11. Practice sheet for the lecture: Combinatorics (DS I)

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Due dates: 04.-06. July
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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(1) For which of the parameter sets does a design exist? Either show that there is none or present one. (This exercise gives 2 points.)

(a)	$S(2, 5, 125)$	(d)	$S(2, 7, 36)$
(b)	$S_2(4, 7, 13)$	(e)	$S_3(2, 4, 20)$
(c)	$S(2, 6, 16)$	(f)	$S_3(3, 5, 21)$

[Hint to (f): Consider the edge set of K_7.]

(2) Let $(\mathcal{P}, \mathcal{B})$ be a $S_\lambda(t, k, v)$ design.

(a) Let $p \in \mathcal{P}$ and $\mathcal{B}^p := \{B : p \notin B \in \mathcal{B}\}$ be the set of blocks, which do not contain p. Show that $(\mathcal{P} \setminus \{p\}, \mathcal{B}^p)$ is a design. What are its parameters?

(b) Consider the *complement* of $S_\lambda(t, k, v)$, i.e., replace each block by its complement. Prove that the complement of $S_\lambda(t, k, v)$ is a t-design. Determine its parameters.

(3) Let $(\mathcal{P}, \mathcal{B}) = S(2, n+1, n^2+n+1)$ be a projective plane and fix $B \in \mathcal{B}$. Show that

$$\Big(\mathcal{P} \setminus B, \big\{C \setminus B \mid C \in \mathcal{B} \setminus \{B\}\big\}\Big)$$

is a resolvable $S(2, n, n^2)$ design.

(4) Let q be a prime power. For every $k, n \in \mathbb{N}, k \leq n$, construct the following design:

$$S_\lambda\left(2, \begin{bmatrix} k \\ 1 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q\right) \text{ with } \lambda = \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q.$$

(5) A *perfect matching* of K_{2n} is a set of n disjoint edges. A *resolution* of K_n is a partition of its edge set into perfect matchings.

(a) Show that two disjoint perfect matchings of K_6 determine a unique resolution. [Hint: Note that the union of two perfect matchings of K_6 is a cycle.]

(b) Consider $K_6 = (V, E)$. Let $\mathcal{P}$ denote the set of perfect matchings and $\mathcal{R}$ the set of resolutions. For $e \in E$ we define $B_e := \{v \in V | v \in e\} \cup \{P \in \mathcal{P} | e \in P\}$ and for each $R \in \mathcal{R}$, we define $B_R := \{P \in \mathcal{P} | P \in R\}$. Conclude that the following is a $S(2, 5, 21)$:

$$(\mathcal{P} \cup V, \{B_e | e \in E\} \cup \{B_R | R \in \mathcal{R}\})$$