}{\omega\rho_0}\Big(-\frac{\mathrm{i}\eta_0}{\omega\rho_0}\operatorname{curl}_{2D}\mathbf{curl}_{2D}\Big)^{j/2}\operatorname{curl}_{2D}\mathbf{f}, & j\text{ is even},\end{cases}\qquad \text{in } \Omega. \tag{2.11}
$$

These terms are well-defined by the regularity assumption on $\operatorname{curl}_{2D}\mathbf{f}$ in Sec. 2.2.

When the far field velocity is computed we may obtain *a-posteriori* the far field pressure by

$$
p^j=-\frac{\mathrm{i}\rho_0c^2}{\omega}\operatorname{div}\mathbf{v}^j,\qquad \text{in } \Omega. \tag{2.12}
$$

The far field terms $\mathbf{v}^j$, p^j are well-defined as stated in the following:

Lemma 2.3 (Existence and uniqueness of $\mathbf{v}^j$, p^j). *Let $\partial\Omega$ be C^∞ and for any $m\in\mathbb{N}_0$ the right hand side of* (2.9a) $\mathbf{f}\in(H_0(\operatorname{div},\Omega))'\cap(H^m(\widetilde{\Omega}))^2$ *in some neighbourhood $\widetilde{\Omega}\subset\Omega$ of $\partial\Omega$, i. e., $\partial\Omega\subset\partial\widetilde{\Omega}$, and $\operatorname{curl}_{2D}\mathbf{f}\in H^m(\Omega)$. Then* (2.9) *provides a unique solution $\mathbf{v}^j\in(H^1(\Omega))^2$ and* (2.12) *a unique function $p^j\in L^2(\Omega)$ for any integer $j\geq 0$.*

The proof will be given in Sec. 4.4.

Remark 2.2. In fact the velocity and the pressure need higher regularity close to the wall, such that the differential operators can be applied. This is assured by the assumption of C^∞ boundary and of more regular source term $\mathbf{f}$.

2.4.2. *ksAn alternative characterisation of the far field terms*

It is easy to verify that the far field pressure terms p^j defined by (2.9) solve the Helmholtz equation

$$
\Delta p^j+\frac{\omega^2}{c^2}p^j=\operatorname{div}\mathbf{f}\cdot\delta_{j=0}+(\eta_0+\eta_0')\frac{\mathrm{i}\omega}{\rho_0c^2}\Delta p^{j-2}, \tag{2.13a}
$$

which has to be completed by boundary conditions

$$
\nabla p^j\cdot\mathbf{n}=\sum_{\ell=1}^{j}J_\ell p^{j-\ell}+K_j(\mathbf{f}), \tag{2.13b}
$$

which will be derived to any order in the Appendix. Here, J_ℓ and K_ℓ are tangential differential operators acting on pressure terms of lower orders or the source term, respectively.

Let us introduce the boundary conditions (2.13b) up to $j = 2$ which are given by

$$\nabla p^0 \cdot \mathbf{n} = \quad \mathbf{f} \cdot \mathbf{n},$$

$$\nabla p^1 \cdot \mathbf{n} = -(1+\mathrm{i})\sqrt{\frac{\eta_0}{2\omega\rho_0}}\big(\partial_t^2 p^0 + \partial_t(\mathbf{f} \cdot \mathbf{n}^\perp)\big), \tag{2.14}$$

$$\nabla p^2 \cdot \mathbf{n} = -(1+\mathrm{i})\sqrt{\frac{\eta_0}{2\omega\rho_0}}\partial_t^2 p^1 - \frac{\mathrm{i}\eta_0}{2\omega\rho_0}\left(\partial_t(\kappa\partial_t p^0) + \partial_t(\kappa\mathbf{f} \cdot \mathbf{n}^\perp)\right) \tag{2.15}$$
$$+ \frac{\mathrm{i}\omega(\eta_0+\eta_0')}{\rho_0 c^2}\mathbf{f} \cdot \mathbf{n} - \frac{\mathrm{i}\eta_0}{\omega\rho_0}\,\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{f} \cdot \mathbf{n}.$$

When the far field pressure is computed we may obtain *a-posteriori* the far field velocity by

$$\mathbf{v}^j = \frac{\mathrm{i}}{\rho_0\omega}(\mathbf{f} \cdot \delta_{j=0} - \nabla p^j) - \frac{\eta_0+\eta_0'}{\rho_0^2 c^2}\nabla p^{j-2} - \frac{\mathrm{i}\eta_0}{\rho_0\omega}\,\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{v}^{j-2}, \quad \text{in } \Omega, \tag{2.16}$$

where $\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{v}^{j-2}$ is given by (2.11) as expression of $\mathbf{f}$.

In this way the far field terms $\mathbf{v}^j$, p^j are well-defined as well, where a higher regularity of $\mathbf{f}$ has to be assumed.

Lemma 2.4 (Existence and uniqueness of $\mathbf{v}^j$, p^j). *Let $\partial\Omega$ be C^∞ and for any $m \in \mathbb{N}_0$ the right hand side of* (2.9a) $\mathbf{f} \in (L^2(\Omega))^2 \cap (H^m(\widetilde{\Omega}))^2$ *be in some neighbourhood $\widetilde{\Omega} \subset \Omega$ of $\partial\Omega$, i. e., $\partial\Omega \subset \partial\widetilde{\Omega}$, and* $\mathrm{curl}_{2D}\,\mathbf{f} \in H^m(\Omega)$. *Then,* (2.13) *provide a unique solution $p^j \in H^1(\Omega)$ and* (2.16) *a unique function $\mathbf{v}^j \in (H^1(\Omega))^2$ for any integer $j \geq 0$.*

The proof uses elements of the proof of Lemma 2.3 and will be let to the reader.

Remark 2.3. Note, that the boundary conditions for $\mathbf{v}^j$ and p^j are local since G_ℓ and J_ℓ in (2.9b) and (2.13b) are differential operators on Γ.

3. Formal asymptotic expansion

3.1. *Decomposition into far and near fields*

In (2.6a) we have introduced the two-scale expansion ansatz, which expresses an approximation to the exact solution as a two-fold *decomposition*, first

- into far field terms, which model the macroscopic picture of the solution,
- which are corrected in the neighbourhood of the boundary by near field terms,

and second

- into terms of different order of magnitude, measured in terms of the small parameter ε.

The far field terms will be defined in physical coordinates in the whole domain Ω where we assume $\partial\Omega$ to be C^∞. Inserting the expansion (2.6a) into the system (2.4) for $\mathbf{v}^\varepsilon$, p^ε for a particular coordinate $\mathbf{x} \in \Omega$ and letting ε tend to zero, the near field terms concentrate closer and closer to the wall and vanish – even their higher derivatives – on $\mathbf{x}$. Collecting terms of the same order in ε the *far field equations* results, which can be written as (2.9a), and (2.12) follows. The far field equations will be completed by boundary conditions, which we will specify in Sec. 3.3. As a matter of fact the far field expansion can only fulfil one of the two boundary conditions and has to be corrected by the near field expansion close to the walls.

To separate the two scales we use the technique of multiscale expansion, which defines the near field terms in a local normalised coordinate system (2.1) such that they decay away from the wall and are set to zero there where the local coordinate system is not defined. The decay property requires the near field terms and their higher derivatives to vanish on a fixed point $\mathbf{x}$ for ε tending to 0, or more precisely, that there exists for any $\mathbf{x} \in \Omega$ some $\alpha > 0$ such that for any differential operator $D^i_{\boldsymbol{\beta}}$ of degree $i = 0, 1, 2$ in the direction $\boldsymbol{\beta} \in \{1, 2\}^i$ it holds

$$\lim_{\varepsilon \to 0} \mathrm{e}^{\alpha s(\mathbf{x})/\varepsilon} D^i_{\boldsymbol{\beta}} \begin{pmatrix} \mathbf{v}^j_{\mathrm{BL},\varepsilon}(\mathbf{x}) \\ q^j_{\mathrm{BL},\varepsilon}(\mathbf{x}) \end{pmatrix} = \mathbf{0}. \tag{3.1}$$

3.2. *Deriving the near field equations*

The near field terms are defined in two steps, first inside an s_1-neighbourhood of the boundary for some $s_1 < s_0$, not depending on ε. Then, they will be continuously extended into the subdomain $s_1 < s < s_0$ such that they and their derivatives vanish at $s = s_0$.

We begin by the definition for $s < s_1$. The standard way would be to take the asymptotic expansion ansatz (2.6a) as "educated guess" with near field terms $\mathbf{v}^j_{\mathrm{BL},\varepsilon} = \mathbf{u}^j(t, \frac{s}{\varepsilon})$ and $p^j_{\mathrm{BL},\varepsilon} = q^j(t, \frac{s}{\varepsilon})$ for some functions $\mathbf{u}^j(t, S)$, $q^j(t, S)$ not depending on ε. The terms are then chosen such that the ansatz solves the system (2.4) with zero source term for any order in ε.

However, we are going to use the special structure of (2.4) to show that

- the near field pressure terms $p^j_{\mathrm{BL},\varepsilon}$ vanish at any order and, hence, the near field velocity terms are divergence free by (2.4b), and so
- the near field velocity can be modelled as

$$\mathbf{v}^j_{\mathrm{BL},\varepsilon} := \varepsilon\, \mathbf{curl}_{2D}\, \phi^j(t, \tfrac{s}{\varepsilon}) \tag{3.2}$$

  where ϕ^j shall not depend on ε. We take here the weighted operator $\varepsilon\, \mathbf{curl}_{2D}$ as it is of order 1 in ε.

Absence of near field pressure Applying the divergence to (2.4a), inserting (2.4b) and using the operator identity $\operatorname{div}\nabla\operatorname{div} = \operatorname{div}\Delta$ we get a Helmholtz equation for the pressure

$$\Big(1 - \varepsilon^2(\eta_0 + \eta_0')\frac{\mathrm{i}\omega}{\rho_0 c^2}\Big)\Delta p^\varepsilon + \frac{\omega^2}{c^2}p^\varepsilon = \operatorname{div}\mathbf{f}. \tag{3.3}$$

Note, that the far field pressure $\sum_{j=0}^N \varepsilon^j p^j$, where p^j are defined by (2.12) or equivalently by (2.13a), solves (3.3) with source term $\operatorname{div}\mathbf{f}$ up to a residual of order ε^{N+2}. Hence, the correcting near field $\sum_{j=0}^N \varepsilon^j p^j_{\mathrm{BL},\varepsilon}$ has to solve (3.3) with zero source term as best as possible in terms of orders in ε. Inserting the near field pressure with the ansatz $p^j_{\mathrm{BL},\varepsilon} = q^j(t, \frac{s}{\varepsilon})$ we obtain for order 0

$$\partial_S^2 q^0(t,S) = 0 \qquad\qquad \Rightarrow \quad q^0 \equiv 0 \tag{3.4a}$$

by the decay condition (3.1). For any higher order j we have the relation

$$\partial_S^2 q^j(t,S) = A_\varepsilon^3 \sum_{\ell=1}^{5} F_\ell(q^{j-\ell}) \tag{3.4b}$$

for some differential operators F_ℓ in t, S, and the near field pressure terms vanish by induction in j. The only solution to this equation, which is decaying exponentially as assumed in (3.1) is $p^j_{\mathrm{BL},\varepsilon} = q^j \equiv 0$ — there is no near field pressure. The equations (3.4) are very different from usual near field equations, as not any of the two functions in the kernel of the second order differential operator ∂_S^2 fulfil the decay property (3.1).

The absence of the near field pressure has in view of (2.4b) the consequence that the near field velocity is divergence free, *i. e.*, $\operatorname{div}\mathbf{v}^j_{\mathrm{BL},\varepsilon} = 0$ for all j, and we can express $\mathbf{v}^j_{\mathrm{BL},\varepsilon}$ by (3.2).

