Computing the Chow Variety of Quadratic Space Curves

Peter Bürgisser[1], Kathlén Kohn[1], Pierre Lairez[1], Bernd Sturmfels[1,2]

[1]Institute of Mathematics, Technische Universität Berlin
[2]Department of Mathematics, University of California, Berkeley

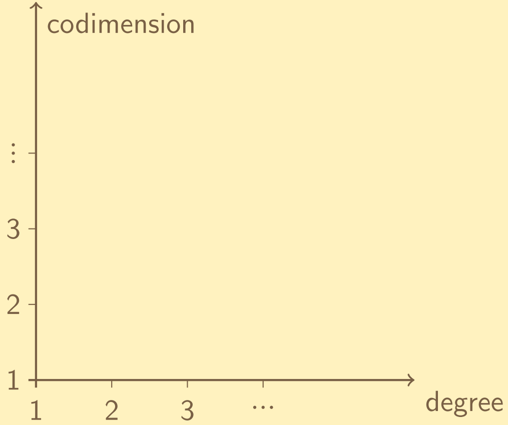
2015 – 11 – 12

Section 1

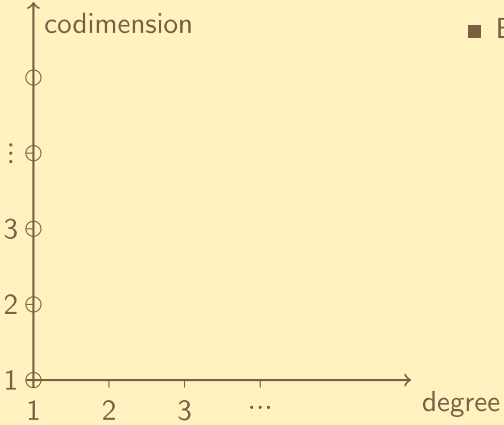
Problem Description

General question: How to parameterize subvarieties of $\mathbb{P}^{n-1}$ with fixed degree and dimension?

General question: How to parameterize subvarieties of $\mathbb{P}^{n-1}$ with fixed degree and dimension?

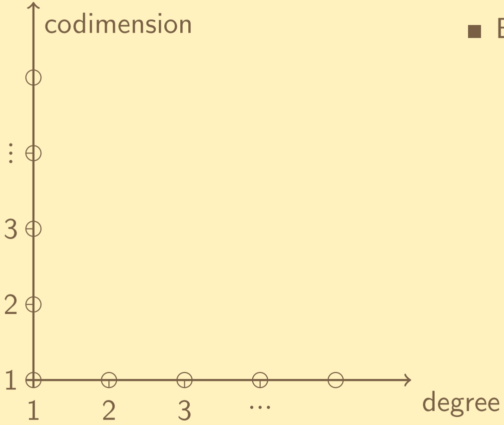


General question: How to parameterize subvarieties of $\mathbb{P}^{n-1}$ with fixed degree and dimension?



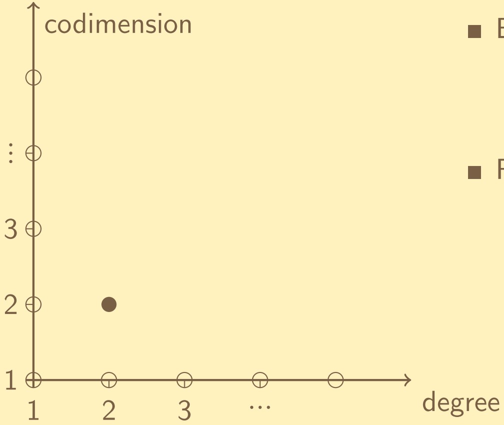
- Easy cases:
  - degree 1: Grassmannian

General question: How to parameterize subvarieties of $\mathbb{P}^{n-1}$ with fixed degree and dimension?



- Easy cases:
  - degree 1: Grassmannian
  - codimension 1: hypersurfaces with single equation

General question: How to parameterize subvarieties of $\mathbb{P}^{n-1}$ with fixed degree and dimension?



