5. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 17. May 2017
Due dates: 23.-25. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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(1) The Bell numbers are defined by $B(n) := \sum_k S(n,k)$.

(a) What does $B(n)$ count? Show the identity by a combinatorial argument:

$$B(n+1) = \sum_i \binom{n}{i} B(i)$$

(b) Let $F(z) := \sum \frac{B(n)}{n!} z^n$ be the exponential generating function of the Bell numbers $B(n)$. Show that there exists a function $f(z)$ such that

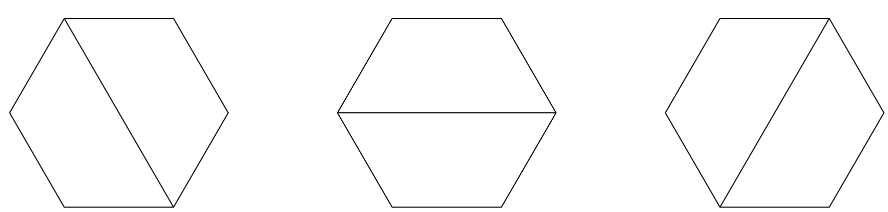
$$F'(z) = f(z)Fz).$$

(c) Solve the differential equation and deduce a closed formula for $F(z)$.

(*) Give a combinatorial argument, which proves that the number of partitions of $[n]$, such that no two consecutive numbers appear in the same block, is $B(n-1)$.

(2) Let t_n denote the number of binary trees with n inner vertices. We say a vertex in a binary tree is an *inner* vertex if it has children. Use the symbolic method to determine the generating function.

(3) A set of chords of a convex $2n$-gon is a *quadrangulation* if no two chords intersect and all faces are quadrangles. Let a_n denote the number of quadrangulations of a convex $2n$-gon. Use the symbolic method to find the generating function $A(x) = \sum_{n\geq 0} a_n x^n$. The figure shows the 3 quadrangulations of a 6-gon:



(4) Let $\mathcal{B}$ be a set and $\mathcal{A} = \mathcal{P}(\mathcal{B})$, the powerset of $\mathcal{B}$ (all finite subsets of $\mathcal{B}$). Show that the generating function A of $\mathcal{A}$ can be expressed as

$$A(z) = \prod_{n\geq 1} (1+z^n)^{b_n}$$

Recall that $A(z) = \sum_{a\in\mathcal{A}} x^{|a|} = \sum_{n\geq 0} a_n x^n$ and, likewise, $B(z) = \sum_{b\in\mathcal{B}} x^{|b|} = \sum_{n\geq 0} b_n x^n$. Think about the right weight for $a \in \mathcal{A}$.

0.6) We will also discuss exercise 6 from sheet 0.