On Artin's L-Functions

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Let $K/\mathbb{Q}$ be a finite normal extension with the Galois group G. The derivatives of any order of Artin L-functions to finitely many distinct characters of G are linearly independent over the field $\mathbb{C}$ of complex numbers. As a consequence of this it follows that the L-functions to the irreducible characters are algebraically independent over $\mathbb{C}$, which improves a classical result of Artin on multiplicative independence, and that the Dedekind zeta functions of finitely many distinct Galois number fields are linearly independent over $\mathbb{C}$.

Let χ be a character of G, let $L_{ur}(s, \chi, K/\mathbb{Q})$ be the unramified part of the corresponding Artin L-function, and let

$$L_{ur}(s, \chi, K/\mathbb{Q})^{\frac{1}{\chi(1)}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for $Re(s) > 1$. Then it holds:
(i) The coefficients a_n are algebraic numbers of the field $\mathbb{Q}(e^{\frac{2\pi i}{|G|}})$ and $|a_n| \leq 1$ for every $n \geq 1$;
(ii) If χ does not contain the trivial character then the summatory function $\sum_{n \leq x} a_n$ is $\mathbf{o}(x)$ as $x \to \infty$.