- Easy cases:
  - degree 1: Grassmannian
  - codimension 1: hypersurfaces with single equation
- First non-trivial case:
  - degree 2
  - codimension 2
  - $n = 4$

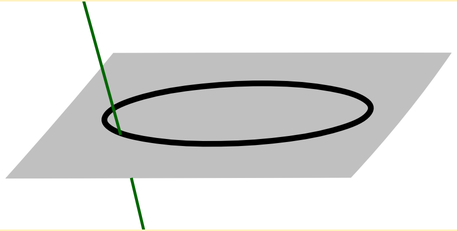
- $X \subseteq \mathbb{P}^3$ subvariety of dimension 1
- Grassmannian $G(2,4)$: Lines in $\mathbb{P}^3$

- $X \subseteq \mathbb{P}^3$ subvariety of dimension 1
- Grassmannian $G(2,4)$: Lines in $\mathbb{P}^3$

$L \in G(2,4)$

- $X \subseteq \mathbb{P}^3$ subvariety of dimension 1
- Grassmannian $G(2,4)$: Lines in $\mathbb{P}^3$

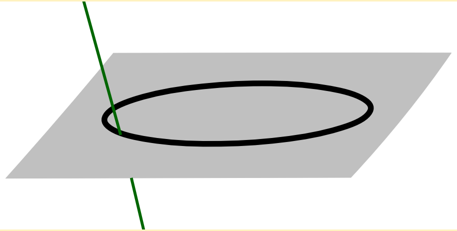
$\Rightarrow \{L \in G(2,4) \mid X \cap L \neq \emptyset\}$ is **hypersurface** in $G(2,4)$



Chow Variety

- $X \subseteq \mathbb{P}^3$ subvariety of dimension 1
- Grassmannian $G(2,4)$: Lines in $\mathbb{P}^3$

$\Rightarrow \{L \in G(2,4) \mid X \cap L \neq \emptyset\}$ is **hypersurface** in $G(2,4)$
$\Rightarrow \{L \in G(2,4) \mid X \cap L \neq \emptyset\}$ is defined by single polynomial in Plücker coordinates, called **Chow form**

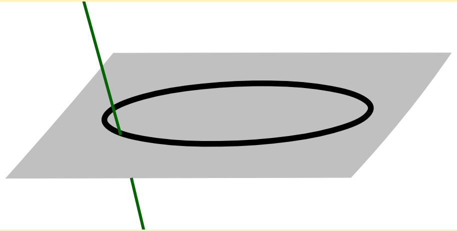


- $X \subseteq \mathbb{P}^3$ subvariety of dimension 1
- Grassmannian $G(2,4)$: Lines in $\mathbb{P}^3$

$\Rightarrow \{L \in G(2,4) \mid X \cap L \neq \emptyset\}$ is **hypersurface** in $G(2,4)$

$\Rightarrow \{L \in G(2,4) \mid X \cap L \neq \emptyset\}$ is defined by single polynomial in Plücker coordinates, called **Chow form**

$\Rightarrow$ Goal: Which polynomials in Plücker coordinates are Chow forms?



Chow Variety

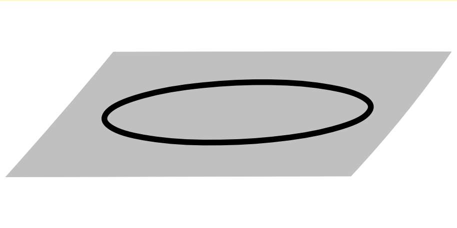
- $X \subseteq \mathbb{P}^3$ subvariety of dimension 1
- Grassmannian $G(2,4)$: Lines in $\mathbb{P}^3$

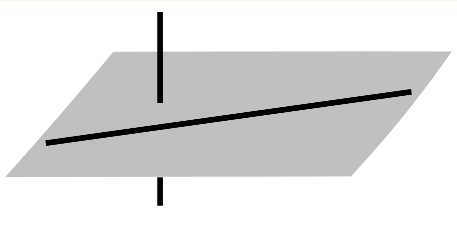
$\Rightarrow \{L \in G(2,4) \mid X \cap L \neq \varnothing\}$ is **hypersurface** in $G(2,4)$

$\Rightarrow \{L \in G(2,4) \mid X \cap L \neq \varnothing\}$ is defined by single polynomial in Plücker coordinates, called **Chow form**

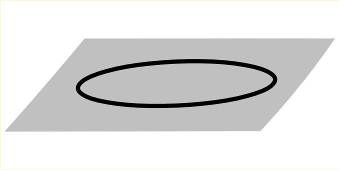
$\Rightarrow$ Goal: Which polynomials in Plücker coordinates are Chow forms?

