2. Practice sheet for the lecture: Combinatorics (DS I)

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Due dates: 2.-4. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI17.html`

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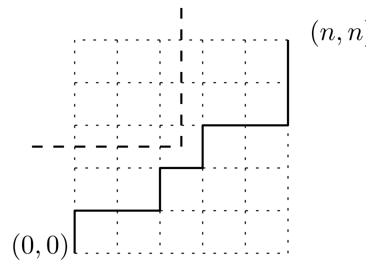
(1) Let $n \in \mathbb{N}$ be odd. Count the number of grid paths from $(0,0)$ to (n,n) with 'up' and 'right' steps, which avoid all grid points (i,j), such that $i > \frac{n}{2} > j$. The figure shows a path for $n = 5$.



(2) Let $\left(\!\binom{n}{k}\!\right)_g$ be the number of multisets with k elements (counted with multiplicity) of an n-element ground set, where each element occurs at most g times.
Show the following identities:

(a)
$$\left(\!\!\binom{m+n}{k}\!\!\right)_g = \sum \left(\!\!\binom{m}{l}\!\!\right)_g \left(\!\!\binom{n}{k-l}\!\!\right)_g$$

(b)
$$(1 + x + \cdots + x^g)^n = \sum_{k=0}^{g \cdot n} \left(\!\!\binom{n}{k}\!\!\right)_g x^k.$$

(3) Let $x^{\underline{n}} := (x)_n$ denote the falling factorials and $x^{\overline{n}} := x \cdot (x+1) \cdots (x+n-1)$ the *raising factorials.* Deduce the following equation from Vandermonde's identity:

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} \quad \left[\text{Hint: } \binom{-x}{k} = (-1)^k \binom{x+k-1}{k}\right]$$

(4) A permutation $\pi \in S_n$ is a transposition if exactly the position of two elements is switched, i.e. in cycle notation there are $n-2$ fixpoints and a cycle of length 2.

a) Show that each permutation $\pi \in S_n$ is a composition of transpositions. We denote the minimum number of transpositions needed to express π by $t(\pi)$.

b) As in the lectured, $c(\pi)$ denotes the number of cycles of π. Show that

$$t(\pi) + c(\pi) = n$$

(5) What is the expected number of left-right-maxima in a random permutation?

(6) The names of 100 prisoners are placed in 100 wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the prisoners are led into the room; each may look in at most 50 boxes, but may not change anything in the room and is permitted no further communication with the others.
The prisoners have a chance to plot their strategy in advance and they are going to need it, because unless *every single prisoner finds his own name* all will subsequently be executed. Find a strategy for them which has a probability of succes exceeding 30%.