Outerplanar graph drawings with few slopes

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Abstract

We prove that every outerplanar graph with maximum degree $\Delta \geqslant 4$ has a straight-line outerplanar drawing with at most $\Delta - 1$ distinct edge slopes. This bound is tight: for every $\Delta \geqslant 4$ there is an outerplanar graph of maximum degree Δ which requires at least $\Delta - 1$ distinct edge slopes for its outerplanar straight-line drawing.

1 Introduction

A *straight-line drawing* of a graph G is a mapping of the vertices of G to distinct points in the plane and of the edges of G to straight-line segments connecting the points representing their end-vertices and passing through no other points representing vertices. If it leads to no confusion, in notation and terminology, we make no distinction between a vertex and the corresponding point, and between an edge and the corresponding segment. The *slope* of an edge in a straight-line drawing is the family of all straight lines parallel to the segment representing this edge. The *slope number* of a graph G, introduced by Wade and Chu [8], is the smallest number s such that there is a straight-line drawing of G using s slopes.

Since at most two edges at each vertex can use the same slope, $\lceil \frac{\Delta}{2} \rceil$ is a lower bound for the slope number of a graph with maximum degree Δ. In general, graphs with maximum degree $\Delta \geqslant 5$ may have arbitrarily large slope number, see [1, 7]. If the maximum degree of a graph is at most 3 then the slope number is at most 4 as shown by Mukkamala and Szegedy [6], improving a result of Keszegh et al. [4]. The question whether the slope number of graphs with maximum degree 4 is bounded by a constant remains open.

The situation is different for *planar* straight-line drawings. It is well known that every planar graph admits a planar straight-line drawing [5]. The *planar slope number* of a planar graph G is the smallest number s such that there is a planar straight-line drawing of G using s slopes. In [3] Keszegh et al. show that the planar slope number is bounded by a function of maximum degree. Their bound is exponential and their proof is non-constructive. Jelínek et al. [2] give an upper bound for the planar slope number of planar graphs of treewidth at most 3, which is $O(\Delta^5)$.

In the present paper we consider drawings of outerplanar graphs. As outerplanar graphs have treewidth at most 2, they admit a planar drawings with $O(\Delta^5)$ slopes. A straight-line drawing of a graph G is *outerplanar* if it is planar and all vertices of G lie on the outer face. The *outerplanar slope number* of an outerplanar graph G is the smallest number s such that there is an outerplanar straight-line drawing of G using s slopes. We provide a tight bound for the outerplanar slope number in terms of the maximum degree.

Theorem 1 *The outerplanar slope number of every outerplanar graph with maximum degree $\Delta \geqslant 4$ is at most $\Delta - 1$.*

This result is sharp, as witnessed by a long cycle where each vertex is made adjacent to $\Delta - 2$ additional independent vertices. The tight bounds for the outerplanar slope number with respect to the maximum degree Δ are therefore: 1 for $\Delta = 1$, 3 for $\Delta \in \{2, 3\}$, and $\Delta - 1$ for $\Delta \geqslant 4$.

The proofs of Theorem 1 are somewhat different for $\Delta = 4$ and $\Delta \geqslant 5$. In the following we sketch the ideas of the proof for $\Delta \geqslant 5$.

2 Bubbles

Suppose we are given an outerplanar drawing of a connected graph G with maximum degree $\Delta \geqslant 5$. This drawing determines the cyclic ordering of edges around every vertex. We produce an outerplanar straight-line drawing of G with few edge slopes which preserves this ordering at every vertex. Our construction is inductive: it composes the entire drawing of G from drawings of subgraphs of G that we call bubbles.

We distinguish the *outer face* of G (the one that is unbounded in the given drawing of G and contains all vertices on the boundary) from the *inner faces*. The edges on the boundary of the former are *outer edges*, while all remaining ones are *inner edges*. A *snip* is a simple closed counterclockwise-oriented curve γ which

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- passes through some pair of vertices u and v of G (possibly being the same vertex) and through no other vertex of G,
- on the way from v to u goes entirely through the outer face of G,
- on the way from u to v (considered only if $u \neq v$) goes through inner faces of G possibly crossing some inner edges of G, each at most once.