Equations for the divergence-free near field velocity In the following we will derive conditions on the near field functions ϕ^j, such that the near field velocity expansion $\sum_{j=0}^N \varepsilon^{j+1}\,\mathbf{curl}_{2D}\,\phi^j(t,\frac{s}{\varepsilon})$ inserted into (2.4a) leaves a residual as small as possible in powers of ε. For a divergence free velocity $\mathbf{v} = \varepsilon\,\mathbf{curl}_{2D}\,\phi$ and vanishing pressure the residual is simply given by

$$\begin{aligned} R^\varepsilon(\mathbf{v}, p = 0) &:= -\mathrm{i}\omega\rho_0\mathbf{v} + \varepsilon^2\eta_0\,\mathbf{curl}_{2D}\operatorname{curl}_{2D}\mathbf{v} \\ &= \varepsilon\,\mathbf{curl}_{2D}(-\mathrm{i}\omega\rho_0\phi + \varepsilon^2\eta_0\operatorname{curl}_{2D}\mathbf{curl}_{2D}\,\phi). \end{aligned}$$

Since for a function $\phi(t,S)$ with $S = \frac{s}{\varepsilon}$ we have

$$\operatorname{curl}_{2D}\mathbf{curl}_{2D}\,\phi = -\varepsilon^{-2}\partial_S^2\phi + \varepsilon^{-1}\kappa A_\varepsilon\partial_S\phi + A_\varepsilon^2\partial_t^2\phi + \varepsilon\kappa' S A_\varepsilon^3\partial_t\phi,$$

where κ is the curvature on the boundary $\partial\Omega$ given by

$$\kappa(t) := \frac{x_1'(t)x_2''(t) - x_2'(t)x_1''(t)}{(x_1'(t)^2 + x_2'(t)^2)^{3/2}}, \qquad \text{and} \qquad A_\varepsilon = A_\varepsilon(t,S) = \frac{1}{1 - \varepsilon\kappa(t)S},$$

we can write the residual R^ε as

$$R^\varepsilon(\varepsilon\,\mathbf{curl}_{2D}\,\phi(t,\tfrac{s}{\varepsilon}),p=0) = \varepsilon\,\mathbf{curl}_{2D}\,A_\varepsilon^3\Big(\mathrm{i}\omega\rho_0\phi + \eta_0\partial_S^2\phi - \sum_{\ell=1}^{3}\varepsilon^\ell C_\ell(\phi)\Big), \quad (3.5)$$

where we use the functions

$$\begin{aligned}
C_1(\phi) &= \kappa\big(3\,\mathrm{i}\omega\rho_0 S + 3\eta_0 S\partial_S^2 + \eta_0\partial_S\big)\phi,\\
C_2(\phi) &= -\eta_0\partial_t^2\phi - \kappa^2\big(3\,\mathrm{i}\omega\rho_0 S^2 + 3\eta_0 S^2\partial_S^2 + 2\eta_0 S\partial_S\big)\phi,\\
C_3(\phi) &= \big(\mathrm{i}\omega\rho_0\kappa^3 S^3 + \eta_0\kappa^3 S^3\partial_S^2 + \eta_0\kappa^3 S^2\partial_S + \eta_0\kappa S\partial_t^2 - \eta_0\kappa' S\partial_t\big)\phi.
\end{aligned}$$

Now, inserting the near field velocity expansion into (3.5) we get

$$\begin{aligned}
R^\varepsilon(\sum_{j=0}^{N}\varepsilon^j\mathbf{v}^j_{\mathrm{BL},\varepsilon},0) &= \varepsilon\,\mathbf{curl}_{2D}\,A_\varepsilon^3\sum_{j=0}^{N}\varepsilon^j\Big(\mathrm{i}\omega\rho_0\phi^j + \eta_0\partial_S^2\phi^j - \sum_{\ell=1}^{3}\varepsilon^\ell C_\ell(\phi^j)\Big) =\\
&= \varepsilon\,\mathbf{curl}_{2D}\,A_\varepsilon^3\Bigg(\sum_{j=0}^{N}\varepsilon^j\Big(\mathrm{i}\omega\rho_0\phi^j + \eta_0\partial_S^2\phi^j - \sum_{\ell=1}^{3}C_\ell(\phi^{j-\ell})\Big)\\
&\qquad - \varepsilon^{N+1}\sum_{j=0}^{2}\sum_{\ell=1+j}^{3}\varepsilon^{\ell-1-j}C_\ell(\phi^{N-j})\Bigg)
\end{aligned}$$

Since $\varepsilon\,\mathbf{curl}_{2D}$ is an operator of order 1, the residuals are all (at least) of order ε^{N+1} if the terms of the expansions ϕ^j, $j=0,\dots,N$ satisfy the near field equation

$$\mathrm{i}\omega\rho_0\phi^j + \eta_0\partial_S^2\phi^j = \sum_{\ell=1}^{3}C_\ell(\phi^{j-\ell}). \quad (3.6a)$$

The second differential operator $\mathrm{i}\omega + \eta_0\partial_S^2$ has a kernel of dimension one if we allow only for exponentially decreasing functions by demanding

$$\lim_{S\to\infty}\phi^j(t,S)\,\mathrm{e}^{\alpha S} = 0, \quad (3.6b)$$

which is equivalent to (3.1). The single function in the kernel is fixed by the homogeneous Dirichlet boundary condition for the tangential component $(\mathbf{v}^j_{\mathrm{BL},\varepsilon}(t,0) + \mathbf{v}^j(t,0))\cdot\mathbf{e}_t(t) = 0$, see (2.4c), where the terms of same order in ε are collected. This condition turns out to be an inhomogeneous Neumann boundary condition for ϕ^j

$$\partial_S\phi^j(t,0) = -v_t^j(t) := -\mathbf{v}^j(t,0)\cdot\mathbf{e}_t(t). \quad (3.6c)$$

Definition of the near field velocity into the whole domain The terms $\phi^j(t,S)$ are defined for any $S\in\mathbb{R}^+$, but are only used for $0\le S\le\frac{s_0}{\varepsilon}$ to define $\mathbf{v}^j_{\mathrm{BL},\varepsilon}$ in the s_0-neighbourhood of the boundary. We define the near field velocity in the whole domain by

$$\mathbf{v}^j_{\mathrm{BL},\varepsilon}(\mathbf{x}) := \varepsilon\,\mathbf{curl}_{2D}\left(\phi^j(t,\tfrac{s}{\varepsilon})\chi(\mathbf{x})\right),$$

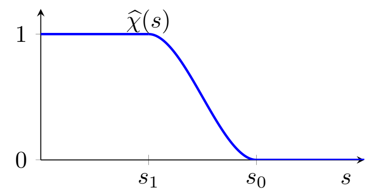

Fig. 2. An example of a cut-off function $\chi(\mathbf{x}) = \widehat{\chi}(s)$, which is used to add smoothly truncated near fields at $O(1)$ distance to the wall to the far fields.

where the cut-off function $\chi(\mathbf{x}) \in C^2(\Omega)$ (see Fig. 2) is zero for distances larger s_0 and a function of the distance to the boundary $\chi(\mathbf{x}) = \widehat{\chi}(s(\mathbf{x}))$ otherwise. In this form the near field velocity terms are divergence-free in the whole domain and fulfil (with non-existing near field pressure) (2.4b) exactly.

Remark 3.1. Note, that the family of functions ϕ^j is intrinsic and thus essential to achieving the approximation result (2.7) of $\mathbf{v}^{\varepsilon,N}$ in the form (2.6a) to $\mathbf{v}^\varepsilon$. In contrast, the cut-off function χ is arbitrary and (2.7) holds for any of those cut-off functions. Replacing cut-off functions leads to an exponentially small change of the solution.

The near field term of order 0 It is easy to see that for $j = 0$ the unique solution of (3.6) is given by

$$\phi^0(t,S) = \frac{1}{\lambda_0}\, v_t^0(t)\, \mathrm{e}^{-\lambda_0 S}, \quad \text{where } \lambda_0 = (1-\mathrm{i})\sqrt{\frac{\omega\rho_0}{2\eta_0}}. \tag{3.7}$$

This is the dominating boundary layer term close to the wall and with $\eta = \eta_0\varepsilon^2$ and $s = S/\varepsilon$ we observe the boundary layer thickness to be of order δ as given in (2.3).

The near field term of order 1 The unique solution of (3.6) for $j = 1$ is given by

$$\phi^1(t,S) = \frac{1}{\lambda_0}\left(v_t^1(t) + \frac{\kappa(1+\lambda_0 S)}{2\lambda_0}\, v_t^0(t)\right)\mathrm{e}^{-\lambda_0 S}. \tag{3.8}$$

Existence and uniqueness of the near field terms We have observed that the near field terms up to order 1 are polynomials in S multiplied with a function, which is exponentially decreasing in S. With the near field terms of lower order on the right hand sides of the near field equations (3.6), which is a simple example of a linear second-order ODE with constant coefficient, the terms of higher order have the same form as well (see the Appendix for more details). Here, the tangential variable t is a parameter which is easily transported by the equations. Let us introduce the function space

$$\Pi(\lambda, X) := \Big\{ u(t,S) = \mathrm{e}^{-\lambda S} \sum_{j=0}^{J} a_j(t) S^j, \text{ for some } J \in \mathbb{N}_0, \\ a_j \in \mathbb{C}, \|a_j\|_X < \infty, j = 0, \dots, J \Big\},$$

where X is related to the smoothness in tangential direction. For functions depending on ε we call them member of $\Pi(\lambda, X)$ if the coefficients a_j are bounded independently of ε.

The only data to the system of near field equations (3.6) are the tangential traces v_t^j of the far field terms on the boundary. So, the near field terms ϕ^j exist only in some Sobolev space $H^s(\Omega)$, $s \geq 1$, if the tangential traces of the far field velocities are smooth enough.

Lemma 3.1 (Existence and uniqueness of the near field equations). *Let for some non-negative integer N $v_t^j \in H^{s-j+N}(\Gamma)$ for $j = 0, \dots, N$ and some $s \in \mathbb{R}$. Then, (3.6) for all $j \leq N$ has a unique solution in the form $\phi^j \in \Pi(\lambda_0, H^s(\Gamma))$. Hence, $\varepsilon\, \mathbf{curl}_{2D}\, \phi^j \in \Pi(\lambda_0, H^{s-1}(\Gamma))^2$.*

3.3. *Far field boundary conditions*

Boundary conditions for the far field velocity The normal trace $\mathbf{v}^{\varepsilon,N} \cdot \mathbf{n}$ vanishes up to order N in ε, *cf.* (2.4c), the normal trace of the far field terms has to fulfil condition

$$\mathbf{v}^j(t,0) \cdot \mathbf{n} = \mathbf{v}^{j-1}_{\mathrm{BL},\varepsilon}(t,0) \cdot \mathbf{e}_s(t) = \partial_t \phi^{j-1}(t,0), \tag{3.9}$$

which can be expressed in terms of v_t^0, v_t^1, … . Up to order 2 this is

$$\begin{aligned} &\mathbf{v}^0(t,0) \cdot \mathbf{n} = 0, \qquad \mathbf{v}^1(t,0) \cdot \mathbf{n} = -\frac{1}{\lambda_0}\, \partial_t v_t^0(t), \\ &\mathbf{v}^2(t,0) \cdot \mathbf{n} = -\frac{1}{\lambda_0}\Big(\partial_t v_t^1(t) + \frac{1}{2\lambda_0}\partial_t(\kappa v_t^0(t))\Big). \end{aligned} \tag{3.10}$$

We can express these terms, if they are smooth enough, in terms of their natural Neumann traces which is $\operatorname{div} \mathbf{v}^0$, $\operatorname{div} \mathbf{v}^1$, …. Using (2.9a) and (2.11) we can express

$$v_t^j(t) = -\frac{c^2}{\omega^2}\partial_t \operatorname{div} \mathbf{v}^j + \frac{\mathrm{i}(\eta_0 + \eta_0')}{\omega\rho_0}\partial_t \operatorname{div} \mathbf{v}^{j-2}(t,0) \\ + \frac{\mathrm{i}}{\omega\rho_0}\left(-\frac{\mathrm{i}\eta_0}{\omega\rho_0}\, \mathbf{curl}_{2D} \operatorname{curl}_{2D}\right)^{j/2} \mathbf{f} \cdot \mathbf{e}_t \cdot \delta_{j \text{ is even}}. \tag{3.11}$$

Inserting these expressions for v_t^j into (3.10) results in (2.10) and will also be used for the derivation of the boundary conditions (2.9b) to any order in the Appendix.

Boundary conditions for the far field pressure Applying div to (2.9a), inserting (2.12) and using the fact that $\operatorname{div} \mathbf{curl}_{2D} \equiv 0$, we get the Helmholtz equation for the pressure (2.13a). Inserting (2.12) directly in (2.9a), we have

$$\nabla p^j \cdot \mathbf{n} = \mathrm{i}\omega\rho_0 \mathbf{v}^j \cdot \mathbf{n} + (\eta_0 + \eta_0') \frac{\mathrm{i}\omega}{\rho_0 c^2} \nabla p^{j-2} \cdot \mathbf{n} \\ - \eta_0 \, \mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{v}^{j-2} \cdot \mathbf{n} + \mathbf{f} \cdot \mathbf{n} \cdot \delta_{j=0}, \quad (3.12)$$

and using (2.10) and (2.12) leads to (2.14). We refer to the Appendix for the derivation of the boundary conditions to any order.