- Chow variety $G(2,2,4)$: Chow forms of 1-dimensional subvarieties of degree 2 in $\mathbb{P}^3$





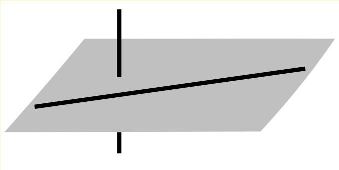
Chow variety $G(2,2,4)$: Chow forms of 1-dimensional subvarieties of degree 2 in $\mathbb{P}^3$



$G(2,2,4)$ has 2 irreducible components, corresponding to:

- planar quadrics
- pairs of lines

$\Rightarrow G(2,2,4) = V(P_{\text{ChowConic}}) \cup V(P_{\text{ChowLines}})$

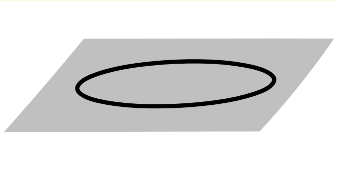


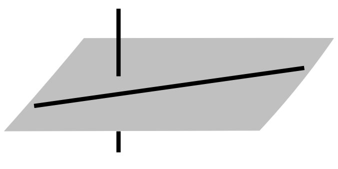
Chow variety $G(2,2,4)$: Chow forms of 1-dimensional subvarieties of degree 2 in $\mathbb{P}^3$

$G(2,2,4)$ has 2 irreducible components, corresponding to:

- planar quadrics
- pairs of lines

$\Rightarrow G(2,2,4) = V(P_{\text{ChowConic}}) \cup V(P_{\text{ChowLines}})$





Wanted: $I_{G(2,2,4)} = P_{\text{ChowConic}} \cap P_{\text{ChowLines}}$

- Our point of departure: Book by Gel'fand, Kapranov, Zelevinsky
- They describe equations that discriminate Chow forms among all hypersurfaces in the Grassmannian
  1. Find coisotropic hypersurfaces
  2. Find Chow forms among coisotropic hypersurfaces

How to find $I_{G(2,2,4)}$

- Our point of departure: Book by Gel'fand, Kapranov, Zelevinsky
- They describe equations that discriminate Chow forms among all hypersurfaces in the Grassmannian
  1. Find coisotropic hypersurfaces
  2. Find Chow forms among coisotropic hypersurfaces
- Here:
  - Check and refine equations from Gel'fand, Kapranov, Zelevinsky
  - Explicit computational solution for $G(2,2,4)$

How to find $I_{G(2,2,4)}$

- Our point of departure: Book by Gel'fand, Kapranov, Zelevinsky
- They describe equations that discriminate Chow forms among all hypersurfaces in the Grassmannian
  1. Find coisotropic hypersurfaces
  2. Find Chow forms among coisotropic hypersurfaces
- Here:
  - Check and refine equations from Gel'fand, Kapranov, Zelevinsky
  - Explicit computational solution for $G(2,2,4)$:
    Chow forms are quadrics in the Plücker coordinates of $G(2,4)$

Section 2

Coisotropic Quadrics

- Represent points in $G(2,4)$ by Plücker coordinates $\boldsymbol{p} = (p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$:
  - For a line in $\mathbb{P}^3$, p_{ij} is ij-minor of a 2×4-matrix whose rows span the line
  - Plücker relation: $\mathcal{R} := p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$

- Represent points in $G(2,4)$ by Plücker coordinates $\boldsymbol{p} = (p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$:
  - For a line in $\mathbb{P}^3$, p_{ij} is ij-minor of a 2×4-matrix whose rows span the line
  - Plücker relation: $\mathcal{R} := p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$
- Write quadrics in $G(2,4)$ as

$$Q(\boldsymbol{p}) \;=\; \boldsymbol{p} \cdot \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_1 & c_6 & c_7 & c_8 & c_9 & c_{10} \\ c_2 & c_7 & c_{11} & c_{12} & c_{13} & c_{14} \\ c_3 & c_8 & c_{12} & c_{15} & c_{16} & c_{17} \\ c_4 & c_9 & c_{13} & c_{16} & c_{18} & c_{19} \\ c_5 & c_{10} & c_{14} & c_{17} & c_{19} & c_{20} \end{pmatrix} \cdot \boldsymbol{p}^T$$

