ON ARTIN'S L-FUNCTIONS. III: ONE DIMENSIONAL CHARACTERS

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Let $K/\mathbb{Q}$ be a finite Galois extension with the Galois group G, and let χ be a nontrivial irreducible character of G. Artin's conjecture predicts that the L-function $L(s, \chi, K/\mathbb{Q})$ is holomorphic in the whole complex plane ([1], P. 105).

Let $\chi_1, \ldots, \chi_r$ be the irreducible nontrivial characters of G. The corresponding L-functions $L(s, \chi_1), \ldots, L(s, \chi_r)$ are algebraically independent over $\mathbb{C}$ ([2], Corollary 4, P. 183). Let $\mathcal{A} := \mathbb{C}[L(s, \chi_1), \ldots, L(s, \chi_r)]$ be the $\mathbb{C}$-Algebra generated by the meromorphic functions $L(s, \chi_1), \ldots, L(s, \chi_r)$. It is isomorphic to the algebra of polynomials in r variables over $\mathbb{C}$. Let $\mathcal{O}(\mathbb{C})$ be the $\mathbb{C}$-algebra of holomorphic functions in $\mathbb{C}$. Artin's conjecture is:

$$\mathcal{A} \subseteq \mathcal{O}(\mathbb{C}).$$

Let $\mathcal{S}$ be the set of all subgroups of G. For a subgroup $H \in \mathcal{S}$ let $\hat{H}_0^1$ be the set of all non-trivial one dimensional complex characters of H, that is, the set of all non-constant group homomorphisms of H in the multiplicative group $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. For a subgroup $H \in \mathcal{S}$ and a character $\varphi \in \hat{H}_0^1$ let $\varphi^G := \mathrm{Ind}_H^G$ be the induced character of G. The Artin L-function $L(s, \varphi^G, K/\mathbb{Q})$ is holomorphic, being equal to the Hecke L-function $L(s, \varphi, K_2/K_1)$ of the abelian extension K_2/K_1, K_1 the fixed field of H, K_2 the fixed field of $\mathrm{Ker}\, \varphi \subseteq H$, so:

$$\mathcal{B} \subseteq \mathcal{H} \subseteq \mathcal{A},$$

where $\mathcal{B}$ is the $\mathbb{C}$-subalgebra of $\mathcal{A}$ generated by the functions $L(s, \varphi^G, K/\mathbb{Q})$, $H \in \mathcal{S}$, $\varphi \in \hat{H}_0^1$, and $\mathcal{H} := \mathcal{A} \cap \mathcal{O}(\mathbb{C})$. How large is the algebra $\mathcal{B}$?

Theorem 1. *The finitely generated $\mathbb{C}$-algebra $\mathcal{B}$ is of Krull dimension r. The quotient field of $\mathcal{B}$ equals the quotient field of $\mathcal{A}$.*

P r o o f: Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. By ([3], P. 209) there exist subgroups $H_1, \ldots, H_l$ of G, non-trivial one dimensional characters φ_i of H_i, $i = 1, \ldots, l$ and integers $m_1, \ldots, m_l$ such that

$$\chi = m_1 \varphi_1^G + \ldots + m_l \varphi_l^G.$$

It follows that

$$L(s, \chi) = L(s, \varphi_1^G)^{m_1} \cdot \ldots \cdot L(s, \varphi_l^G)^{m_l} \in \mathcal{B},$$

hence $\mathcal{A}$ is contained in the quotient field of $\mathcal{B}$. Since $\mathcal{B}$ is contained in $\mathcal{A}$ it follows that the quotient field of $\mathcal{B}$ equals the quotient field of $\mathcal{A}$. The Krull dimension of the finitely generated $\mathbb{C}$-algebra $\mathcal{B}$ equals the transcendence degree of its quotient field, that is, the trancendence degree of the quotient field of $\mathcal{A}$, which is r. $\square$

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Theorem 2. *The following assertions are equivalent:*
(a) $\mathcal{B} = \mathcal{A}$.
(b) *G is M-group.*

P r o o f:
(a)$\Rightarrow$(b): Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. Since $L(s,\chi) \in \mathcal{B}$ there exist subgroups $H_1, \ldots, H_l$ of G, one dimensional non-trivial irreducible characters φ_j of H_j, $j = 1, \ldots, l$ and a polynomial

$$P(X_1, \ldots, X_l) = \sum_{i_1 \geq 0, \ldots, i_l \geq 0} a_{i_1 \ldots i_l} X_1^{i_1} \ldots X_l^{i_l} \in \mathbb{C}[X_1, \ldots, X_l]$$

such that

$$L(s,\chi) = P(L(s,\varphi_1^G), \ldots, L(s,\varphi_l^G)),$$

that is

$$L(s,\chi) = \sum_{i_1 \geq 0, \ldots, i_l \geq 0} a_{i_1 \ldots i_l} L(s, i_1\varphi_1^G + \ldots + i_l\varphi_l^G).$$