Every snip γ defines a *bubble* H in G as the subgraph of G induced on the vertices lying on or inside γ. Note that H is a connected subgraph of G as γ crosses no outer edges. The oriented simple path P from u to v in H going counterclockwise along the boundary of the outer face of H is called the *root-path* of H. If $u = v$ then the root-path consists of that single vertex only. The *roots* of H are the vertices u and v together with all vertices of H incident to the edges crossed by γ. Note that vertices of H not being roots cannot have edges to $G - H$. Note also that the root-path and the roots of H do not depend on the particular snip γ used to define H. The order of roots along the root-path gives the *root-sequence* of H. A bubble with k roots is called a *k-bubble*. A special role in our proof is played by 1- and 2-bubbles.

Bubbles admit a natural decomposition, which is the base of our recursive drawing.

Lemma 2 *Let H be a bubble with root-path $P = v_1 \dots v_k$. Every component of $H - \{v_1, \dots, v_k\}$ is adjacent to either one vertex among $v_1, \dots, v_k$ or two consecutive vertices from $v_1, \dots, v_k$. Moreover, there is at most one component adjacent to v_i and v_{i+1} for all $1 \leqslant i < k$.*

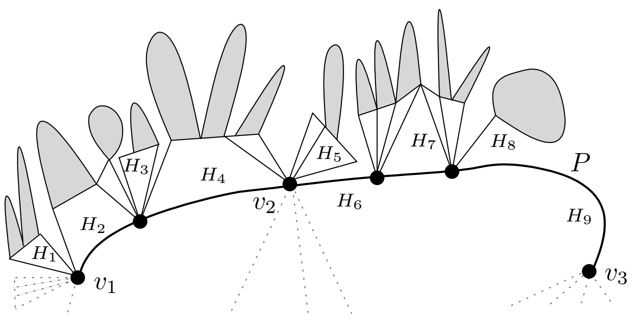

Figure 1: A 3-bubble H with root path P (drawn thick), root-sequence (v_1, v_2, v_3) (connected to the remaining graph by dotted edges), and splitting sequence into v- and e-bubbles $(H_1, \dots, H_9)$. For example, (H_2, H_3, H_4) is a 2-bubble.

Lemma 2 allows us to assign each component of $H - \{v_1, \dots, v_k\}$ to a vertex of P or an edge of P so that every edge is assigned at most one component. For a component C assigned to a vertex v_i, the graph induced on $C \cup \{v_i\}$ is called a *v-bubble*. If P consists of a single vertex with no component assigned to it, we consider that vertex alone to be a v-bubble. For a component C assigned to an edge $v_i v_{i+1}$, the graph induced on $C \cup \{v_i, v_{i+1}\}$ is called an *e-bubble*. If no component is assigned to an edge of P then we consider that edge alone an e-bubble. All v-bubbles of v_i in H are naturally ordered by their clockwise arrangement around v_i in the drawing. All this leads to a decomposition of the bubble H into a sequence $(H_1, \dots, H_b)$ of v- and e-bubbles such that the naturally ordered v-bubbles of v_1 precede the e-bubble of $v_1 v_2$, which precedes the naturally ordered v-bubbles of v_2, and so on. We call this sequence the *splitting sequence* of H and write $H = (H_1, \dots, H_b)$. Note that v- and e-bubbles are special kinds of 1- and 2-bubbles respectively. Every 1-bubble is a bouquet of v-bubbles. The splitting sequence of a 2-bubble may consist of several v- and e-bubbles. For an illustration see Figure 1.