  - $Q(\boldsymbol{p}) \in V := \mathbb{C}[\boldsymbol{p}]_2 / \mathbb{C}\mathcal{R}$

- Represent points in $G(2,4)$ by Plücker coordinates $\boldsymbol{p} = (p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$:
  - For a line in $\mathbb{P}^3$, p_{ij} is ij-minor of a 2×4-matrix whose rows span the line
  - Plücker relation: $\mathcal{R} := p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$
- Write quadrics in $G(2,4)$ as

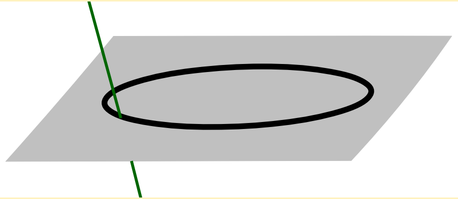
$$Q(\boldsymbol{p}) \;=\; \boldsymbol{p} \cdot \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_1 & c_6 & c_7 & c_8 & c_9 & c_{10} \\ c_2 & c_7 & c_{11} & c_{12} & c_{13} & c_{14} \\ c_3 & c_8 & c_{12} & c_{15} & c_{16} & c_{17} \\ c_4 & c_9 & c_{13} & c_{16} & c_{18} & c_{19} \\ c_5 & c_{10} & c_{14} & c_{17} & c_{19} & c_{20} \end{pmatrix} \cdot \boldsymbol{p}^T$$

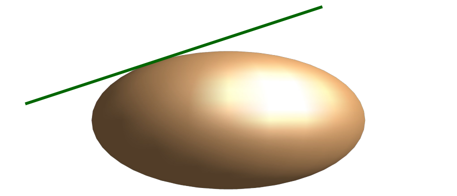
  - $Q(\boldsymbol{p}) \in V := \mathbb{C}[\boldsymbol{p}]_2 / \mathbb{C}\mathcal{R}$
  - $\boldsymbol{c} = (c_0, c_1, \ldots, c_{20})$ homogeneous coordinates on $\mathbb{P}^{19} = \mathbb{P}(V)$

$\Rightarrow G(2,2,4) \subseteq \mathbb{P}^{19}$

Coisotropic Ideal

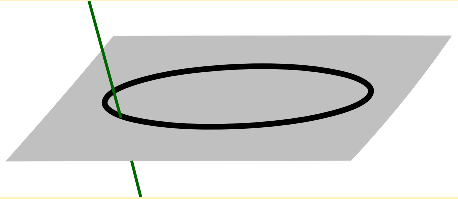
- Irreducible hypersurface $Z \subseteq G(2,4)$ is coisotropic if it is
  - the Chow form of a quadratic space curve, OR
  - the Hurwitz form of a quadric surface, i.e., all lines tangent to surface

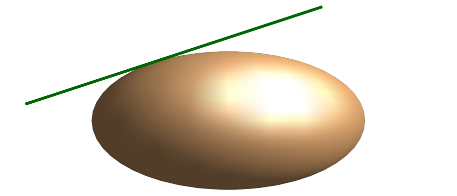




Coisotropic Ideal

- Irreducible hypersurface $Z \subseteq G(2,4)$ is coisotropic if it is
  - the Chow form of a quadratic space curve, OR
  - the Hurwitz form of a quadric surface, i.e., all lines tangent to surface



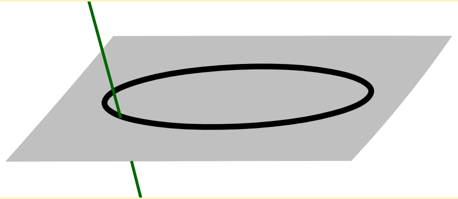


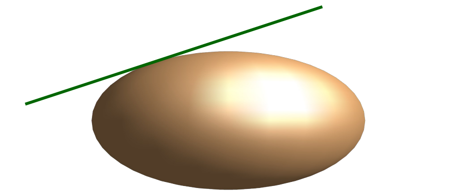
- $\{Q(\boldsymbol{p}) = 0\}$ coisotropic iff there exist $s, t \in \mathbb{C}$ such that

$$\frac{\partial Q}{\partial p_{01}} \cdot \frac{\partial Q}{\partial p_{23}} - \frac{\partial Q}{\partial p_{02}} \cdot \frac{\partial Q}{\partial p_{13}} + \frac{\partial Q}{\partial p_{03}} \cdot \frac{\partial Q}{\partial p_{12}} = s \cdot Q + t \cdot \mathcal{R}$$

for $\mathcal{R} := p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$ [Cayley, 1860]

