4. Practice sheet for the lecture: Combinatorics (DS I)

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Due dates: 16.-18. May

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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(1) Prove the following identities for the Fibonacci numbers f_n by bijection.

(a) $f_n + f_{n-1} + \sum_{k=0}^{n-2} 2^{n-k-2} f_k = 2^n$

(b) $f_{2n-1} = \sum_{k=1}^{n} \binom{n}{k} f_{k-1}$

(2) Consider walks in the plane starting in $(0,0)$ where each step is $R : (x,y) \to (x+1,y)$ or $U_a : (x,y) \to (x, y+a)$ with a a positive integer. There are 5 walks that contain a point on the line $x + y = 2$, namely: $RR, RU_1, U_1R, U_1U_1, U_2$. Let a_n denote the number of walks that contain a point on the line $x + y = n$. Express a_n in terms of Fibonacci numbers. [Hint: Look at small numbers and make a good guess.]

(3) In the lecture we saw a proof of Binet's formula based on generating functions. An alternative approach comes from linear algebra:

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \implies \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ where } A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Derive Binet's Formula by diagonalizing the matrix A.

(4) Let a_k denote the number of words of length k over the alphabet $\{u, l, r\}$ with no l and r consecutive, i.e. lr and rl do not appear. These words can be interpreted as grid paths of length k which go up, left and right, and do not intersect themselves.

(a) Find a linear recursion for a_k.

(b) Express the generating function as a rational function.

(c) Find a closed form for a_k.

(5) For fixed $s \in \mathbb{N}$, find a recursion for the sequence $(a_n(s))_{n \geq 0}$ where

$$a_n(s) = (1 + \sqrt{s})^n + (1 - \sqrt{s})^n.$$

(6) In how many ways can you pay n Dollar with 1\$, 5\$ and 10\$ notes? Find a generating function and compute the number of ways to pay 50 Dollar.