4. Justification of the asymptotic expansion

4.1. *Preliminaries*

We start by introducing some notation and properties of the considered equations. Besides the usual Sobolev spaces $H^m(\Omega)$ and $H^{m+1/2}(\partial\Omega)$, $m \in \mathbb{Z}$ we define the Hilbert spaces

$$H(\operatorname{div}, \Omega) := \{\mathbf{v} \in L^2(\Omega) : \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$$
$$H(\operatorname{curl}_{2D}, \Omega) := \{\mathbf{v} \in L^2(\Omega) : \operatorname{curl}_{2D} \mathbf{v} \in L^2(\Omega)\}$$

where for the differential operator $D \in \{\operatorname{div}, \operatorname{curl}_{2D}\}$ we have the semi-norm and norm

$$|\mathbf{v}|_{H(D,\Omega)} := \|D\mathbf{v}\|_{L^2(\Omega)}, \qquad \|\mathbf{v}\|^2_{H(D,\Omega)} := \|\mathbf{v}\|^2_{L^2(\Omega)} + |\mathbf{v}|^2_{H(D,\Omega)}.$$

Furthermore, we define for some $g \in H^{-1/2}(\partial\Omega)$

$$H_g(\operatorname{div}, \Omega) := \{\mathbf{v} \in H(\operatorname{div}, \Omega) : \mathbf{v} \cdot \mathbf{n} = g \text{ on } \partial\Omega\},$$

and

$$H_0(\operatorname{div} 0, \Omega) := \{\mathbf{v} \in H(\operatorname{div}, \Omega) : \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$
$$H(\operatorname{curl}_{2D} 0, \Omega) := \{\mathbf{v} \in H(\operatorname{curl}_{2D}, \Omega) : \operatorname{curl}_{2D} \mathbf{v} = 0\}.$$

Furthermore, for vector fields $\mathbf{u}, \mathbf{v} \in H(\operatorname{curl}_{2D}, \Omega)$ we will use the integration by parts formula

$$\int_\Omega \operatorname{curl}_{2D} \mathbf{u} \, v \, \mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n}^\perp \, v \, \mathrm{d}\sigma(\mathbf{x}) + \int_\Omega \mathbf{u} \cdot \mathbf{curl}_{2D} \, v \, \mathrm{d}\mathbf{x}. \qquad (4.1)$$

Note, that $\mathbf{u} \cdot \mathbf{n}^\perp$ and $\operatorname{curl}_{2D} \mathbf{u}$ are the Dirichlet and Neumann traces for $\mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{u}$ and $\mathbf{u} \cdot \mathbf{n}$ and $\operatorname{div} \mathbf{u}$ for $\mathbf{grad} \operatorname{div} \mathbf{u}$, as we see by twice applying integration by parts

$$\int_\Omega \mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{u}\cdot\mathbf{v}\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{u}\cdot\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{v}\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \underbrace{(\mathrm{curl}_{2D}\,\mathbf{u})}_{\text{Neumann trace}}\ \mathbf{v}\cdot\mathbf{n}^\perp - \underbrace{(\mathbf{u}\cdot\mathbf{n}^\perp)}_{\text{Dirichlet trace}}\ \mathrm{curl}_{2D}\,\mathbf{v}\mathrm{d}\sigma(\mathbf{x}),$$

$$\int_\Omega \mathbf{grad}\,\mathrm{div}\,\mathbf{u}\cdot\mathbf{v}\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{u}\cdot\mathbf{grad}\,\mathrm{div}\,\mathbf{v}\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \underbrace{(\mathrm{div}\,\mathbf{u})}_{\text{Neumann trace}}\ (\mathbf{v}\cdot\mathbf{n}) - \underbrace{(\mathbf{u}\cdot\mathbf{n})}_{\text{Dirichlet trace}}\ \mathrm{div}\,\mathbf{v}\,\mathrm{d}\sigma(\mathbf{x}).$$

For C^∞ smooth domains in two dimensions the spaces $(H^1(\Omega))^2 \cap H_0(\mathrm{div},\Omega)$ and $H_0(\mathrm{div},\Omega)\cap H(\mathrm{curl}_{2D},\Omega)$ are equivalent, meaning that there exists two constants $0 < C_1 < C_2$, such that for any $\mathbf{v}\in H_0(\mathrm{div},\Omega)\cap H(\mathrm{curl}_{2D},\Omega)$ it holds

$$C_1\|\mathbf{v}\|^2_{(H^1(\Omega))^2} \le \|\mathbf{v}\|^2_{H(\mathrm{div},\Omega)\cap H(\mathrm{curl}_{2D},\Omega)} := \|\mathbf{v}\|^2_{L^2(\Omega)} + \|\,\mathrm{div}\,\mathbf{v}\|^2_{L^2(\Omega)} + \|\,\mathrm{curl}_{2D}\,\mathbf{v}\|^2_{L^2(\Omega)} \le C_2\|\mathbf{v}\|^2_{(H^1(\Omega))^2}.$$

This follows from the fact that a gradient of a vector field can be bounded by its divergence, its rotation and its normal trace[32,21,15],

$$\|\,\mathbf{grad}\,\mathbf{v}\|_{L^2(\Omega)} \le C\left(\|\,\mathrm{div}\,\mathbf{v}\|_{L^2(\Omega)} + \|\,\mathrm{curl}_{2D}\,\mathbf{v}\|_{L^2(\Omega)} + \|\mathbf{v}\cdot\mathbf{n}\|_{H^{1/2}(\partial\Omega)}\right), \tag{4.2}$$

since so called Neumann fields do not exist in two dimension.

4.2. *Eigensolutions and well-posedness of the limit equations*

We claim that in the limit $\varepsilon\to 0$ the far field terms $\mathbf{v}^j$ satisfy a system of the form

$$\begin{aligned} \nabla\,\mathrm{div}\,\mathbf{v} + \frac{\omega^2}{c^2}\mathbf{v} &= \mathbf{f}, \quad \text{in } \Omega,\\ \mathbf{v}\cdot\mathbf{n} &= g, \quad \text{on } \partial\Omega, \end{aligned} \tag{4.3}$$

and for particular choice of $\mathbf{f}$ and g the system (4.3) coincides with the far field equations (2.9). The source term in the first equation is not necessarily the original source term $\mathbf{f}$.

The weak formulation for (4.3) is: Seek $\mathbf{v}\in H_g(\mathrm{div},\Omega)$ such that for all

$$\int_\Omega \mathrm{div}\,\mathbf{v}\,\mathrm{div}\,\mathbf{v}' - \frac{\omega^2}{c^2}\mathbf{v}\cdot\mathbf{v}'\mathrm{d}\mathbf{x} = f(\mathbf{v}'), \qquad \forall \mathbf{v}'\in H_0(\mathrm{div},\Omega) \tag{4.4}$$

where $f(\mathbf{v}') = -\int_\Omega \mathbf{f}\cdot\mathbf{v}'\mathrm{d}\mathbf{x}$. We would like to extend the source terms by allowing the functional f to include distributions, *e. g.*, $\int_\Gamma \mathrm{f}\,\mathbf{v}'\cdot\mathbf{n}\,\mathrm{d}\sigma(\mathbf{x})$ for some function $\mathrm{f}\in H^{1/2}(\Gamma)$ on an one-dimensional submanifold $\Gamma\subset\Omega$ with normal vector $\mathbf{n}$.

The system (4.3) has a similar form as the Helmholtz equation, where the Laplace operator is replaced by the operator $\nabla\,\mathbf{grad}$. In contrast to the vectorial Helmholtz equation with the vector Laplace operator there is only one boundary condition, which is consistent with the variational formulation (4.4) in $H(\mathrm{div}, \Omega)$. Unlike for the (vectorial) Helmholtz equation the Sobolev space $H(\mathrm{div}, \Omega)$ is not compactly embedded in $L^2(\Omega)$ and the proof for well-posedness (see Lemma 4.2) uses the Fredholm alternative after a Helmholtz decomposition of $H_0(\mathrm{div}, \Omega)$. To do so, the eigenvalues of (4.3) have to be excluded. Let us now specify these eigenvalues.

Lemma 4.1. *The positive part of the spectrum of* (4.3) *with homogeneous source terms consists only of eigenvalues* $\frac{\omega^2}{c^2}$ *which coincide with the Neumann eigenvalues of* $-\Delta$.

Proof. Let $\mathbf{w}$ be a solution of (4.3) for vanishing source terms. For $\frac{\omega^2}{c^2} > 0$ it is evident that $\mathrm{curl}_{2D}\,\mathbf{w} = 0$. We use the decomposition (see Sec. 3.e in Ref. 1) for functions $\mathbf{w}' \in (L^2(\Omega))^2$

$$\mathbf{w}' = \nabla\phi' + \mathbf{w}'_c,$$

where $\phi' \in H^1(\Omega)$ and $\mathbf{w}'_c \in H_0(\mathrm{div}\,0, \Omega)$. The decomposition is orthogonal since

$$\int_\Omega \nabla\phi' \cdot \mathbf{w}'_0 \mathrm{d}\mathbf{x} = -\int_\Omega \phi' \,\mathrm{div}\,\mathbf{w}'_0 \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \phi' \mathbf{w}_c \cdot \mathbf{n} \mathrm{d}\sigma(\mathbf{x}) = 0.$$

Due to the orthogonality of the two parts $\nabla\phi$ and $\mathbf{w}_c$ of $\mathbf{w}$ they fulfil (4.3) independently. Then, with $\mathrm{div}\,\mathbf{w}_c = 0$ it follows $\mathbf{w}_c = \mathbf{0}$. The scalar potential ϕ has to fulfil

$$\begin{aligned} \nabla\big(\Delta\phi + \tfrac{\omega^2}{c^2}\phi\big) &= \mathbf{0}, \\ \nabla\phi \cdot \mathbf{n} &= 0. \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} \Delta\phi + \tfrac{\omega^2}{c^2}\phi &= C, \\ \nabla\phi \cdot \mathbf{n} &= 0. \end{aligned} \tag{4.5}$$

for some constant $C \in \mathbb{C}$. Note, that (4.5) is Fredholm with index 0, and we can apply the Fredholm alternative:

(i) If $\frac{\omega^2}{c^2}$ is not a Neumann eigenvalue of $-\Delta$ the system (4.5) has a unique solution $\phi = \frac{c^2}{\omega^2}C$ and so $\mathbf{w} = \nabla\phi = \mathbf{0}$. Hence, $\frac{\omega^2}{c^2}$ is not an eigenvalue of (4.3). The system is uniquely solvable.

(ii) If $\frac{\omega^2}{c^2}$ is a Neumann eigenvalue of $-\Delta$ the system (4.5) is not uniquely solvable, and so (4.3). This means that $\frac{\omega^2}{c^2}$ is an eigenvalue and $\nabla\phi$ an non-trivial solution of (4.3). Contrarily, $\mathbf{w} = \nabla\phi + \mathbf{w}_c$ can only be in the kernel of (4.3), if $\mathbf{w}_c = \mathbf{0}$ and ϕ solves (4.5). □

Now, we are in the position to state the well-posedness, where we begin with the case of vanishing normal trace, $g = 0$. The case for general normal trace will be considered in Lemma 4.4. The statement of the lemma is applicable for a wide class of functionals f, where the bound simplifies for the common case of curl_{2D}-free source terms $\mathbf{f}$. For this let us introduce the notation $f \circ \mathbf{curl}_{2D} : L^2(\Omega) \to \mathbb{C}$ for functionals, which are defined as $(f \circ \mathbf{curl}_{2D})(\psi) = f(\mathbf{curl}_{2D}\,\psi)$.