- Irreducible hypersurface $Z \subseteq G(2,4)$ is coisotropic if it is
  - the Chow form of a quadratic space curve, OR
  - the Hurwitz form of a quadric surface, i.e., all lines tangent to surface





- $\{Q(\boldsymbol{p}) = 0\}$ coisotropic iff there exist $s, t \in \mathbb{C}$ such that

$$\frac{\partial Q}{\partial p_{01}} \cdot \frac{\partial Q}{\partial p_{23}} - \frac{\partial Q}{\partial p_{02}} \cdot \frac{\partial Q}{\partial p_{13}} + \frac{\partial Q}{\partial p_{03}} \cdot \frac{\partial Q}{\partial p_{12}} = s \cdot Q + t \cdot \mathcal{R}$$

for $\mathcal{R} := p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$ [Cayley, 1860]
$\rightsquigarrow$ **coisotropic ideal** I_{Coiso}

- $V(I_{\text{Coiso}}) \subseteq \mathbb{P}^{19}$ represents all coisotropic hypersurfaces in $G(2,4)$ of degree 2

Proposition (Bürgisser, K., Lairez, Sturmfels**)**

I_{Coiso} is intersection of 3 prime ideals and thus radical:

$$I_{\mathrm{Coiso}} \;=\; P_{\mathrm{Hurwitz}} \cap P_{\mathrm{ChowLines}} \cap P_{\mathrm{Squares}}$$

- *$V(P_{\mathrm{Hurwitz}})$: Hurwitz forms of quadric surfaces in $\mathbb{P}^3$*
- *$V(P_{\mathrm{ChowLines}})$: Chow forms of pairs of lines in $\mathbb{P}^3$*
- *$V(P_{\mathrm{Squares}})$: quadrics that are squares modulo Plücker relation*

Proposition (Bürgisser, K., Lairez, Sturmfels**)**

I_{Coiso} is intersection of 3 prime ideals and thus radical:

$$I_{\mathrm{Coiso}} = P_{\mathrm{Hurwitz}} \cap P_{\mathrm{ChowLines}} \cap P_{\mathrm{Squares}}$$

- $V(P_{\mathrm{Hurwitz}})$*: Hurwitz forms of quadric surfaces in* $\mathbb{P}^3$
- $V(P_{\mathrm{ChowLines}})$*: Chow forms of pairs of lines in* $\mathbb{P}^3$
- $V(P_{\mathrm{Squares}})$*: quadrics that are squares modulo Plücker relation*

- Geometric perspective: P_{Squares} extraneous
  $\Rightarrow$ correct ideal for coisotropic variety:

$$P_{\mathrm{Hurwitz}} \cap P_{\mathrm{ChowLines}} = (I_{\mathrm{Coiso}} : P_{\mathrm{Squares}})$$

Proposition (Bürgisser, K., Lairez, Sturmfels)

I_{Coiso} is intersection of 3 prime ideals and thus radical:

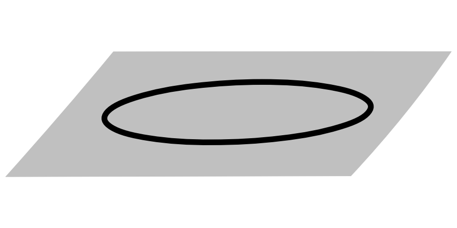
$$I_{\mathrm{Coiso}} = P_{\mathrm{Hurwitz}} \cap P_{\mathrm{ChowLines}} \cap P_{\mathrm{Squares}}$$

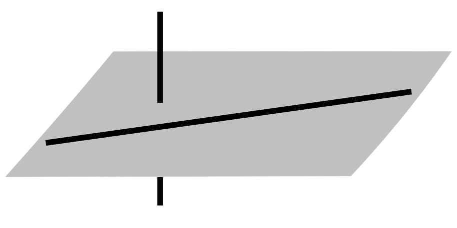
	I	P_{Hurwitz}	$P_{\mathrm{ChowLines}}$	P_{Squares}
codimension	10	10	11	14
degree	92	92	140	32
minimally generated	175 cubics	20 quadrics	265 cubics	84 quadrics