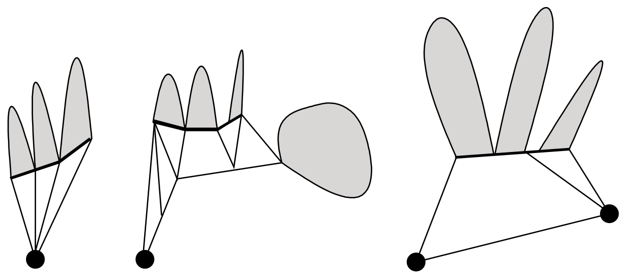

Figure 2: Three ways of obtaining smaller bubbles from v- and e-bubbles. The new root-path is drawn thick.

The general structure of the induction in our proof is covered by the following (see Figure 2):

Lemma 3

3.1. Let H be a v-bubble rooted at v and let $v_1, \dots, v_k$ be the neighbors of v in H in clockwise order. Then $H - v$ is a k-bubble with root-sequence $v_1, \dots, v_k$.

3.2. Let H be a v-bubble rooted at v. Let P be an induced path going from v counterclockwise along the outer face of H to a vertex w of H, such that the internal vertices of P are not cut-vertices of H. Let H' be the component of $H - P$ adjacent to both v and w. Then H' is a bubble with roots being the neighbors of P in H'.

3.3. Let H be an e-bubble with root-edge uv. Suppose that $u_1, \dots, u_k, v$ are the neighbors of u in H in clockwise order and $u, v_1, \dots, v_\ell$ are the neighbors of v in H in clockwise order. Then $H -$

$\{u, v\}$ is a bubble with root-sequence $u_1, \dots, u_k$, $v_1, \dots, v_\ell$, where u_k and v_1 may coincide.

3 Bounding regions

Depending on the maximum degree Δ of G we define the set S of $\Delta - 1$ slopes to consist of the horizontal slope and the slopes of vectors $\mathbf{f}_1, \dots, \mathbf{f}_{\Delta-2}$ where

$$\mathbf{f}_i = (-\tfrac{1}{2} + \tfrac{i-1}{\Delta-3}, 1) \quad \text{for } i = 1, \dots, \Delta - 2.$$

An important property of S is that it cuts the horizontal segment from $(-\frac{1}{2}, 1)$ to $(\frac{1}{2}, 1)$ into $\Delta - 3$ segments of equal length $\frac{1}{\Delta-3}$. We construct an outerplanar straight-line drawing of G using only slopes from S and preserving the given cyclic ordering of edges at each vertex of G.

The essential tool in proving that our construction does not make bubbles overlap are bounding regions. Their role is to bound the regions occupied by bubbles. The bounding region for a bubble is parametrized by ℓ and r which depend on the degrees of the roots in the bubble. Let v be a point in the plane. For a vector x let $R(v; x) = \{v + \alpha x : \alpha \geqslant 0\}$. We define $LB(v; \ell)$ to be the set consisting of v and of all points p such that $p_y \geqslant v_y$. If $\ell > 0$ then we furthermore require that p lies

- to the right of $R(v; \mathbf{f}_{\Delta-2})$ if $\ell = \Delta - 1$,
- to the right of $R(v; \mathbf{f}_\ell + \frac{1}{\Delta-4}\mathbf{f}_1)$ if $2 \leqslant \ell \leqslant \Delta - 2$,
- on or to right of $R(v; \mathbf{f}_1)$ if $\ell = 1$.

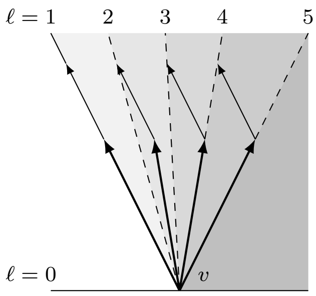

Figure 3: Boundaries of $LB(v; \ell)$ for $\Delta = 6$. Vectors $\mathbf{f}_i$ at v are indicated by thick arrows. Vectors $\frac{1}{2}\mathbf{f}_1$ at $v + \mathbf{f}_i$ are indicated by thin arrows. Note that $\mathbf{f}_3$ lies on the boundary of $LB(v; 4)$.