Remark 4.1. If $f(\mathbf{v}') = \int_\Omega \mathbf{f} \cdot \mathbf{v}' \mathrm{d}\mathbf{x}$, then $\operatorname{curl}_{2D} \mathbf{f} \in L^2(\Omega)$ is sufficient to have $f \circ \mathbf{curl}_{2D} \in L^2(\Omega)$, since

$$\|f \circ \mathbf{curl}_{2D}\|_{L^2(\Omega)} = \sup_{\psi \in C_0^\infty(\Omega)} \frac{|\int_\Omega \mathbf{f} \cdot \mathbf{curl}_{2D}\, \psi \mathrm{d}\mathbf{x}|}{\|\psi\|_{L^2(\Omega)}}$$

$$= \sup_{\psi \in C_0^\infty(\Omega)} \frac{|\int_\Omega \operatorname{curl}_{2D} \mathbf{f} \psi \mathrm{d}\mathbf{x}|}{\|\psi\|_{L^2(\Omega)}} \leq \|\operatorname{curl}_{2D} \mathbf{f}\|_{L^2(\Omega)},$$

and $f \circ \mathbf{curl}_{2D} = 0$ for curl_{2D}-free sources. As second example, for $f(\mathbf{v}') = \int_\Gamma \mathrm{f} \mathbf{v}' \cdot \mathbf{n} \mathrm{d}\sigma(\mathbf{x})$ with $\mathrm{f} \in H^{3/2}(\Gamma)$ it is $f \circ \mathbf{curl}_{2D} \in L^2(\Omega)$ since

$$\|f \circ \mathbf{curl}_{2D}\|_{L^2(\Omega)} = \sup_{\psi \in C_0^\infty(\Omega)} \frac{|\int_\Gamma \mathrm{f}\, \mathbf{curl}_{2D}\, \psi \cdot \mathbf{n} \mathrm{d}\sigma(\mathbf{x})|}{\|\psi\|_{L^2(\Omega)}}$$

$$= \sup_{\psi \in C_0^\infty(\Omega)} \frac{|\int_\Gamma \mathrm{f} \nabla \psi \cdot \mathbf{n}^\perp \mathrm{d}\sigma(\mathbf{x})|}{\|\psi\|_{L^2(\Omega)}} \leq \|\mathrm{f}\|_{H^{3/2}(\Gamma)}.$$

Lemma 4.2. *Let* $\frac{\omega^2}{c^2}$ *be distinct from the Neumann eigenvalues of* $-\Delta$*,* $f \in (H_0(\operatorname{div}, \Omega) \cap H(\operatorname{curl}_{2D}, \Omega))'$ *whereas* $f \circ \mathbf{curl}_{2D} \in L^2(\Omega)$*, and* $g = 0$*. Then,* (4.4) *has a unique solution* $\mathbf{v} \in H_0(\operatorname{div}, \Omega) \cap H(\operatorname{curl}_{2D}, \Omega)$*, and there exists a constant* $C > 0$ *such that*

$$\|\mathbf{v}\|_{H_0(\operatorname{div},\Omega) \cap H(\operatorname{curl}_{2D},\Omega)} \leq C(\|f\|_{(H_0(\operatorname{div},\Omega) \cap H(\operatorname{curl}_{2D}\, 0,\Omega))'} + \|f \circ \mathbf{curl}_{2D}\|_{L^2(\Omega)}).$$

If furthermore $\partial\Omega$ *is* C^∞ *then*

$$\|\mathbf{v}\|_{(H^1(\Omega))^2} \leq C(\|f\|_{(H_0(\operatorname{div},\Omega) \cap H(\operatorname{curl}_{2D}\, 0,\Omega))'} + \|f \circ \mathbf{curl}_{2D}\|_{L^2(\Omega)}).$$

Proof. The uniqueness of $\mathbf{v}$ follows by assumption and Lemma 4.1.

In the remaining we prove existence and stability in three steps.

(i) Testing (4.4) with $\mathbf{v}' = \mathbf{curl}_{2D}\, \psi'$ for $\psi' \in C_0^\infty(\Omega)$ we find

$$f(\mathbf{curl}_{2D}\, \psi') = -\frac{\omega^2}{c^2} \int_\Omega \mathbf{v} \cdot \mathbf{curl}_{2D}\, \psi' \mathrm{d}\mathbf{x} = -\frac{\omega^2}{c^2} \int_\Omega \operatorname{curl}_{2D} \mathbf{v}\, \psi' \mathrm{d}\mathbf{x}$$

and so $\operatorname{curl}_{2D} \mathbf{v}$ can be identified with the functional $f \circ \mathbf{curl}_{2D}$ which is by assumption bounded in $L^2(\Omega)$. Hence,

$$\|\operatorname{curl}_{2D} \mathbf{v}\|_{L^2(\Omega)} = \frac{c^2}{\omega^2} \|f \circ \mathbf{curl}_{2D}\|_{L^2(\Omega)},$$

and $\mathbf{v} \in H_0(\operatorname{div}, \Omega) \cap H(\operatorname{curl}_{2D}, \Omega)$.

(ii) Now, we use the unique Helmholtz decomposition [21] for $\mathbf{v} \in H_0(\operatorname{div}, \Omega)$ as sum of a curl-free part $\mathbf{v}_0 \in \mathcal{N}(\operatorname{curl}_{2D}) := H_0(\operatorname{div}, \Omega) \cap H(\operatorname{curl}_{2D} 0, \Omega)$ and a soleinodal part $\mathbf{v}_c = \mathbf{curl}_{2D}\, \psi$, $\psi \in H_0^1(\Omega)$, where in fact ψ has a higher regularity since $\operatorname{curl}_{2D} \mathbf{v} = \operatorname{curl}_{2D} \mathbf{v}_c \in L^2(\Omega)$. Note, that $\psi \equiv 0$ on $\partial\Omega$ implies $\mathbf{curl}_{2D}\, \psi \cdot \mathbf{n} = \nabla\psi \cdot \mathbf{n}^\perp \equiv 0$ and so in fact $\mathbf{curl}_{2D}\, H_0^1(\Omega) \subset H_0(\operatorname{div}, \Omega)$.

Functions in $\mathcal{N}(\mathrm{curl}_{2D})$ and in $\mathbf{curl}_{2D}\, H_0^1(\Omega)$ possess the property to be mutually orthogonal. Since any function $\mathbf{v}_0' \in \mathcal{N}(\mathrm{curl}_{2D})$ is a gradient of a scalar function $\phi_0' \in H^1(\Omega)\backslash\mathbb{C}$, we have for any $\mathbf{v}_c' \in \mathbf{curl}_{2D}\, H_0^1(\Omega)$

$$\int_\Omega \mathbf{v}_c' \cdot \mathbf{v}_0' \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{v}_c' \cdot \nabla \phi_0' \mathrm{d}\mathbf{x} = -\int_\Omega \operatorname{div} \mathbf{v}_c' \phi_0' \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{v}_c' \cdot \mathbf{n}\, \phi_0' \,\mathrm{d}\sigma(\mathbf{x}) = 0.$$

Testing (4.4) now with $\mathbf{v}' = \mathbf{curl}_{2D}\, \psi'$, $\psi' \in H_0^1(\Omega)$ arbitrarily, using the Helmholtz decomposition and the orthogonality we get

$$\int_\Omega \mathbf{curl}_{2D}\, \psi \cdot \mathbf{curl}_{2D}\, \psi' \mathrm{d}\mathbf{x} = f(\mathbf{curl}_{2D}\, \psi'),$$

which uniquely defines $\psi \in H_0^1(\Omega)$ and so $\mathbf{v}_c \in L^2(\Omega)$. By the Lax-Milgram lemma it holds for some constant $C > 0$

$$\|\mathbf{v}_c\|_{L^2(\Omega)} \le \|\psi\|_{H^1(\Omega)} \le C \|f \circ \mathbf{curl}_{2D}\|_{L^2(\Omega)}.$$

(iii) Finally, testing (4.4) with $\mathbf{v}' \in \mathcal{N}(\mathrm{curl}_{2D})$ we get a variational formulation: Seek $\mathbf{v}_0 \in \mathcal{N}(\mathrm{curl}_{2D})$ such that

$$(\mathbf{v}_0, \mathbf{v}')_{H(\mathrm{div},\Omega)} - (1 + \tfrac{\omega^2}{c^2}) \langle \mathbf{v}_0, \mathbf{v}' \rangle_{L^2(\Omega)} = f(\mathbf{v}') \quad \forall \mathbf{v}' \in \mathcal{N}(\mathrm{curl}_{2D}). \quad (4.6)$$

The first term is the $H(\mathrm{div},\Omega)$-inner product which is $\mathcal{N}(\mathrm{curl}_{2D})$-elliptic and the associated operator is the identity $\mathbf{I}$. Since $\mathcal{N}(\mathrm{curl}_{2D})$ as a subspace of $(H^1(\Omega))^2$ is compactly embedded in $(L^2(\Omega))^2$ by the Rellich-Kondrachov theorem, see Chapter 2 in Ref. 4, the operator $\mathbf{K}$ associated to the bilinear form $\langle \mathbf{v}_0, \mathbf{v}' \rangle_{L^2(\Omega)}$ is compact. Hence, the sum $\mathbf{I}+\mathbf{K}$ is a Fredholm operator with index $(\mathbf{I}+\mathbf{K}) = 0$ [26], *i. e.*, the dimension of its kernel coincides with the codimension of its range. Since we have uniqueness of $\mathbf{v}$ and so of $\mathbf{v}_0$ the Fredholm alternative [26] implies existence of a solution $\mathbf{v}_0$ with

$$\|\mathbf{v}_0\|_{H(\mathrm{div},\Omega)} \le C \|\mathbf{f}\|_{(H_0(\mathrm{div},\Omega) \cap H(\mathrm{curl}_{2D}\, 0,\Omega))'}.$$

Using (4.2) the stability result for C^∞ boundary follows. This completes the proof. □

Lemma 4.3. *For any* $g \in H^{-1/2}(\partial\Omega)$ *there exists a function* $\mathbf{E}(g) \in H_g(\mathrm{div},\Omega) \cup H(\mathrm{curl}_{2D}\, 0, \Omega)$ *and a constant* $C > 0$ *such that* $\|\mathbf{E}(g)\|_{H(\mathrm{div},\Omega)} \le C\|g\|_{H^{-1/2}(\partial\Omega)}$ *and for all* $\mathbf{v}' \in H_0(\mathrm{div},\Omega)$ *it holds* $\int_\Omega \operatorname{div} \mathbf{E}(g) \operatorname{div} \mathbf{v}' \mathrm{d}\mathbf{x} = 0$*. If furthermore* $\partial\Omega$ *is* $C^{1,1}$ *then* $\|\mathbf{E}(g)\|_{(H^1(\Omega))^2} \le C\|g\|_{H^{1/2}(\partial\Omega)}$.

Proof. Let $\phi(g) \in H^1(\Omega)\backslash\mathbb{C}$ be the unique solution of

$$\begin{aligned} \Delta\phi(g) &= \frac{1}{|\Omega|} \langle g, 1 \rangle_{L^2(\Omega)} \qquad &&\text{in } \Omega, \\ \nabla\phi(g) \cdot \mathbf{n} &= g &&\text{on } \partial\Omega, \end{aligned}$$

and by the Lax-Milgram lemma we have $\|\phi(g)\|_{H^1(\Omega)} \le C\|g\|_{H^{-1/2}(\partial\Omega)}$ for some $C > 0$.

We define $\mathbf{E}(g) = \nabla\phi(g)$ which is curl_{2D}-free in Ω, and obviously $\|\mathbf{E}(g)\|_{(L^2(\Omega))^2} \leq C\|g\|_{H^{-1/2}(\partial\Omega)}$. Since $\|\operatorname{div}\mathbf{E}(g)\|_{L^2(\Omega)} = \frac{1}{\sqrt{|\Omega|}}|\langle g, 1\rangle_{L^2(\Omega)}|$ the first part of the lemma follows. Since $\operatorname{div}\mathbf{E}(g) = \Delta\phi(g)$ is constant in Ω the Stokes theorem implies the second part.

With Theorem 4.18 in Ref. 19 we conclude in $\|\mathbf{E}(g)\|_{(H^1(\Omega))^2} \leq \|\phi(g)\|_{H^2(\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)}$. □

Lemma 4.4. *Let* $\frac{\omega^2}{c^2}$ *be distinct from the Neumann eigenvalues of* $-\Delta$, $f \in (H_0(\operatorname{div},\Omega) \cap H(\mathrm{curl}_{2D}\,0,\Omega))'$ *whereas* $f \circ \mathbf{curl}_{2D} \in L^2(\Omega)$ *and* $g \in H^{-1/2}(\partial\Omega)$. *Then,* (4.4) *has a unique solution* $\mathbf{v} \in H(\operatorname{div},\Omega) \cap H(\mathrm{curl}_{2D},\Omega)$, *and there exists a constant* $C > 0$ *such that*

$$\|\mathbf{v}\|_{H(\operatorname{div},\Omega)\cap H(\mathrm{curl}_{2D},\Omega)} \leq C\Big(\|f\|_{(H_0(\operatorname{div},\Omega)\cap H(\mathrm{curl}_{2D}\,0,\Omega))'} + \|f \circ \mathbf{curl}_{2D}\|_{(H_0^1(\Omega))'} + \|g\|_{H^{-1/2}(\partial\Omega)}\Big).$$

If furthermore $\partial\Omega$ *is* C^∞ *then*

$$\|\mathbf{v}\|_{(H^1(\Omega))^2} \leq C(\|f\|_{(H_0(\operatorname{div},\Omega)\cap H(\mathrm{curl}_{2D}\,0,\Omega))'} + \|f \circ \mathbf{curl}_{2D}\|_{(H_0^1(\Omega))'} + \|g\|_{H^{1/2}(\partial\Omega)}).$$

Proof. By linearity of (4.4) and in view of Lemma 4.2 we can restrict us to the case $f = 0$. Let $\mathbf{E}(g)$ be the extension as defined in Lemma 4.3. Then, the difference $\mathbf{w} := \mathbf{v} - \mathbf{E}(g) \in H_0(\operatorname{div},\Omega)$ satisfies

$$\int_\Omega \operatorname{div}\mathbf{w}\cdot\operatorname{div}\mathbf{v}' - \frac{\omega^2}{c^2}\mathbf{w}\cdot\mathbf{v}'\mathrm{d}\mathbf{x} = \frac{\omega^2}{c^2}\int_\Omega \mathbf{E}(g)\cdot\mathbf{v}'\mathrm{d}\mathbf{x} =: f_g(\mathbf{v}'), \quad \forall\mathbf{v}' \in H_0(\operatorname{div},\Omega),$$

where $f_g \circ \mathbf{curl}_{2D} = 0$ as $\mathrm{curl}_{2D}\,\mathbf{E}(g) = 0$ (see Remark 4.1) and $\|f_g\|_{L^2(\Omega)} \leq \|g\|_{H^{-1/2}(\partial\Omega)}$. This variational formulation fulfils the assumptions of Lemma 4.2 and there exists a unique $\mathbf{w}$ which bounded in $H(\operatorname{div},\Omega) \cap H(\mathrm{curl}_{2D},\Omega)$ by $\|g\|_{H^{-1/2}(\partial\Omega)}$. Since the same bound holds for the L^2-norms of $\mathbf{E}(g)$ and $\operatorname{div}\mathbf{E}(g)$ and $\mathrm{curl}_{2D}\,\mathbf{E}(g) = 0$, we obtain first statement of the lemma. Using the statements in Lemma 4.2 and Lemma 4.3 for C^∞ boundaries we conclude in the second statement of the lemma. □

4.3. *Stability of the original problem*

We have stated in Lemma 2.1 the stability of the solution of the original singularly perturbed system (2.4). That is, that the acoustic velocity is bounded by the source term in the $H(\operatorname{div},\Omega)$-norm with a constant independent of ε and in the $H(\mathrm{curl}_{2D},\Omega)$-(semi-)norm like ε^{-1}, whereas the pressure is bounded with a constant independent of ε.