Section 3

Chow Forms

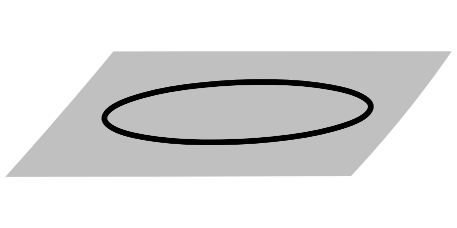
$$G(2,2,4) = V(P_{\text{ChowConic}}) \cup V(P_{\text{ChowLines}})$$

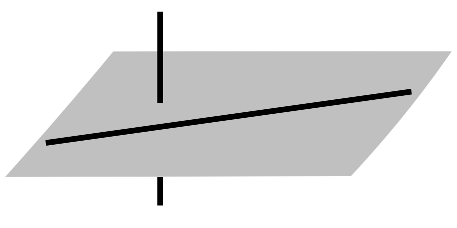




$G(2,2,4)$

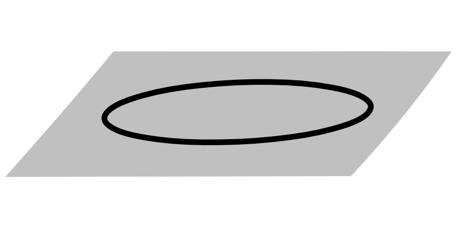
$$V(I_{\mathrm{Coiso}}) \supseteq G(2,2,4) = V(P_{\mathrm{ChowConic}}) \cup V(P_{\mathrm{ChowLines}})$$

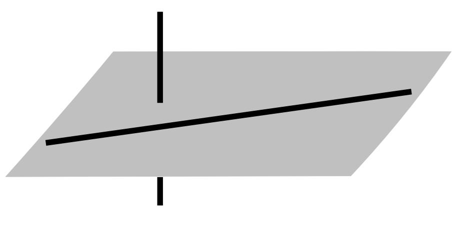




$G(2,2,4)$

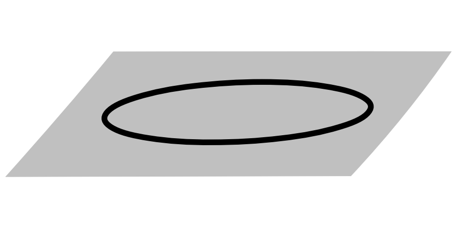
$$V(I_{\text{Coiso}}) \supseteq G(2,2,4) = V(P_{\text{ChowConic}}) \cup V(P_{\text{ChowLines}})$$

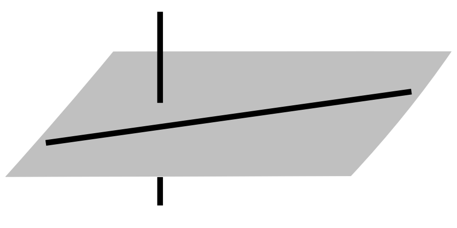




$$I_{\text{Coiso}} = P_{\text{Hurwitz}} \cap P_{\text{ChowLines}} \cap P_{\text{Squares}}$$

$$V(I_{\mathrm{Coiso}}) \supseteq G(2,2,4) = V(P_{\mathrm{ChowConic}}) \cup V(P_{\mathrm{ChowLines}})$$

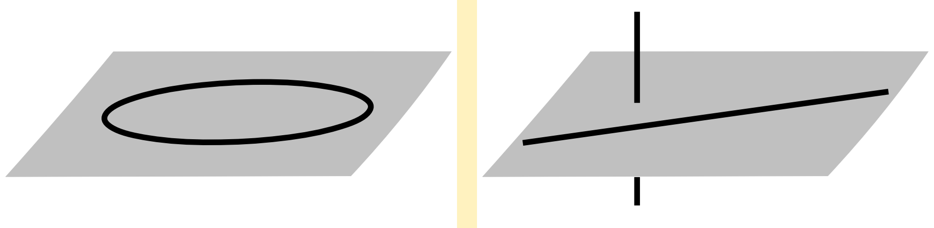




$$I_{\mathrm{Coiso}} = P_{\mathrm{Hurwitz}} \cap P_{\mathrm{ChowLines}} \cap P_{\mathrm{Squares}}$$

$\Rightarrow P_{\mathrm{ChowConic}}$ missing from this decomposition?

$$V(I_{\mathrm{Coiso}}) \supseteq G(2,2,4) = V(P_{\mathrm{ChowConic}}) \cup V(P_{\mathrm{ChowLines}})$$



$$I_{\mathrm{Coiso}} \;=\; P_{\mathrm{Hurwitz}} \cap P_{\mathrm{ChowLines}} \cap P_{\mathrm{Squares}}$$

$\Rightarrow P_{\mathrm{ChowConic}}$ missing from this decomposition?

