6. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 24. May 2017
Due dates: 29. May and 01. June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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(1) Let $\mathcal{G}$ and $\mathcal{H}$ be structures which are counted by the exponential generating functions $G(x)$ and $H(x)$ with $H(0) = 0$. Show that $G(H(x))$ has the following combinatorial interpretation. A composite $\mathcal{G} \circ \mathcal{H}$ structure on $[n]$ consists of
- a partition P of $[n]$,
- for each block $B \in P$, a $\mathcal{H}$-structure labeled with elements in B, and
- a $\mathcal{G}$-structure on the set of blocks.
Hint: Consider multinomial coefficients for integers $k_1, k_2, \ldots, k_l, n$ with $\sum_i k_i = n$

$$\binom{n}{k_1, k_2, \ldots, k_l} := \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_l!}$$

A possible application of this composition principle appears by counting unordered labeled rooted trees on an n-element set. To ensure $H(0) = 0$, the empty tree is not counted.

(2) The q–binomials fulfill the equation

$$\sum_{i=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \cdot q^{(k+1)(n-i)} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

for all $n \geq k \geq 0$. Prove this via the lattice path model for q–binomials.

(3) Show that the q–binomials fulfill the equation

$$\sum_{k \geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix} z^k = \prod_{i=0}^{n} \frac{1}{1 - q^i z}$$

(4) Find and prove a q-analogue of Vandermonde's identity.

(5) The q–binomials fulfill the equation

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ m-k \end{bmatrix}$$

for all $n \geq m \geq k \geq 0$. Prove this via the model on $\mathbb{F}_q$ subspaces for q–binomials.