Polynomial Time Recognition of Uniform Cocircuit Graphs

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Abstract
We present an algorithm which takes a graph as input and decides in polynomial time if the graph is the cocircuit graph of a uniform oriented matroid. In the affirmative case the algorithm returns the set of signed cocircuits of the oriented matroid.

Keywords: Oriented matroid, cocircuit graph, recognition algorithm, polynomial algorithm.

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1 Introduction

The cocircuit graph is a natural combinatorial construction associated with oriented matroids. In the case of pseudoline-arrangements, i.e., uniform rank 3 oriented matroids, its vertices are the intersection points of the lines and two points share an edge if they are connected by a line segment which does not intersect other lines. More generally, the Topological Representation Theorem of Folkman and Lawrence [4] says that every oriented matroid can be represented as an arrangement of pseudospheres. The cocircuit graph is the 1-skeleton of this arrangement.

For the uniform case, Montellano-Ballesteros and Strausz [6] provide a graph theoretical characterization of cocircuit graphs. Cordovil, Fukuda and Guedes de Oliveira [2] show that a uniform oriented matroid is basically determined by its cocircuit graph (up to isomorphism and reorientation).

After introducing basic notions of oriented matroids we provide an algorithm inspired by [2] which given a graph $G$ decides in polynomial time if $G$ is the cocircuit graph of a uniform oriented matroid. In the affirmative case the algorithm returns the set of signed cocircuits of the oriented matroid.

Given a finite ground set $E$ we define a signed set $X \subseteq E$ as an underlying set $X$ with a bipartition $(X^+, X^-)$ into a positive and a negative part. The parts may be empty. By $X^0$ we refer to the zero-support $E \setminus X$. Reorienting $X$ on $A \subseteq E$ yields the signed set $A \cdot X := ((X^+ \setminus A) \cup (X^- \cap A), (X^- \setminus A) \cup (X^+ \cap A))$. By $-X$ we refer to $E \setminus X$. Given signed sets $X,Y$ we denote by $S(X,Y) := (X^+ \cap Y^-) \cup (X^- \cap Y^+)$ their separator.

We define an oriented matroid as a pair $M = (E, C^*)$ of a ground set $E$ and a multiset $C^*$ of signed sets called cocircuits satisfying the following axioms:

(C1) $\emptyset \notin C^*$
(C2) $X \in C^* \Rightarrow -X \in C^*$
(C3) $X,Y \in C^*$ and $X = Y \Rightarrow X = \pm Y$
(C4) $X,Y \in C^*$ and $X \neq Y$ and $e \in S(X,Y) \Rightarrow$ there is $Z \in C^*$ with $Z^+ \subseteq X^+ \cup Y^+$ and $Z^- \subseteq X^- \cup Y^-$ and $e \in Z^0$.

The cocircuit graph $G_M$ of an oriented matroid $M = (E, C^*)$ has as vertex set $C^*$ and $X,Y \in C^*$ share an edge if and only if $S(X,Y) = \emptyset$ and the symmetric difference $|X^0 \Delta Y^0| = 2$.

A uniform oriented matroid $M = (E, C^*)$ of order $n$ and rank $r$ is an oriented matroid with $|E| = n$ and $C^*$ having exactly the subsets of size $n - r + 1$ as underlying sets. For more about oriented matroids, see [1].
2 The Algorithm

We are given a (simple, connected) graph $G = (V, A)$ and want to test in polynomial time whether $G$ is the cocircuit graph of a uniform oriented matroid. In the affirmative case we assign a signed set $X(v)$ to every vertex $v$ such that $X(V)$ is the set of cocircuits of $\mathcal{M}$ with $G = G_\mathcal{M}$. The algorithm has the following structure:

(A) Determine the parameters $n$ and $r$
(B) Construct the set of great cycles $\mathcal{G}$ (defined below)
(C) Assign zero-supports $X^0$ to vertices and great cycles
(D) Assign signed sets $X(v) = (X^+(v), X^-(v))$ to vertices
(E) Check the cocircuit axioms

Each of these steps works if $G$ is the cocircuit graph of a uniform oriented matroid and outputs that $G$ is not, otherwise. The main idea of step (C) is contained in [2].

(A) Determine the parameters. Cocircuit graphs are regular and antipodal. These properties are checked here.