This shall be proved in the following. With the absorbing terms due to the viscosity the solution can be bounded, however, with a constant like ε^{-2}. Taking the small absorbing terms on the right hand side of a system of the form of the limit system for $\varepsilon \to 0$, we cannot directly apply the well-posedness result of the last section as boundary terms appear by integration by parts as a consequence that this system is posed in $H_0(\mathrm{div},\Omega)$ in which the tangential component (of the test functions) does not vanish in general. The estimation of these boundary terms is a key point in the proof.

Proof. [Proof of Lemma 2.1] Similarly to (2.2a) we can write a boundary value problem for $\mathbf{v}^\varepsilon$ only. This is

$$\Big(1-\varepsilon^2\frac{\mathrm{i}(\eta_0+\eta_0')\omega}{\rho_0 c^2}\Big)\nabla\,\mathrm{div}\,\mathbf{v}^\varepsilon+\frac{\omega^2}{c^2}\,\mathbf{v}^\varepsilon+\varepsilon^2\frac{\mathrm{i}\eta_0\omega}{\rho_0c^2}\;\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\,\mathbf{v}^\varepsilon=\frac{\mathrm{i}\omega}{\rho_0c^2}\,\mathbf{f}, \tag{4.7}$$

completed by homogeneous Dirichlet boundary conditions. The associated variational formulation is: Seek $\mathbf{v}^\varepsilon\in(H^1_0(\Omega))^2$ such that for all $\mathbf{v}'\in(H^1_0(\Omega))^2$

$$\int_\Omega\Big(1-\varepsilon^2\frac{\mathrm{i}(\eta_0+\eta_0')\omega}{\rho_0c^2}\Big)\mathrm{div}\,\mathbf{v}^\varepsilon\,\mathrm{div}\,\overline{\mathbf{v}}'-\frac{\omega^2}{c^2}\,\mathbf{v}^\varepsilon\cdot\overline{\mathbf{v}}'\\-\varepsilon^2\frac{\mathrm{i}\eta_0\omega}{\rho_0c^2}\;\mathrm{curl}_{2D}\,\mathbf{v}^\varepsilon\;\mathrm{curl}_{2D}\,\overline{\mathbf{v}}'\,\mathrm{d}\mathbf{x}=\frac{\mathrm{i}\omega}{\rho_0c^2}\;\langle\mathbf{f},\mathbf{v}'\rangle_{L^2(\Omega)}\,. \tag{4.8}$$

This formulation can alternatively be written as

$$\mathsf{a}_\varepsilon(\mathbf{v}^\varepsilon,\mathbf{v}')-(1+\frac{\omega^2}{c^2})\,\langle\mathbf{v}^\varepsilon,\mathbf{v}'\rangle_{L^2(\Omega)}=\frac{\mathrm{i}\omega}{\rho_0c^2}\;\langle\mathbf{f},\mathbf{v}'\rangle_{L^2(\Omega)}$$

with the sesquilinear form

$$\mathsf{a}_\varepsilon(\mathbf{v},\mathbf{v}')=\int_\Omega\Big(1-\varepsilon^2\frac{\mathrm{i}(\eta_0+\eta_0')\omega}{\rho_0c^2}\Big)\mathrm{div}\,\mathbf{v}\,\mathrm{div}\,\overline{\mathbf{v}}'+\mathbf{v}\cdot\overline{\mathbf{v}}'\\-\varepsilon^2\frac{\mathrm{i}\eta_0\omega}{\rho_0c^2}\;\mathrm{curl}_{2D}\,\mathbf{v}\;\mathrm{curl}_{2D}\,\overline{\mathbf{v}}'\,\mathrm{d}\mathbf{x},$$

which corresponds for ε small enough to an isomorphism $\mathbf{A}_\varepsilon:(\mathbf{A}_\varepsilon\mathbf{v},\mathbf{v}')_\varepsilon=\mathsf{a}_\varepsilon(\mathbf{v},\mathbf{v}')$. The subspace $(H^1_0(\Omega))^2$ of $(H^1(\Omega))^2$ is compactly embedded in $L^2(\Omega)$ by the Rellich-Kondrachov theorem (Chapter 2 in Ref. 4) and $\mathbf{K}:(\mathbf{K}\mathbf{v},\mathbf{v}')_\varepsilon=-(1+\frac{\omega^2}{c^2})\,\langle\mathbf{v}^\varepsilon,\mathbf{v}'\rangle_{L^2(\Omega)}$ is a compact perturbation of $\mathbf{A}_\varepsilon$. Hence, the sum $\mathbf{A}_\varepsilon+\mathbf{K}$ is a Fredholm operator with $\mathrm{index}\,(\mathbf{A}_\varepsilon+\mathbf{K})=0$ [26], *i. e.*, the dimension of its kernel coincides with the codimension of its range, and by the Fredholm alternative [26] uniqueness implies the existence.

Now, we are going to show stability, and so uniqueness. This will be done in four steps, (i) the proof of a non-optimal stability result for $\mathbf{v}^\varepsilon$, (ii) an estimate of $\mathbf{v}^\varepsilon$ by $\mathbf{f}$ and the Neumann trace $\mathrm{curl}_{2D}\,\mathbf{v}^\varepsilon$, (iii) an estimate of the Neumann trace $\mathrm{curl}_{2D}\,\mathbf{v}^\varepsilon$ by $\mathbf{f}$, which leads to the desired estimate for $\mathbf{v}^\varepsilon$ and the $L^2(\Omega)$-norm of p^ε, and (iv) the estimate of ∇p^ε.

(i) Testing (4.8) with $\mathbf{v}'=\mathbf{v}^\varepsilon$ and taking the imaginary part we get

$$\|\,\mathrm{div}\,\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)}+\|\,\mathrm{curl}_{2D}\,\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)}\le C\varepsilon^{-2}|\,\langle\mathbf{f},\mathbf{v}^\varepsilon\rangle_{L^2(\Omega)}\,| \tag{4.9}$$

with a constant $C > 0$ independent of ε. Now, taking the real part we can assert that

$$\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} \le C(|\langle \mathbf{f}, \mathbf{v}^\varepsilon\rangle_{L^2(\Omega)}| + \|\operatorname{div}\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^2\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)}). \tag{4.10}$$

Summing (4.9) and (4.10) and using the Cauchy-Schwarz and Young's inequality we get the non-optimal stability estimate

$$\|\mathbf{v}^\varepsilon\|_{H(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega)} \le C\,\varepsilon^{-1}\|\mathbf{f}\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'}, \tag{4.11}$$

and so uniqueness. This result is independent of the regularity of the boundary and holds also for the resonant case where $\frac{\omega^2}{c^2}$ is a Neumann eigenvalue of $-\Delta$. In the same way and using (4.2) we obtain for C^∞ boundaries the estimate (2.5c).

(ii) We can rewrite the system for $\mathbf{v}^\varepsilon$ as

$$\begin{aligned}\nabla\operatorname{div}\mathbf{v}^\varepsilon + \frac{\omega^2}{c^2}\mathbf{v}^\varepsilon = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f} + \varepsilon^2\frac{\mathrm{i}(\eta_0+\eta_0')\omega}{\rho_0 c^2}\nabla\operatorname{div}\mathbf{v}^\varepsilon \\ - \varepsilon^2\frac{\mathrm{i}\eta_0\omega}{\rho_0 c^2}\,\mathbf{curl}_{2D}\operatorname{curl}_{2D}\mathbf{v}^\varepsilon =: \mathbf{f}^\varepsilon, \\ \mathbf{v}^\varepsilon\cdot\mathbf{n} = 0.\end{aligned}$$

Applying Lemma 4.2 we can bound with constants $C_1, C_2 > 0$ independent of ε

$$\begin{aligned}\|\mathbf{v}^\varepsilon\|_{H(\operatorname{div},\Omega)} &\le C_1\Big(\|\mathbf{f}\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'} \\ &\quad + \varepsilon^2\|\nabla\operatorname{div}\mathbf{v}^\varepsilon\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'} \\ &\quad + \varepsilon^2\|\,\mathbf{curl}_{2D}\operatorname{curl}_{2D}\mathbf{v}^\varepsilon\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'}\Big) \\ &= C_1(\|\mathbf{f}\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'} + \varepsilon^2\|\operatorname{div}\mathbf{v}^\varepsilon\|_{L^2(\Omega)} \\ &\quad + \varepsilon^2\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon\|_{L^2(\Omega)} + \varepsilon^2\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon\|_{H^{-1/2}(\partial\Omega)}) \\ &\le C_2(\|\mathbf{f}\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'} + \varepsilon^2\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon\|_{H^{-1/2}(\partial\Omega)})\end{aligned} \tag{4.12}$$

where we used integration by parts for $\nabla\operatorname{div}\mathbf{v}^\varepsilon$ and $\mathbf{curl}_{2D}\operatorname{curl}_{2D}\mathbf{v}^\varepsilon$ leading to a term with the Neumann trace $\operatorname{curl}_{2D}\mathbf{v}^\varepsilon$, and applied then the estimate (4.11).

(iii) Now, for any $w \in H^{1/2}(\partial\Omega)$ we define a function $\mathbf{w}_0 \in \mathcal{N}(\operatorname{curl}_{2D})$ with $\mathbf{w}_0\cdot\mathbf{n}^\perp = w$, which is for all $\mathbf{w}_0' \in \mathcal{N}(\operatorname{curl}_{2D})$ and $\lambda' \in H^{-1/2}(\partial\Omega)$ solution of

$$\begin{aligned}\langle\operatorname{div}\mathbf{w}_0, \operatorname{div}\mathbf{w}_0'\rangle_{L^2(\Omega)} - \tfrac{\omega^2}{c^2}\langle\mathbf{w}_0,\mathbf{w}_0'\rangle_{L^2(\Omega)} \\ + \left\langle\lambda, \mathbf{w}_0'\cdot\mathbf{n}^\perp\right\rangle_{L^2(\partial\Omega)} &= 0, \\ \left\langle\mathbf{w}_0\cdot\mathbf{n}^\perp, \lambda'\right\rangle_{L^2(\partial\Omega)} &= \langle w, \lambda'\rangle_{L^2(\partial\Omega)}\,.\end{aligned}$$

Since the operator related to $\langle \operatorname{div} \mathbf{w}_0, \operatorname{div} \mathbf{w}_0' \rangle_{L^2(\Omega)} - \frac{\omega^2}{c^2} \langle \mathbf{w}_0, \mathbf{w}_0' \rangle_{L^2(\Omega)}$ is Fredholm with index 0 (see proof of Lemma 4.2) and invertible, as we excluded the kernel, the system has by the theory of saddle point problems [5] a unique solution $(\mathbf{w}_0, \lambda)$, where $\lambda = 0$ (since $\operatorname{curl}_{2D} \mathbf{w}_0 = 0$) and for some constant $C > 0$

$$\|\mathbf{w}_0\|_{H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega)} \leq C \|w\|_{H^{1/2}(\partial\Omega)}. \tag{4.13}$$

Now, deriving a variational formulation for (4.7) with test functions $\mathbf{w}_0 \in \mathcal{N}(\operatorname{curl}_{2D})$, that are functions which may have a non-zero tangential component, we get

$$\begin{aligned}\big(1 - \varepsilon^2 \frac{\mathrm{i}(\eta_0 + \eta_0')\omega}{\rho_0 c^2}\big) \langle \operatorname{div} \mathbf{v}^\varepsilon, \operatorname{div} \mathbf{w}_0 \rangle_{L^2(\Omega)} - \frac{\omega^2}{c^2} \langle \mathbf{v}^\varepsilon, \mathbf{w}_0 \rangle_{L^2(\Omega)} \\ + \varepsilon^2 \frac{\mathrm{i}\eta_0\omega}{\rho_0 c^2} \langle \operatorname{curl}_{2D} \mathbf{v}^\varepsilon, \mathbf{w}_0 \cdot \mathbf{n}^\perp \rangle_{L^2(\partial\Omega)} = \frac{\mathrm{i}\omega}{\rho_0 c^2} \langle \mathbf{f}, \mathbf{w}_0 \rangle_{L^2(\Omega)},\end{aligned}$$

and so

$$\varepsilon^2 \langle \operatorname{curl}_{2D} \mathbf{v}^\varepsilon, w \rangle_{L^2(\partial\Omega)} = \frac{1}{\eta_0} \langle \mathbf{f}, \mathbf{w}_0 \rangle_{L^2(\Omega)} + \varepsilon^2 \frac{\eta_0 + \eta_0'}{\eta_0} \langle \operatorname{div} \mathbf{v}^\varepsilon, \operatorname{div} \mathbf{w}_0 \rangle_{L^2(\Omega)}.$$

Taking the supremum over all $w \in H^{1/2}(\partial\Omega)$ and using the estimates (4.11) and (4.13) we find

$$\varepsilon^2 \| \operatorname{curl}_{2D} \mathbf{v}^\varepsilon \|_{H^{-1/2}(\partial\Omega)} \leq C \|\mathbf{f}\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'}.$$

Inserting this expression into (4.12) we get

$$\|\mathbf{v}^\varepsilon\|_{H(\operatorname{div},\Omega)} \leq C \|\mathbf{f}\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'}.$$

and (2.4b) leads the bound of p^ε.

