8. Practice sheet for the lecture: Combinatorics (DS I)

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Due dates: 06.-08. June

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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(1) Symmetric chain decompositions of $\mathcal{B}_n$

(a) Given a symmetric chain C in $\mathcal{B}_n$, is there a symmetric chain decomposition containing C?

(b) Show that the number of chains of length $n-2k$ in a symmetric chain decomposition of $\mathcal{B}_n$ is $\binom{n}{k} - \binom{n}{k-1}$.

(2) Symmetric chain decompositions of $\mathcal{B}_n$ and Catalan numbers

(a) Give a bijection between pairings of brackets of length $2n$ and binary trees with n nodes.

(b) Use (1b) and (2a) to derive the known explicit formula for the Catalan numbers, i.e. $C_n = \frac{1}{n+1}\binom{2n}{n}$.

(3) Let $\mathcal{B}_n^{\vee}$ be the truncation of the Boolean lattice where the maximal and minimal element is deleted. Let $\mathcal{C}$ be a symmetric chain decomposition which is canonical (originating from the bracketing process). Let $\overline{\mathcal{C}}$ be its complement, i.e. for a chain $C \in \mathcal{C}$ the set $\overline{C}$ of complements of sets in C is a set in $\overline{\mathcal{C}}$.

(a) Show that $\overline{\mathcal{C}}$ is a symmetric chain decomposition.

(b) Show that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are *orthogonal*, i.e. $|C \cap D| \leq 1$ for all $C \in \mathcal{C}$ and $D \in \overline{\mathcal{C}}$.

(4) Let $\mathcal{A}$ be a family of k-subsets of $[n]$ and $\mathcal{B}$ be a family of l-subsets of $[n]$ such that $l + k \leq n$ and $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(a) Show that $|\mathcal{A}| < \binom{n-1}{k-1}$ or $|\mathcal{B}| \leq \binom{n-1}{k-1}$.
Hint: Use shadows as in the second proof of the Erdős-Ko-Rado theorem.

(b) Deduce the Erdős-Ko-Rado theorem from (a).

(0.2) We will discuss excercise 2 of sheet 0.