By the linear independence of L-functions corresponding to different characters ([2], Theorem 1, P. 179) it follows that there exist $i_1, \ldots, i_l$ such that

$$\chi = i_1\varphi_1^G + \ldots + i_l\varphi_l^G.$$

Since χ is irreducible there exist $1 \leq j \leq l$ such that

$$\chi = \varphi_j^G,$$

hence G is M-group.
(b)$\Rightarrow$(a): Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. Since G is M-group, there exist a subgroup $H \subseteq G$ and a one dimensional character $\varphi : H \to \mathbb{C}^\times$ such that

$$\chi = \varphi^G.$$

Since χ is not trivial, the character φ is not trivial, so

$$L(s,\chi) = L(s,\varphi^G) \in \mathcal{B}.$$

Hence

$$\mathcal{A} \subseteq \mathcal{B}.$$

□

Let $\mathcal{B}'$ be the integral closure of $\mathcal{B}$ in $\mathcal{A}$. It holds

$$\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathcal{H} \subseteq \mathcal{A}.$$

Theorem 3. *The following assertions are equivalent:*
(a) $\mathcal{B}' = \mathcal{A}$.
(b) *G is quasi M-group: For each irreducible character χ of G there exist a subgroup H of G, a 1-dimensional character $\varphi : H \to \mathbb{C}^\times$ and a number $k \geq 1$ such that $k\chi = \varphi^G$.*

P r o o f:
(a)$\Rightarrow$(b): Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. The element $L(s,\chi)$ of $\mathcal{A}$ satisfies a monic equation with coefficients in $\mathcal{B}$:

$$L(s,\chi)^l + b_{l-1}L(s,\chi)^{l-1} + \ldots + b_1 L(s,\chi) + b_0 = 0, \tag{1}$$

$l \geq 1$, $b_0, \ldots, b_{l-1} \in \mathcal{B}$. Let $\varphi_1^G, \ldots, \varphi_m^G$ be all pairwise distinct characters of G which are obtained by inducing from non-trivial linear characters of subgroups of G, and let $f_1 := L(s,\varphi_1^G), \ldots, f_m := L(s,\varphi_m^G)$. It holds:

$$\mathcal{B} = \mathbb{C}[f_1, \ldots, f_m].$$

Each coefficient b_j, $j = 0, \ldots, l-1$ is a polynomial in $f_1, \ldots, f_m$:

$$b_j = P_j(f_1, \ldots, f_m) = \sum_{t_1 \geq 0, \ldots, t_m \geq 0} a^{(j)}_{t_1 \ldots t_m} f_1^{t_1} \cdot \ldots \cdot f_m^{t_m} =$$

$$= \sum_{t_1 \geq 0, \ldots, t_m \geq 0} a^{(j)}_{t_1 \ldots t_m} L(s, t_1 \varphi_1^G + \ldots + t_m \varphi_m^G),$$

and (1) rewrites as

$$L(s, l\chi) + \sum_{j=0}^{l-1} \sum_{t_1 \geq 0, \ldots, t_m \geq 0} a^{(j)}_{t_1 \ldots t_m} L(s, t_1 \varphi_1^G + \ldots + t_m \varphi_m^G + j\chi) = 0. \tag{2}$$

By the linear independence of L-functions corresponding to different characters ([2], Theorem 1, P. 179) and by (2) it follows that there exist $j \in \{0, \ldots, l-1\}$ and $t_1 \geq 0, \ldots, t_m \geq 0$ such that

$$l\chi = t_1 \varphi_1^G + \ldots + t_m \varphi_m^G + j\chi,$$

that is

$$(l-j)\chi = t_1 \varphi_1^G + \ldots + t_m \varphi_m^G.$$

Since χ is an irreducible character there exist $u \in \{1, \ldots, m\}$ and $1 \leq k \leq l-j$ such that $k\chi = \varphi_u^G$, so G is quasi M-group.
(b)⇒(a): Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. Since G is quasi M-group, there exist a subgroup $H \subseteq G$, a 1-dimensional character $\varphi : H \to \mathbb{C}^\times$ and $k \geq 1$ such that

$$k\chi = \varphi^G.$$

Since χ is not trivial, the character φ is not trivial. It holds

$$L(s, \chi)^k = L(s, k\chi) = L(s, \varphi^G) \in \mathcal{B},$$

so $L(s, \chi) \in \mathcal{B}'$. Hence

$$\mathcal{A} \subseteq \mathcal{B}'.$$

□

It is not known whether there exist quasi M-groups which are not M-groups. By a theorem of Taketa every M-group is solvable. It is not known whether every quasi M-group is solvable.

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