(iv) Multiplying (4.7) with $\nabla \operatorname{div} \overline{\mathbf{v}}^\varepsilon - \varepsilon^2 \frac{\mathrm{i}\eta_0\omega}{\rho_0 c^2} \mathbf{curl}_{2D} \operatorname{curl}_{2D} \overline{\mathbf{v}}^\varepsilon$ and taking the real part we get

$$\begin{aligned}\|\nabla \operatorname{div} \mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} + \left(\frac{\eta_0\omega}{\rho_0 c^2}\right)^2 \|\varepsilon^2 \mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} \\ - \varepsilon^2 \eta_0(\eta_0 + \eta_0') \left(\frac{\omega}{\rho_0 c^2}\right)^2 \operatorname{Re} \big\langle \mathbf{grad} \operatorname{div} \mathbf{v}^\varepsilon, \varepsilon^2 \mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{v}^\varepsilon \big\rangle_{L^2(\Omega)} \\ = \operatorname{Re} \big\langle \mathbf{f} + \frac{\omega^2}{c^2} \mathbf{v}^\varepsilon, \nabla \operatorname{div} \mathbf{v}^\varepsilon \big\rangle_{L^2(\Omega)} \\ + \frac{\eta_0\omega}{\rho_0 c^2} \operatorname{Im} \big\langle \mathbf{f} + \frac{\omega^2}{c^2} \mathbf{v}^\varepsilon, \varepsilon^2 \mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{v}^\varepsilon \big\rangle_{L^2(\Omega)}.\end{aligned}$$

With the Cauchy-Schwarz inequality, Youngs inequality and (2.5a) we find for ε small enough that

$$\|\nabla \operatorname{div} \mathbf{v}^\varepsilon\|_{L^2(\Omega)} + \|\varepsilon^2 \mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{v}^\varepsilon\|_{L^2(\Omega)} \leq C \|\mathbf{f}\|_{L^2(\Omega)},$$

and with (2.4b) the estimate (2.5b) follows. □

Using Lemma 4.3 it is easy to prove the following strengthening of Lemma 2.1.

Corollary 4.1. *Let the assumption of Lemma 2.1 hold and* $\mathbf{v}^\varepsilon$*,* p^ε *satisfy* (2.4a), (2.4b) *and*

$$\mathbf{v}^\varepsilon \cdot \mathbf{n} = g \in H^{-1/2}(\partial\Omega), \qquad \mathbf{v}^\varepsilon \cdot \mathbf{n}^\perp = 0. \tag{4.14}$$

Then, it holds with a constant C *not depending on* ε *that*

$$\begin{aligned} &\|\mathbf{v}^\varepsilon\|_{H(\operatorname{div},\Omega)} + \varepsilon \|\operatorname{curl}_{2D} \mathbf{v}^\varepsilon\|_{L^2(\Omega)} + \|p^\varepsilon\|_{L^2(\Omega)} \qquad (4.15a)\\ &\qquad\qquad \le C \Big(\|\mathbf{f}\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'} + \|g\|_{H^{-1/2}(\partial\Omega)} \Big), \\ &\|\nabla p^\varepsilon\|_{L^2(\Omega)} \le C \Big(\|\mathbf{f}\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \Big). \qquad (4.15b)\end{aligned}$$

If furthermore $\partial\Omega$ *is* C^∞ *then*

$$\|\mathbf{v}^\varepsilon\|_{(H^1(\Omega))^2} \le C\Big(\|\mathbf{f}\|_{(H_0(\operatorname{div},\Omega)\cap H(\operatorname{curl}_{2D},\Omega))'} + \|g\|_{H^{1/2}(\partial\Omega)} \Big). \tag{4.15c}$$

4.4. *Well-posedness of the far field equations*

In the definition of the sequence of far field velocities $\mathbf{v}^j$ and far field pressures p^j terms of lower order appear on the right hand side, see (2.9) and (2.13), to which differential operators are applied. For well-defined far field terms we have therefor to study the regularity. The proof relies on the regularity theory for strongly elliptic operators (see Chapter 4 in Ref. 19). As the present operator $-\nabla \operatorname{div} - \frac{\omega^2}{c^2}$ is not strongly elliptic as $-\nabla \operatorname{div}$ has a kernel of functions $\mathbf{curl}_{2D}\, \psi \in (H^1_0(\Omega))^2$, we will consider the operator $-\Delta - \frac{\omega^2}{c^2}$, which differs from the present operator only by $\mathbf{curl}_{2D} \operatorname{curl}_{2D}$.

For sake of completeness we will reproduce Theorem 4.18 in Ref. 19 for the strongly elliptic operator $-\Delta - \frac{\omega^2}{c^2}$.

Lemma 4.5. *Let* Ω_1*,* Ω_2 *as well as* Γ_1*,* Γ_2 *defined like in Lemma 4.6, and* Γ_2 *being* $C^{r+1,1}$ *for some integer* $r \ge 0$*. Then, the solution* $u \in H^1(\Omega_2)$ *of* $-\Delta u - \frac{\omega^2}{c^2} u = f$*,* $f \in H^r(\Omega_2)$ *satisfies*

$$\begin{aligned} \|u\|_{H^{r+2}(\Omega_1)} &\le C \left(\|u\|_{H^1(\Omega_2)} + \|u\|_{H^{r+3/2}(\Gamma_2)} + \|f\|_{H^r(\Omega_2)} \right), \\ \|u\|_{H^{r+2}(\Omega_1)} &\le C \left(\|u\|_{H^1(\Omega_2)} + \|\partial_n u\|_{H^{r+1/2}(\Gamma_2)} + \|f\|_{H^r(\Omega_2)} \right), \end{aligned}$$

for some positive constant C*.*

Now, we are in the position to transfer the regularity result to the system (4.3).

Lemma 4.6. *Let* Ω_1*,* Ω_2 *be subdomains of* Ω *intersecting the boundary* $\partial\Omega$ *in* Γ_1*,* Γ_2 *respectively and* $(\overline{\Omega}_1 \backslash \Gamma_1) \subset \Omega_2$*,* $\overline{\Gamma}_1 \subset \Gamma_2$*. Suppose, for an integer* $r \ge 0$*, that* Γ_2

is $C^{r+1,1}$. *Assume further that in* (4.3) $\mathbf{v} \in (H^1(\Omega_2))^2$, $\mathbf{f} \in (H^r(\Omega_2))^2$, $\operatorname{curl}_{2D} \mathbf{f} \in H^{r+1}(\Omega_2)$, *and* $g \in H^{r+3/2}(\Gamma_2)$. *Then, there exists a constant* $C > 0$ *such that*

$$\|\mathbf{v}\|_{(H^{r+2}(\Omega_1))^2} \leq C\Big(\|\mathbf{v}\|_{(H^1(\Omega_2))^2} + \|\mathbf{f}\|_{(H^r(\Omega_2))^2} + \|\operatorname{curl}_{2D} \mathbf{f}\|_{H^{r+1}(\Omega_2)} + \|g\|_{H^{r+3/2}(\Gamma_2)}\Big).$$

Proof. First, we rewrite the system (4.3) in terms of the strongly elliptic operator $-\Delta - \frac{\omega^2}{c^2}$

$$\Delta \mathbf{v} + \frac{\omega^2}{c^2}\mathbf{v} = \mathbf{f} - \frac{c^2}{\omega^2}\mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{f}, \qquad \text{in } \Omega, \tag{4.16a}$$

$$\mathbf{v} \cdot \mathbf{n} = g, \qquad \text{on } \partial\Omega, \tag{4.16b}$$

$$\operatorname{curl}_{2D} \mathbf{v} = \frac{c^2}{\omega^2}\operatorname{curl}_{2D} \mathbf{f}, \qquad \text{on } \partial\Omega, \tag{4.16c}$$

where equation (4.16c) which is consistent to (4.16a) is added to obtain two boundary conditions on $\partial\Omega$. With the assumption that $\operatorname{curl}_{2D} \mathbf{f} \in H^{r+1}(\Omega_2)$ the trace of $\operatorname{curl}_{2D} \mathbf{f}$ to Γ_2 is bounded in $H^{r+1/2}(\Gamma_2)$. With its Dirichlet boundary condition (4.16b) we would like to apply the first bound in Lemma 4.5 for the normal component v_s, and since

$$\operatorname{curl}_{2D} \mathbf{v} = \partial_t v_s - \partial_s v_t, \qquad \text{on } \partial\Omega,$$

the second bound in Lemma 4.5 for the tangential component v_t. However, in the curvilinear coordinate system the Laplace operator is not fully decoupled for the tangential and normal component. Inserting its expression and

$$\mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{f} = \partial_s \operatorname{curl}_{2D} \mathbf{f}\, \mathbf{e}_t - A\partial_t \operatorname{curl}_{2D} \mathbf{f}\, \mathbf{e}_s, \qquad \text{in } \Omega,$$

we find that the normal component satisfies

$$\begin{aligned} \Delta v_s + \left(\frac{\omega^2}{c^2} - \kappa^2 A^2\right) v_s &= \mathbf{f} \cdot \mathbf{e}_s - \frac{c^2}{\omega^2} A\partial_t \operatorname{curl}_{2D} \mathbf{f} \\ &\quad - A^2(2\kappa\partial_t + \kappa' A)v_t =: \widetilde{f}_1, && \text{in } \Omega, \\ v_s &= g, && \text{on } \partial\Omega, \end{aligned} \tag{4.17}$$

whereas the tangential component solves

$$\begin{aligned} \Delta v_t + \left(\frac{\omega^2}{c^2} - \kappa^2 A^2\right) v_t &= \mathbf{f} \cdot \mathbf{e}_t + \frac{c^2}{\omega^2}\partial_s \operatorname{curl}_{2D} \mathbf{f} \\ &\quad + A^2(2\kappa\partial_t + \kappa' A)v_s := \widetilde{f}_2, && \text{in } \Omega, \\ \partial_n v_t = -\partial_s v_t &= \operatorname{curl}_{2D} \mathbf{v} - \partial_t v_s \\ &= \frac{c^2}{\omega^2}\operatorname{curl}_{2D} \mathbf{f} - \partial_t v_s =: \widetilde{g}, && \text{on } \partial\Omega. \end{aligned} \tag{4.18}$$

The coupling between v_t and v_s is with at most their first tangential derivative which enable us to prove the regularity step by step.

