3. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 03. May 2017
Due dates: 9.-11. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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(1)
 (a) How many subsets of the set $[n]$ contain at least one odd integer?
 (b) How many multi-subsets of the set $[n]$ have size k?
 (c) Let $\subset$ denote a strict subset relation. For a given $k \in [n]$, how many sequences $(T_1, T_2, \ldots, T_k)$ are there with

$$\emptyset \subset T_1 \subset T_2 \subset \ldots \subset T_k \subset [n] \ ?$$

(2) A composition of n is an ordered set of numbers $(a_1, \ldots, a_k)$ with $a_i \in \mathbb{N}$ and $0 < a_i \leq n$ such that $a_1 + \cdots + a_k = n$. For example, $n = 4$ has the compositions: $1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, $2+2$, $3+1$, $1+3$, 4.
 (a) Prove that n has $\binom{n-1}{k-1}$ compositions into exactly k parts.
 (b) Let $c(n)$ be the number of compositions of n into an even number of even parts. For $n = 8$, these compositions are: $2+2+2+2$, $4+4$, $6+2$, $2+6$. Give a closed formula for $c(n)$.

(3) Polygonal numbers
 (a) What are triangle, square, pentagon, hexagon, and heptagon numbers? Or more general, what are polygonal numbers?
 (*) What are tetrahedral numbers?

(4) Pentagonal number theorem
 (a) Complete the proof of the pentagonal number theorem: Show that *moving slope to front* is an injection from distinct partitions of type II to distinct partitions of type I.
 (b) Let $c_k := \frac{(3k-1)k}{2}$ be the kth pentagonal number. Show that the number of partitions of n can be computed in the following way:

$$p(n) = \sum_k (-1)^{k-1} \big(p(n - c_k) + p(n - c_{-k})\big)$$

 [Hint: Use the pentagonal number theorem & the generating function of $p(n)$.]

(5) Show that the number of partitions of n where no part is divisible by $d \in \mathbb{N} \setminus \{0, 1\}$ equals the number of partitions of n where no d parts have the same size.
[Hint: We proved this statement in the lecture for $d = 2$.]