- Check if $G$ is regular. If so, let $\delta$ denote the degree and set $r := \frac{\delta}{2} + 1$.
- Check if for every $v \in V$ there exists a unique $v^- \in V$ with $\text{dist}(v, v^-) = \text{diam}(G)$. In the affirmative case set $n := \text{diam}(G) + r - 1$.
- Check if $|V| = 2 \binom{n}{r-1}$.
- If any of the above tests results negative, $G$ is not a cocircuit graph.

All the checked properties are necessary conditions for $G$ to be a cocircuit graph of some uniform $\mathcal{M}$. Moreover $n$ and $r$ must be the order and the rank of the oriented matroid of $\mathcal{M}$, respectively. The runtime here is bounded from above by calculating a shortest path matrix for every starting vertex in the second step. This can be solved in $O(V^3)$ by applying Dijkstra’s Algorithm [3] $|V|$ times.

(B) Construct the set of great cycles A set $C \subseteq C^*$ is called a great cycle with zero-support $X^0(C)$ if it consists of the cocircuits whose zero-support contains the $(r - 1)$-set $X^0(C)$. The set of great cycles is denoted by $\mathcal{G}$. The following part of the algorithm computes $\mathcal{G}$.

- For every $v \in V(G)$ and every neighbor $w \in N(v)$, if the edge $\{v, w\}$ is not contained in any $C \in \mathcal{G}$ then
compute a shortest path $P$ from $w$ to $v$,

- set $C' := (\{v, w\}, P, \{v^-, w^-\}, P^-)$,
- add $C'$ to $G$.

- If $G$ is no partition of the edge set then $G$ is not a cocircuit graph.

For the correctness of this step we need:

**Lemma 2.1** The set of great cycles $\mathcal{G}$ of a cocircuit graph $G$ partitions the edge set. Moreover the edge $\{v, w\}$ is contained in the cycle $C'$ constructed in the algorithm.

The lemma tells us in particular that $|\mathcal{G}| = |A| = \frac{\delta}{2} |V| = 4(r - 1) \left(\begin{array}{c} n \\ r \end{array}\right)$ and that $|C| = 2 \text{diam}(G) = 2(n - r + 1)$. Hence the runtime $|V| \delta |C| |V|^2$ is bounded by $4(r - 1)(n - r + 1)|V|^3 \in O(|V|^4)$.

(C) **Assign zero-supports to vertices and great cycles.** In this part we assign zero supports to great cycles and vertices. We use the incidence structure on $G$, i.e., let $J$ be the graph with vertex set $G$ where $C$ and $C'$ share an edge if they have a vertex in common in $G$. Note that $J$ is isomorphic to the Johnson Graph $J(n, n - r + 2)$, see [5] for a definition. During the algorithm we denote by $\mathcal{P}$ the set of cycles with already assigned zero-support. Cycles with all their vertices having a zero-support already are in $\mathcal{T}$. We assign supports starting from a vertex $v$, then all cycles containing $v$. These cycles form a clique in $J$.

By $G_i, \mathcal{P}_i, \mathcal{T}_i$ we denote the respective sets at distance $i$ from this clique. We write $[k]$ for $\{1, \ldots, k\}$.

- Take any $v \in V$ and set $X^0(v) := [r - 1]$.
- For every $C_i$ in the set $\{C_1, \ldots, C_{r-1}\}$ of great cycles containing $v$,
  - set $X^0(C_i) := [r - 1] \setminus \{i\}$,
  - add $C_i$ to $\mathcal{P}_0$.
- Take any $\tilde{C} = (v_1, \ldots, v_{2(n-r+1)})$ from $\mathcal{P}_0$.
- Set $X^0(v_{(n-r+1)+i}) := X^0(v_i) := X^0(C) \cup \{i\}$ for every $i \in [n - r + 1]$.
- Add $\tilde{C}$ to $\mathcal{T}_0$ and $\mathcal{T}_{-1}$.
- Set $i := 0$ and until all $V$ is labelled repeat:
  - T-loop: For every $C \in \mathcal{P}_i$ take $\tilde{C} \in \mathcal{T}_{i-1}$ such that $C \cap \tilde{C} \neq \emptyset$.
    - For every $v \in C$ find a $C' \in \mathcal{G}$ with $v \in C'$ and $C' \cap \tilde{C}$ containing a $w$:
      - set $X^0(v) := X^0(C) \cup (X^0(w) \setminus X^0(\tilde{C}))$,
      - add $C$ to $\mathcal{T}_i$.
  - P-loop: For every $\tilde{C} \in \mathcal{T}_i$ and $v \in \tilde{C}$:
    - If there exists an unlabeled great cycle $C$ containing $v$ look for another
\( w \in C \) with known \( X^0(w) \),  
set \( X^0(C) := X^0(v) \cap X^0(w) \),  
add \( C \) to \( \mathcal{P}_{i+1} \).