For this we introduce subsets $\Omega_{1,\ell}$, $\ell = 0, \dots, r+1$ intersecting $\partial\Omega$ in $\Gamma_{1,\ell}$ with

$$\Omega_{1,0} := \Omega_2, \quad \Omega_{1,r+1} := \Omega_1, \quad \overline{\Omega}_{1,\ell} \subset \Omega_{1,\ell-1}, \quad \overline{\Gamma}_{1,\ell} \subset \Gamma_{1,\ell-1}, \quad \ell = 1, \dots, r+1.$$

Let us assume that $\mathbf{v} \in H^{\ell+1}(\Omega_{1,\ell})$ for some $\ell \in \{0, \dots, r\}$. Then, $\widetilde{f}_1 \in H^{\ell}(\Omega_{1,\ell})$, $g \in H^{\ell+3/2}(\Gamma_{1,\ell}) \subset H^{r+3/2}(\Gamma_2)$, $\widetilde{f}_2 \in H^{\ell}(\Omega_{1,\ell})$, and $\widetilde{g} \in H^{\ell+1/2}(\Gamma_{1,\ell})$, and in view of Lemma 4.5 $\mathbf{v} \in H^{\ell+2}(\Omega_{1,\ell+1})$. Since by assumption $\mathbf{v} \in H^1(\Omega_{1,0})$ the statement of the Lemma follows by induction in ℓ. □

Proof. [Proof of Lemma 2.3] We will prove the lemma by induction in j. Consider first $j = 0$. As $\mathbf{f} \in (H_0(\mathrm{div}, \Omega))'$, $\mathbf{v}^0 \cdot \mathbf{n} = 0$ and $\mathrm{curl}_{2D}\, \mathbf{f} \in L^2(\Omega)$ by assumption, by Lemma 4.2 with $\frac{\omega^2}{c^2}$ distinct from the Neumann eigenvalues of $-\Delta$ there exists a unique solution $\mathbf{v}^0 \in (H^1(\Omega))^2$. So, $\mathrm{div}\, \mathbf{v}^0 \in L^2(\Omega)$ and by (2.12) $p^0 \in L^2(\Omega)$. As $\mathbf{f} \in (H^m(\widetilde{\Omega}))^2$ for any $m \in \mathbb{N}_0$ in a neighbourhood $\widetilde{\Omega}$ of $\partial\Omega$ using Lemma 4.6 and the trace theorem we can bound the trace $\mathrm{div}\, \mathbf{v}^0 \in H^{m-1/2}(\partial\Omega)$ for any $m \in \mathbb{N}_0$.

Now, let $j > 0$ and $\mathbf{v}^\ell \in (H^1(\Omega))^2$, $\mathrm{div}\, \mathbf{v}^\ell \in H^{m-1/2}(\partial\Omega)$ for any $m \in \mathbb{N}_0$ and $\ell = 0, \dots, j-1$. By assumption we have $\mathbf{f} \in (H^m(\widetilde{\Omega}))^2$ and so with (2.11) $\mathrm{curl}_{2D}\, \mathbf{v}^j \in L^2(\Omega) \cap H^{m-1}(\widetilde{\Omega})$ and, as G_ℓ and H_ℓ are differential operators, with (2.9b) $\mathbf{v}^j \cdot \mathbf{n} \in H^{m+1/2}(\partial\Omega)$ for any $m \in \mathbb{N}_0$. So, by Lemma 4.2 there exists a unique solution $\mathbf{v}^j \in (H^1(\Omega))^2$ and by Lemma 4.6 and the trace theorem $\mathrm{div}\, \mathbf{v}^j \in H^{m-1/2}(\partial\Omega)$ for any $m \in \mathbb{N}_0$. Furthermore, (2.12) implies $p^j \in L^2(\Omega)$.

Hence, by complete induction in j the proof is complete. □

4.5. *Estimate of the modelling error*

The near field functions have been defined in (t, s)-coordinates close to the wall which is assumed to be C^∞. Since integrals over the physical domain can be transformed into such over the (t, s)-domain where a factor $1 - \kappa s$ appears, the following equivalence follows easily.

Lemma 4.7. *For any function* $\widehat{u} : [0, \infty] \times \Gamma \to \mathbb{C}$ *with* $u \equiv 0$ *for* $s \geq s_0$ *and* $v(\mathbf{x}) := \widehat{u}(t, s)$ *there exists a constant* $C = C(\|\kappa\|_{L^\infty(\Gamma)} s_0) > 0$ *such that*

$$\|v\|_{L^2(\Omega)} \leq C\, \|\widehat{u}\|_{L^2(\Gamma \times [0,\infty])}.$$

The near field equations have been derived in a neighbourhood of the boundary where the cut-off function $\chi(\mathbf{x})$ is the constant 1. With the cut-off function $\chi(\mathbf{x})$ we can state the following

Lemma 4.8. *For any function* $u \in \Pi(\lambda, L^2(\Gamma))$ *with* $\mathrm{Re}\,\lambda > 0$ *and* $\widehat{u}(t, s) = u(t, \frac{s}{\varepsilon})$ *there exists a constant* $C > 0$ *independent of* ε *such that*

$$\|\widehat{u}\, \widehat{\chi}\|_{L^2(\Gamma \times [0,\infty])} \leq C\, \sqrt{\varepsilon},$$

where $\chi(\mathbf{x}) = \widehat{\chi}(s)$ *is a cut-off function for the near field introduced in Sec. 3.2 and Fig. 2. Furthermore, there exist two positive constants* C *and* γ *independent of* ε *such that such that for* $n = 1, 2, 3$

$$\|\widehat{u}\,\widehat{\chi}^{(n)}\|_{L^2(\Gamma\times[0,\infty])} \leq C\exp(-\gamma/\varepsilon).$$

Proof. Since $u \in \Pi(\lambda, L^2(\Gamma))$ it holds $|u|^2 \in \Pi(2\lambda, L^1(\Gamma))$ and so

$$\int_0^\infty |u(t,S)|^2 \mathrm{d}S = -U(t,0)$$

where $U(t,S) \in \Pi(2\lambda, L^1(\Gamma))$ such that $\partial_S U(t,S) = |u(t,S)|^2$. Since $U(t,0) \in L^1(\Gamma)$

$$\|\widehat{u}\,\widehat{\chi}\|^2_{L^2(\Gamma\times[0,\infty])} = \int_\Gamma \int_0^\infty |\widehat{u}(t,s)|^2 \widehat{\chi}^2 \mathrm{d}s\,\mathrm{d}t \leq \varepsilon \int_\Gamma \int_0^\infty |u(t,S)|^2 \mathrm{d}S\,\mathrm{d}t \leq C\,\varepsilon.$$

The second part of the lemma can be easily proven as in Theorem 2.2 in Ref. 6. □

Now, we can prove the error estimate of the derived expansion.

Proof. [Proof of Lemma 2.2] The proof is in two parts, (i) for the sum of far and near field, (ii) for the far field approximation only.

(i) The asymptotic expansion solution $(\mathbf{v}^{\varepsilon,N}, p^{\varepsilon,N})$ has been defined in (2.6a) and (3.2), where the far field terms $(\mathbf{v}^j, p^j)$ solve (2.9) and the near field terms ϕ^j satisfy (3.6). By its definition (2.6a) and (3.2) $(\mathbf{v}^{\varepsilon,N}, p^{\varepsilon,N})$ solves (2.4b) exactly and $\mathbf{v}^{\varepsilon,N}$ has vanishing tangential component on $\partial\Omega$. The momentum equation (2.4b) is satisfied only up to the residual

$$R_1^\varepsilon(\mathbf{v}^{\varepsilon,N}, p^{\varepsilon,N}) - \mathbf{f}$$

which we are going to estimate in the $L^2(\Omega)$-norm. The contribution of the far field to the residual is

$$\begin{aligned}\mathbf{f}^{\varepsilon,N} &:= R_1^\varepsilon\Big(\sum\nolimits_{j=0}^N \varepsilon^j \mathbf{v}^j, \sum\nolimits_{j=0}^N \varepsilon^j p^j\Big) - \mathbf{f} \\ &= -\varepsilon^{N+1}\left(\eta_0 \Delta(\mathbf{v}^{N-1} + \varepsilon\mathbf{v}^N) - \eta_0' \nabla\,\mathrm{div}(\mathbf{v}^{N-1} + \varepsilon\mathbf{v}^N)\right),\end{aligned}$$

and so with some constant C independent of ε

$$\|\mathbf{f}^{\varepsilon,N}\|_{L^2(\Omega)} \leq C\,\varepsilon^{N+1}. \tag{4.19a}$$

The contribution of the near field is

$$\begin{aligned}\mathbf{f}^{\varepsilon,N}_{BL} := R_1^\varepsilon(\mathbf{v}^{\varepsilon,N}_{BL}, 0) = -\varepsilon\,\mathbf{curl}_{2D}\left(A_\varepsilon^3\,\varepsilon^{N+1} \sum_{j=0}^2 \sum_{\ell=1+j}^3 \varepsilon^{\ell-1-j} C_\ell(\phi^{N-j})\right)\chi(\mathbf{x}) \\ + \varepsilon^M \sum_{j=1}^2 \mathbf{U}_j^\varepsilon(t, \tfrac{s}{\varepsilon})\widehat{\chi}^{(j)}(s),\end{aligned}$$

where $\mathbf{U}_1^\varepsilon, \mathbf{U}_2^\varepsilon, \mathbf{U}_3^\varepsilon \in \Pi(\lambda_0, L^2(\Gamma))$ for all $\varepsilon \leq |\Omega|$, some integer $M \leq N+1$ and with a constant independent of ε. Note, that we use the local coordinates $t = t(\mathbf{x})$, $s = s(\mathbf{x})$, and $S = \frac{s}{\varepsilon}$, and that $\chi(\mathbf{x}) = \widehat{\chi}(s(\mathbf{x}))$ if the (smallest) distance to the wall is less than s_1 and 0 otherwise. Using Lemma 4.7 and 4.8 we can bound (remember, that $\varepsilon\,\mathbf{curl}_{2D}$ is of order 1)

$$\|\mathbf{f}_{BL}^{\varepsilon,N}\|_{L^2(\Omega)} \leq C\,\varepsilon^{N+\frac{3}{2}}. \tag{4.19b}$$

The near field is constructed such that in sum with the far field the tangential velocity component on the boundary is zero. The normal velocity component of $v^{\varepsilon,N}$ is with (3.10) given as

$$\mathbf{v}^{\varepsilon,N} \cdot \mathbf{n} = \varepsilon^N \mathbf{v}^N_{\mathrm{BL},\varepsilon}(t,0) \cdot \mathbf{n} = \varepsilon^{N+1} \partial_t \phi^N(t,0).$$

Since ϕ^N is bounded by derivatives of v_t^0, ..., v_t^N which are bounded by Lemma 4.6, we have

$$\|\mathbf{v}^{\varepsilon,N} \cdot \mathbf{n}\|_{H^{-1/2}(\partial\Omega)} \leq C\,\varepsilon^{N+1}.$$

Hence, by Corollary 4.1 we obtain

$$\|\mathbf{v}^\varepsilon - \mathbf{v}^{\varepsilon,N}\|_{H(\mathrm{div},\Omega)} + \varepsilon\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon - \operatorname{curl}_{2D}\mathbf{v}^{\varepsilon,N}\|_{L^2(\Omega)} + \|p^\varepsilon - p^{\varepsilon,N}\|_{H^1(\Omega)} \leq C\,\varepsilon^{N+1}.$$

We can improve the estimate for $\operatorname{curl}_{2D}\mathbf{v}^{\varepsilon,N}$. By Lemma 4.7 and Lemma 4.8 it is

$$\|(\varepsilon\operatorname{curl}_{2D}\varepsilon\,\mathbf{curl}_{2D}\,\phi^j)\chi\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}$$

and the same bound holds for $\varepsilon\operatorname{curl}_{2D}\varepsilon\,\mathbf{curl}_{2D}(\phi^j\chi)$ which differ from $(\varepsilon\operatorname{curl}_{2D}\varepsilon\,\mathbf{curl}_{2D}\,\phi^j)\chi$ only in terms in $\widehat{\chi}'$ and $\widehat{\chi}''$. Since $\operatorname{curl}_{2D}\mathbf{v}^j$ do not depend on ε for no $j \in \mathbb{N}_0$ we find the desired estimate

$$\begin{aligned}\sqrt{\varepsilon}&\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon - \operatorname{curl}_{2D}\mathbf{v}^{\varepsilon,N}\|_{L^2(\Omega)}\\ &\leq \sqrt{\varepsilon}\Big(\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon - \operatorname{curl}_{2D}\mathbf{v}^{\varepsilon,N+1}\|_{L^2(\Omega)} + \varepsilon^{N+1}\|\operatorname{curl}_{2D}\mathbf{v}^{N+1}\|_{L^2(\Omega)}\\ &\qquad + \varepsilon^{N+1}\|\operatorname{curl}_{2D}\varepsilon\,\mathbf{curl}_{2D}(\phi^{N+1}\chi)\|_{L^2(\Omega)}\Big) \leq C\,\varepsilon^{N+1}.\end{aligned}$$