See Figure 3 for an illustration. Similarly, $RB(v; r)$ consists of v and of all points p such that $p_y \geqslant v_y$. If $r < \Delta - 1$ we furthermore require that p lies

- to the left of $R(v; \mathbf{f}_1)$ if $r = 0$,
- to the left of $R(v; \mathbf{f}_r + \frac{1}{\Delta-4}\mathbf{f}_{\Delta-2})$ if $1 \leqslant r \leqslant \Delta - 3$,
- on or to the left of $R(v; \mathbf{f}_{\Delta-2})$ if $r = \Delta - 2$.

Now, for points u, v in the plane such that $u_y = v_y$ and $u_x \leqslant v_x$ we define bounding regions as follows: if $0 \leqslant \ell, r \leqslant \Delta - 1$ then $B(uv; \ell, r) = LB(u; \ell) \cap RB(v; r)$ and if additionally a height $h > 0$ is specified then $\bar{B}(uv; \ell, r; h) = B(uv; \ell, r) \cap \{p : p_y < u_y + h\}$.

We denote $B(vv; \ell, r)$ simply by $B(v; \ell, r)$ and $\bar{B}(vv; \ell, r; h)$ simply by $\bar{B}(v; \ell, r; h)$. We use $B(v; \ell, r)$ and $\bar{B}(v; \ell, r; h)$ to bound drawings of 1-bubbles H with root v such that $r - \ell + 1 = d_H(v)$. Note that every 1-bubble drawn inside $B(v; \ell, r)$ may be scaled to fit inside $\bar{B}(v; \ell, r; h)$ for any $h > 0$ without changing slopes. We use $\bar{B}(uv; \ell, r; h)$ with $u \neq v$ to bound drawings of 2-bubbles H whose root-path starts at u and ends at v, such that $\ell = \Delta - d_H(u)$ and $r = d_H(v) - 1$. Here H cannot be scaled as the positions of two of its vertices are fixed, so the value of h matters. We also use $\bar{B}(uv; 1, \Delta - 2; h)$ with $u \neq v$ to bound drawings of bubbles with any number of roots and with root-path starting at u and ending at v.

4 The drawing

The following lemma does the main job in the proof of Theorem 1 for $\Delta \geqslant 5$.

Lemma 4 *Suppose $\Delta \geqslant 5$.*

4.1. Let H be a 1-bubble with root v. Suppose that the position of v is fixed. Let ℓ and r be such that $0 \leqslant \ell, r \leqslant \Delta - 1$ and $r - \ell + 1 = d_H(v)$. Then there is a straight-line drawing of H inside $B(v; \ell, r)$.

4.2. Let H be a 2-bubble with first root u and last root v. Suppose that the positions of u and v are fixed on a horizontal line so that u lies to the left of v. Let $\ell = \Delta - d_H(u)$ and $r = d_H(v) - 1$. Then there is a straight-line drawing of H inside $\bar{B}(uv; \ell, r; \frac{\Delta-3}{\Delta-4}|uv|)$ such that the root-path of H is drawn as the segment uv.

4.3. Let H be a k-bubble with roots $v_1, \dots, v_k$ in this order along the root-path. If $k = 1$ then suppose $d_H(v_1) \leqslant \Delta - 2$, otherwise suppose $d_H(v_1), d_H(v_k) \leqslant \Delta - 1$. Suppose that for some $\lambda > 0$ the positions of $v_1, \dots, v_k$ are fixed in this order on a horizontal line so that $|v_1 v_2| = \dots = |v_{k-1} v_k| = \lambda$. Then there is a straight-line drawing of H inside $\bar{B}(v_1 v_k; 1, \Delta - 2; \frac{\Delta-3}{\Delta-4}\lambda)$ such that the root-path of H is drawn as the segment $v_1 v_k$.

The drawings claimed above use only slopes from S and preserve the order of edges around each vertex w of H under the assumption that if there are edges connecting w to $G - H$ then they are drawn in the correct order outside the considered bounding region.

Proof. The proof of all three statements goes by one induction on the size of the bubble H. In the following we only show the induction step for 4.2 in a special case where the root-path of H consists of the single edge uv. This already uses a big part of the ideas for the full proof.