Proposition (Bürgisser, K., Lairez, Sturmfels**)**

$P_{\mathrm{Hurwitz}} \subseteq P_{\mathrm{ChowConic}}$ *and thus* $V(P_{\mathrm{ChowConic}}) \subseteq V(P_{\mathrm{Hurwitz}})$

Squarefree coisotropic quadric Q is a Chow form iff certain differential forms vanish modulo Q [Gel'fand, Kapranov, Zelevinsky]

Squarefree coisotropic quadric Q is a Chow form iff certain differential forms vanish modulo Q [Gel'fand, Kapranov, Zelevinsky]

$\rightsquigarrow$ $\boldsymbol{I_{G(2,2,4)}}$

$\Rightarrow V(I_{G(2,2,4)})$ equals $G(2,2,4)$ up to non-reduced surfaces

Squarefree coisotropic quadric Q is a Chow form iff certain differential forms vanish modulo Q [Gel'fand, Kapranov, Zelevinsky]

$\rightsquigarrow$ $\boldsymbol{I_{G(2,2,4)}}$

$\Rightarrow V(I_{G(2,2,4)})$ equals $G(2,2,4)$ up to non-reduced surfaces

Proposition (Bürgisser, K., Lairez, Sturmfels**)**

Let $\mathfrak{m} := \langle c_0, c_1, \ldots, c_{20} \rangle$ *be the irrelevant ideal, then*

$$\sqrt{I_{G(2,2,4)}} = (I_{G(2,2,4)} : \mathfrak{m}) = P_{\mathrm{ChowConic}} \cap P_{\mathrm{ChowLines}} \cap P_{\mathrm{Squares}}$$

Squarefree coisotropic quadric Q is a Chow form iff certain differential forms vanish modulo Q [Gel'fand, Kapranov, Zelevinsky]
$\rightsquigarrow$ $\boldsymbol{I_{G(2,2,4)}}$
$\Rightarrow V(I_{G(2,2,4)})$ equals $G(2,2,4)$ up to non-reduced surfaces

Proposition (Bürgisser, K., Lairez, Sturmfels**)**

Let $\mathfrak{m} := \langle c_0, c_1, \ldots, c_{20} \rangle$ *be the irrelevant ideal, then*

$$\sqrt{I_{G(2,2,4)}} = (I_{G(2,2,4)} : \mathfrak{m}) = P_{\mathrm{ChowConic}} \cap P_{\mathrm{ChowLines}} \cap P_{\mathrm{Squares}}$$

Recall: $$I_{\mathrm{Coiso}} = P_{\mathrm{Hurwitz}} \cap P_{\mathrm{ChowLines}} \cap P_{\mathrm{Squares}}$$

$$P_{\mathrm{Hurwitz}} \subseteq P_{\mathrm{ChowConic}}$$

Future Directions

- Compute ideals of Chow varieties with higher degree and/or dimension
- Which Chow forms are Hurwitz forms, and which Hurwitz forms are Chow forms?
- Compute volume of ε-tubes around coisotropic hypersurfaces
- Generalize Cayley's differential characterization of coisotropy

$$\frac{\partial Q}{\partial p_{01}} \cdot \frac{\partial Q}{\partial p_{23}} - \frac{\partial Q}{\partial p_{02}} \cdot \frac{\partial Q}{\partial p_{13}} + \frac{\partial Q}{\partial p_{03}} \cdot \frac{\partial Q}{\partial p_{12}} = s \cdot Q + t \cdot \mathcal{R}$$

  from $G(2,4)$ to $G(2,n)$
- Catanese, 2014: Hypersurface $Z \subseteq G(2,4)$ coisotropic iff Z selfdual $\Rightarrow$ Generalize to $G(2,n)$

Thank you!

For our computations, check
www3.math.tu-berlin.de/algebra/static/pluecker/