12. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 05. July 2017
Due dates: 11.-13. July
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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(1) In the lecture, we deduced the formula $N(\emptyset) = \sum_B (-1)^{|B|} N_{\geq}(B)$. Show that it implies the following *inclusion-exclusion formula.* For $A_1, \ldots, A_r \subseteq X$, it holds:

$$\left| X - \bigcup_{i=1}^{r} A_i \right| = \sum_{I \subseteq [r]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|$$

Hint: For $I \subset [r]$, consider $F_{=}(I) := \{x \in X \colon (x \in A_i \forall i \in I) \wedge (x \notin A_j \forall j \notin I)\}$.

(2) How many permutations of the 26 letters of the English alphabet do not contain any of the strings red, blue, or cyan?

(3) Let (a_k) and (b_k) be sequences with

$$b_n = \sum_{k \leq n} \binom{n}{k} a_k.$$

Find a representation of the a-sequence in terms of the b-sequence.

(4) Suppose $f(X)$ is known for all $X \subseteq [n]$ and we want to compute

$$g(X) := \sum_{Y \supseteq X} f(Y)$$

(a) Note that $g(X)$ can be computed by adding $2^{n-|X|}$ values. What is the total number of operations needed to compute g for all X independently (with this approach)?

(b) The fast zeta transform allows to compute g faster. For $i \in [n]$, we define

$$g_0(X) = f(X)$$
$$g_i(X) = \begin{cases} g_{i-1}(X) + g_{i-1}(X+i), & i \notin X \\ g_{i-1}(X), & i \in X. \end{cases}$$

Prove that

$$g_i(X) = \sum_{Y : Y - [i] \subseteq X \subseteq Y} f(Y).$$

How many operations are needed to compute g with this approach?

(5) Count the number of triples (p_0, p_1, p_2), where p_i are vertex-disjoint lattice paths of length 6 from $A_i = (i, 2-i)$ to $B_i = (3+i, 5-i)$.

(0.5) We will discuss excercise 5 of sheet 0.