In the considered case the splitting sequence of H consists of some v-bubbles $X_1, \ldots, X_p$ rooted at u, followed by an e-bubble Y, followed by some v-bubbles $Z_1, \ldots, Z_q$ rooted at v. We start by drawing Y. Define $\ell' = \Delta - d_Y(u)$ and $r' = d_Y(v) - 1$. Suppose that Y is not the single edge uv as otherwise there is nothing more from Y to draw. Let $u_{\ell'}, \ldots, u_{\Delta-2}, v$ be the neighbors of u in Y in the clockwise order. Let $u, v_1, \ldots, v_{r'}$ be the neighbors of v in Y in the clockwise order. It follows from 3.3 that $Y - \{u, v\}$ is a bubble with roots $u_{\ell'}, \ldots, u_{\Delta-2}, v_1, \ldots, v_{r'}$, where $u_{\Delta-2}$ and v_1 may coincide. Define

$$\alpha = \begin{cases} |uv| & \text{if } u_{\Delta-2} = v_1, \\ \frac{\Delta-3}{\Delta-2}|uv| & \text{if } u_{\Delta-2} \neq v_1. \end{cases}$$

Put each vertex u_i at point $u + \alpha\mathbf{f}_i$ and each vertex v_i at point $v + \alpha\mathbf{f}_i$. Note that if $u_{\Delta-2}$ and v_1 coincide then they are correctly put to the same point. The points u_i and v_i split the horizontal segment $u_{\ell'}v_{r'}$ into segments of length $\frac{\alpha}{\Delta-3}$. Induction hypothesis 4.3 applied to the bubble $Y - \{u, v\}$ yields its drawing inside $\bar{B}(u_{\ell'}v_{r'}; 1, \Delta - 2; \frac{\alpha}{\Delta-4})$ having all additional properties stated in 4.3. It follows easily from the definition of bounding regions that

$$\bar{B}(u_{\ell'}v_{r'}; 1, \Delta - 2; \tfrac{\alpha}{\Delta-4}) \subset \bar{B}(uv; \ell, r; \tfrac{\Delta-3}{\Delta-4}|uv|).$$

This already completes the proof if $H = Y$. Now, suppose there are some v-bubbles $X_1, \ldots, X_p$ starting the splitting sequence of H. By induction hypothesis 4.1 the 1-bubble $X = (X_1, \ldots, X_p)$ can be drawn properly inside $B(u; \ell, \ell' - 1)$. We scale this drawing so that it fits inside $\bar{B}(u; \ell, \ell' - 1; \alpha)$. The latter bounding region is contained in $\bar{B}(uv; \ell, r; \frac{\Delta-3}{\Delta-4}|uv|)$. Moreover, it lies entirely below the horizontal line going through the points u_i and v_i, and entirely to the left of the edge $uu_{\ell'}$. Therefore, it does not overlap with the drawing of Y. The 1-bubble $Z = (Z_1, \ldots, Z_q)$ is drawn similarly on the other side. The resulting drawing of H clearly fulfills all the requirements. If Y is the single edge uv then we draw X and Z as above choosing $\alpha = |uv|$, which makes their bounding regions disjoint. □

To prove Theorem 1 for $\Delta \geqslant 5$, fix the position of any vertex v of G and apply 4.1 to the graph G considered as a 1-bubble with root v.

5 Further comments

The set of slopes used to prove Theorem 1 is rather special. A natural question arises: Depending on Δ, what is the smallest number s, such that any set S of s slopes allows for an outerplanar drawing using only slopes from S? We can only provide the following upper bound.

Theorem 5 *Every set S of $2\Delta - 4$ slopes may be used to draw any outerplanar graph G with maximum degree Δ using only slopes from S.*

Another problem we would like to mention is the following: Are there a function f and a polynomial p such that every outerplanar graph with maximum degree Δ and n vertices admits an outerplanar straight-line drawing on integer coordinates inside a $p(n) \times p(n)$ grid while using at most $f(\Delta)$ slopes?

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