(ii) We can use the triangle inequality and (2.7) to bound

$$\|\mathbf{v}^\varepsilon - \sum_{j=0}^N \varepsilon^j\mathbf{v}^j\|_{H(\mathrm{div},\Omega\setminus\overline{\Omega}_\delta)} + \sqrt{\varepsilon}\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon - \operatorname{curl}_{2D}\sum_{j=0}^N \varepsilon^j\mathbf{v}^j\|_{L^2(\Omega\setminus\overline{\Omega}_\delta)} \leq C\,\varepsilon^{N+1}$$

since the $L^2(\Omega \setminus \overline{\Omega}_\delta)$-norm of $\mathbf{v}_{BL}^{\varepsilon,N}$ decays faster than any order in ε, which can be shown analogously to the estimates with cut-off functions in Lemma 4.8. We can improve the result in the same way as in (i) as $\mathbf{v}^j$ and p^j do not depend on ε and so

$$\begin{aligned}\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon - \operatorname{curl}_{2D}\sum_{j=0}^{N}\varepsilon^j\mathbf{v}^j\|_{L^2(\Omega\setminus\overline{\Omega}_\delta)} &\le \|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon - \operatorname{curl}_{2D}\sum_{j=0}^{N+1}\varepsilon^j\mathbf{v}^j\|_{L^2(\Omega\setminus\overline{\Omega}_\delta)}\\ &\quad + \varepsilon^{N+1}\|\operatorname{curl}_{2D}\mathbf{v}^{N+1}\|_{L^2(\Omega\setminus\overline{\Omega}_\delta)} \le C\,\varepsilon^{N+1}.\end{aligned}$$

This completes the proof. □

5. Conclusion

In this article the multiscale behaviour of the acoustic velocity and pressure in viscous gases inside a bounded two-dimensional domain have been studied using the multiscale expansion, this for the case that the frequency is not an eigenfrequency of the limit problem of zero viscosity. The linearised Navier-Stokes equations are decoupled in equations for the velocity and pressure, where the pressure equation lacks a boundary condition. With the technique of multiscale expansion we could define a sequence of terms approximating the velocity and pressure for small viscosities, this separately inside and outside a $O(\sqrt{\eta})$-neighbourhood of the boundary. The derivation and mathematical justification of the expansion include curvature effects. We have derived the terms of the velocity and pressure expansion explicitly up to order 2, where the far field terms of the velocity and the pressure fulfil each partial differential equations in the whole domain. This differential equations include as boundary conditions a given normal component of the velocity (Dirichlet trace) or the normal derivative of the pressure (Neumann trace), respectively. The Neumann trace of the velocity or the Dirichlet trace of the pressure defines then the near field terms, which decay exponentially away from the boundary.

The asymptotic expansion to any order is rigorously justified by a stability and error analysis.

The presented study is a starting point to derive impedance conditions of higher order and can be extended to the case of resonances of the limit problem in bounded domains, to unbounded domains or to domains with non-smooth boundaries.

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Appendix A. Derivation of the boundary conditions to any order

The definition of the impedance boundary conditions is closely related to the solution of the ordinary equations (3.6) for the near field terms $\phi^j(t,S)$. We can decompose (3.6) into two problems, an ODE with a source term ψ, say, and homogeneous

Neumann data,

$$\eta_0(\lambda_0^2 + \partial_S^2)\phi(t,S) = \psi(t,S), \tag{A.1a}$$

$$\partial_S\phi(t,0) = 0, \tag{A.1b}$$

and into an ODE without source term and with Neumann boundary data $-v_t^j(t)$. For ψ of the form $\psi(t,S) = q(t,S)\mathrm{e}^{-\lambda_0 S}$ for some polynomial q in S the problem (A.1) has a unique (decaying) solution ϕ of the form $\phi(t,S) = p(t,S)\,\mathrm{e}^{-\lambda_0 S}$ for some polynomial p in S. Let L be the solution operator which maps the source term (A.1) to its solution, or equivalently, the polynomial q to the polynomial p.

Using L we can write the near field terms to any order as expression of the tangential trace of the far field terms as given in the following

Lemma A.1. *The near field terms of any order $j \in \mathbb{N}_0$ can be expressed as*

$$\phi^j(t,S) = \sum_{\ell=0}^{j}(E_\ell v_t^{j-\ell})(t,S), \tag{A.2}$$

where $E_j = 0$ for $j < 0$, $E_0 v = \frac{1}{\lambda_0}\mathrm{e}^{-\lambda_0 S}v$ and for $j > 0$ $E_j : C^\infty(\Gamma) \to C^\infty(\Gamma \times [0,\infty))$ are differential operators in t given by the recursion relation

$$(E_j v)(t,S) = \sum_{\ell=1}^{3}(L(C_\ell(E_{j-\ell}v)))(t,S). \tag{A.3}$$

Furthermore, $\phi^j(t,S)$ are polynomials in S multiplied by $\mathrm{e}^{-\lambda_0 S}$.

Proof. We verify the two conditions defining ϕ^j. First, the Neumann boundary conditions is fulfiled,

$$\partial_S\phi^j(t,0) = \sum_{\ell=0}^{j}\partial_S(E_\ell v_t^{j-\ell})(t,0) = \partial_S(E_0 v_t^j)(t,0) = -v_t^j(t,0),$$

since $\partial_S(L\psi)(t,0) = 0$ for any ψ by (A.1b). Second, with $i\omega\rho_0 + \eta_0\partial_S^2 = \eta_0(\lambda_0^2 + \partial_S^2)$ the ordinary differential equation is satisfied,

$$\begin{aligned}
&i\omega\rho_0\phi^j(t,S) + \eta_0\partial_S^2\phi^j(t,S) - \sum_{m=1}^{3}C_m(\phi^{j-m})(t,S)\\
&\qquad = \sum_{\ell=0}^{j}\eta_0(\lambda_0^2 + \partial_S^2)(E_\ell v_t^{j-\ell})(t,S) - (\sum_{m=1}^{3}C_m(\sum_{\ell=0}^{j-m}E_\ell v_t^{j-m-\ell}))(t,S)\\
&\qquad = \sum_{\ell=0}^{j}\eta_0(\lambda_0^2 + \partial_S^2)\sum_{m=1}^{3}(L(C_m(E_{\ell-m}v_t^{j-\ell})))(t,S)\\
&\qquad\quad - \sum_{m=1}^{3}(C_m(\sum_{\ell=0}^{j-m}E_\ell v_t^{j-m-\ell}))(t,S)\\
&\qquad = \sum_{m=1}^{3}(C_m(\sum_{\ell=0}^{j}E_{\ell-m}v_t^{j-\ell} - \sum_{\ell=0}^{j-m}E_\ell v_t^{j-m-\ell}))(t,S) = 0,
\end{aligned}$$

where we used $E_{\ell-m} = 0$ for $\ell < m$ and simple resorting.

That ϕ^j are polynomials in S times $\mathrm{e}^{-\lambda_0 S}$ follows by recursion in j and the properties of the solution operator L. □

Now, we are able to express the normal trace of the far field velocity by lower order tangential velocities to any order (see (3.10) up to order 2).

Corollary A.1. *The normal trace of the far field velocity of any order $j \in \mathbb{N}_0$ can be written as*

$$\mathbf{v}^j(t,0)\cdot\mathbf{n} = \sum_{\ell=1}^{j}(D_\ell v_t^{j-\ell})(t), \tag{A.4}$$

where $D_\ell = 0$ for $\ell \le 0$, $D_1(t) = -\frac{1}{\lambda_0}\partial_t$ and $D_\ell : C^\infty(\Gamma) \to C^\infty(\Gamma)$ for $\ell \ge 2$ are the differential operators on Γ

$$(D_\ell v)(t) = \partial_t(E_{\ell-1}v)(t,0). \tag{A.5}$$

Proof. Since $\mathbf{v}^j(t,0)\cdot\mathbf{n} = \sum_{\ell=1}^{j}(D_\ell v_t^{j-\ell})(t) = \sum_{\ell=0}^{j-1}\partial_t(E_\ell v_t^{j-1-\ell})(t) = \partial_t\phi^{j-1}(t)$ we have equivalence to (3.9). □

Now, we can represent the operators G_ℓ and H_ℓ in the expressions of the normal (Dirichlet) trace of the far field velocity in terms of the Neumann trace and the source on the boundary, see (2.9b) to any order. For this we use the Kronecker symbol δ_τ with truth values τ where $\delta_\tau = 1$ if τ is true and 0 otherwise.

Corollary A.2. *The relation* (2.9b) *holds with*

$$(G_\ell v)(t) = \Big(\big(-\frac{c^2}{\omega^2}E_{\ell-1} + \frac{\mathrm{i}(\eta_0+\eta_0')}{\omega\rho_0}E_{\ell-3}\big)\partial_t^2 v\Big)(t,0),$$

$$H_j(\mathbf{f}) = -\frac{\mathrm{i}}{\omega\rho_0}\sum_{\ell=1}^{j}E_{\ell-1}\Big(-\frac{\mathrm{i}\eta_0}{\omega\rho_0}\,\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\Big)^{\frac{j-\ell}{2}}\partial_t(\mathbf{f}\cdot\mathbf{n}^\perp)\cdot\delta_{j-\ell \textit{ is even}}.$$

Proof. Using (3.11), (A.5) and $\mathbf{e}_t = -\mathbf{n}^\perp$ we can write

$$\begin{aligned}
\mathbf{v}^j(t,0)\cdot\mathbf{n} &= \sum_{\ell=1}^{j}(D_\ell v_t^{j-\ell})(t) = \sum_{\ell=1}^{j}\partial_t(E_{\ell-1}v_t^{j-\ell})(t,0),\\
&= \sum_{\ell=1}^{j}\partial_t\Big(-\frac{c^2}{\omega^2}E_{\ell-1}\partial_t\,\mathrm{div}\,\mathbf{v}^{j-\ell} + \frac{\mathrm{i}(\eta_0+\eta_0')}{\omega\rho_0}E_{\ell-1}\partial_t\,\mathrm{div}\,\mathbf{v}^{j-\ell-2}\\
&\qquad - \frac{\mathrm{i}}{\omega\rho_0}E_{\ell-1}\Big(-\frac{\mathrm{i}\eta_0}{\omega\rho_0}\,\mathbf{curl}_{2D}\,\mathrm{curl}_{2D}\Big)^{\frac{j-\ell}{2}}\mathbf{f}\cdot\mathbf{n}^\perp\cdot\delta_{j-\ell\text{ is even}}\Big)(t,0)\\
&= \sum_{\ell=1}^{j}(G_\ell\,\mathrm{div}\,\mathbf{v}^{j-\ell})(t) + H_j(\mathbf{f}),
\end{aligned}$$

which is (2.9b). Here, we resorted the sum and used the fact that differential operators in t commute. □

Finally, we show the expressions for the boundary conditions of the far field pressure.

Lemma A.2. *The relation* (2.13b) *holds with*

$$J_\ell = -\frac{\omega^2}{c^2} \sum_{m=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \left(\frac{\mathrm{i}(\eta_0 + \eta_0')\omega}{\rho_0 c^2} \right)^m G_{\ell-2m},$$

$$K_j(\mathbf{f}) = \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} \left(\frac{\mathrm{i}(\eta_0 + \eta_0')\omega}{\rho_0 c^2} \right)^m \Bigg(\mathrm{i}\omega\rho_0 H_{j-2m}(\mathbf{f}) + \left(-\frac{\mathrm{i}\eta_0}{\omega\rho_0} \,\mathbf{curl}_{2D}\, \mathrm{curl}_{2D} \right)^{j-2m} \mathbf{f}\cdot\mathbf{n}\cdot \delta_{j-2m\geq 2}\delta_{j \text{ is even}} \Bigg).$$

Proof. Inserting (2.9b) and (2.12) into (3.12) we get

$$\nabla p^j \cdot \mathbf{n} = -\frac{\omega^2}{c^2} \sum_{\ell=1}^{j} G_\ell p^{j-\ell} + \mathrm{i}\omega\rho_0 H_j(\mathbf{f}) + (\eta_0 + \eta_0') \frac{\mathrm{i}\omega}{\rho_0 c^2} \nabla p^{j-2} \cdot \mathbf{n} + \left(-\frac{\mathrm{i}\eta_0}{\omega\rho_0} \,\mathbf{curl}_{2D}\, \mathrm{curl}_{2D} \right)^{j/2} \mathbf{f}\cdot\mathbf{n}\cdot\delta_{j \text{ is even}}.$$

Here we use the convention $p^j \equiv 0$ for $j < 0$. This boundary condition depends only on far field pressure terms of lower orders and the source term $\mathbf{f}$ on the boundary. For $j = 0$ with $H_0(\mathbf{f}) = 0$ we have the well-known limit condition (2.14) for the far field pressure. Now, inserting the expression for $\nabla p^{j-2} \cdot \mathbf{n}$ into that for $\nabla p^j \cdot \mathbf{n}$ and continuing with $\nabla p^{j-2m} \cdot \mathbf{n}$ up to $m = \lfloor \frac{j-1}{2} \rfloor$ and using $\nabla p^0 \cdot \mathbf{n} = \mathbf{f} \cdot \mathbf{n}$ leads to (2.13b) with operators J_ℓ and K_j defined in the lemma. □

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