10. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 20. June 2017
Due dates: 27.-29. June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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(1) Consider necklaces with 12 beads of at most three different colors

(a) How many different necklaces exist? Necklace equivalence is given by the dihedral group.

(b) How many different necklaces with 3 red, 4 green, and 5 blue beats exist?

(2) Consider red-blue-face-colorings of the platonic solids with rotational symmetries.

(a) Determine the number of differently colored dodecahedra.

(b) Determine the number of differently colored icosahedra.

(c) Determine the number of differently colored soccer balls. Note that a soccer ball is a truncated icosahedron.

(3) A graph $G = (V, E)$ is *isomorphic* to a graph $H = (V', E')$, if a relabeling of the vertices of G yields the graph H, i.e., if there exists a bijection $\Phi : V \to V'$ such that the mapping $\Psi((v, w)) := (\Phi(v), \Phi(w))$ is a bijection from E to E'.

(a) Count non–isomorphic graphs on four vertices using a Polya theorem.

(b) Count non–isomorphic graphs on four vertices when loops are allowed. *Loops* are edges starting and ending at the same vertex.

(c) Let $g_{n,k}$ be the number of non-isomorphic graphs on n vertices with k edges (no loops). Let G be the symmetric group on the vertices, which acts on $\binom{[n]}{2}$ by $\pi(\{i, j\}) := \{\pi(i), \pi(j)\}$. Prove

$$\sum_{k=0}^{\binom{n}{2}} g_{n,k} x^k = P_G\Big(1 + x, 1 + x^2, \ldots 1 + x^{\binom{n}{2}}\Big).$$

(4) Let $\mathcal{T}$ be the family of rooted ternary trees. A *rooted ternary tree* is a rooted tree where every vertex has at most 3 children. Two rooted ternary trees are equivalent, if one can be obtained from the other by permuting the subtrees of its vertices. Let G the symmetric group S_3 with its standard action on three elements and $\mathcal{F} = \mathcal{T}^{[3]}$. Consider the action of G on $\mathcal{F}$. For $f \in \mathcal{F}$, we define $w(f) = w(T_a) \cdot w(T_b) \cdot w(T_c)$, where $w(T) = x^{\#\text{ vertices of T}}$. For an orbit F of $\mathcal{F}$, we define $w(F) = w(f)$ for $f \in F$.

(a) Determine $\sum_{F \text{ orbit}} w(F)$.

(b) Let t_n denote the number of rooted ternary trees on n vertices and denote its generating function by $T(x) := \sum t_n x^n = 1+x+x^2+2x^3+4x^4+8x^5+17x^6+\ldots$. Conclude that

$$T(x) = 1 + x \cdot P_G\Big(T(x), T(x^2), T(x^3)\Big).$$

(0.4) We will discuss excercise 4 of sheet 0.