- If any vertex receives several different labels \( G \) is not a cocircuit graph.
- Increase \( i \) by one.

The runtime is bounded by the T-loop \( |G|(|C|\delta + |C|\delta|C|) \in O(V^3) \).

For the existence of \( C' \) in the T-loop we need:

**Lemma 2.2** Let \( C \in \mathcal{G}_i \) and \( C' \in \mathcal{G}_{i-1} \) be intersecting. Then for every \( v \in C \) there is a \( C'' \in \mathcal{G}_i \) that contains \( v \) and intersects with \( C' \).

For the existence of \( w \) in the P-loop we need:

**Lemma 2.3** For every \( C \in \mathcal{G}_i \) there are at least two \( C', C'' \in \mathcal{G}_{i-1} \) which intersect with \( C \).

(D) **Assign signed sets to vertices.** We will label the vertices with signed sets. As in (C) we start with cycles \( \mathcal{S}_0 \) which form a clique in \( J \). Then we label cycles at increasing distance of \( \mathcal{S}_0 \). Since we calculated zero-supports in (C) we set \( \overline{X}(v) := E \setminus X^0(v) \) for every \( v \in V \).

- Take any \( v \in V \) and set \( X(v) := (\overline{X}(v), \emptyset) \).
- For all great cycles \( C \ni v \) and \( w, w' \in N(v) \cap C \) set \( X(w) := (X^+(v) \cap \overline{X}(w), X(w) \setminus X^+(v)) \) and \( X(w') := (\overline{X}(w), \emptyset) \).
- Add all \( C \in \mathcal{G} \) containing \( v \) to \( \mathcal{S}_0 \).
- Set \( i := 0 \) and repeat until all \( V \) is signed:
  - For every great cycle \( C \in \mathcal{S}_i \)
    - get two signed non-antipodal vertices \( v, w \in C \) and take \( u \in N(v) \cap C \) such that \( 0 < d(u, w) < d(u, w^-) \), i.e., \( u, v, w \) lie on the same half of \( C \),
    - sign \( u \) identically to \( v \) on \( \overline{X}(v) \cap \overline{X}(u) \) and take the signing of \( X(w) \) on \( e = \overline{X}(u) \setminus \overline{X}(v) \).
    - This way we sign all vertices in \( C \).
    - If a vertex receives two different labels \( G \) is not a cocircuit graph.
    - Add all unsigned cycles \( C' \cap C \neq \emptyset \) to \( \mathcal{S}_{i+1} \).
  - Increase \( i \) by one.

The runtime is \( |G||C|^2 \in O(|V|^4) \). We need Lemma 2.3 for the existence of \( v, w \in C \). For the correctness of the signing we have:

**Lemma 2.4** Take a great cycle \( C \) of a uniform cocircuit graph, \( X, Y \in C \), \( X \neq Y \) and let \( (X = X_0, \ldots, X_k = Y) \) be the shortest \((X, Y)\)-path in \( C \) and \( e \in X \cap Y \). Then \( e \in S(X, Y) \) if and only if there is \( X^0 \ni e \).
(E) Check the cocircuit axioms. Here we check if the signed sets we assigned to the vertices satisfy (C4) of the cocircuit axioms. The other axioms are satisfied by construction.

- For every two non-antipodal vertices \( u, v \in V \) and every element in the separator of their signed labels \( e \in S(X(u), X(v)) \) check if there exists a \( w \in V \) with \( X(w)^+ \subseteq X(u)^+ \cup X(v)^+ \) and \( X(w)^- \subseteq X(u)^- \cup X(v)^- \) and \( e \in X(w)^0 \).

Here the runtime is \( |V|^3(n - r + 1) \in O(|V|^4) \). The correctness is obvious. The last step of the algorithm could also be done by checking whether the constructed \( C^* \) is a certain metrical and antipodal embedding into the \((r - 1)\)-dual of the \( n \)-cube, see [6].

Thus we have the following result:

**Theorem 2.5** The preceding algorithm checks if a given graph \( G = (V, A) \) is the cocircuit graph of a uniform oriented matroid \( M \) and constructs such \( M \) in the affirmative case in time \( O(|V|^4) \).

**References**


