The number of S_4–fields with given discriminant

Jürgen Klüners
klueners@mathematik.uni-kassel.de

Question

Definition: $N_{S_4}(d)$ is the number of quartic S_4–fields with (absolute) discriminant d.

Conjecture: $\forall \varepsilon > 0 : N_{S_4}(d) = O_\varepsilon(d^\varepsilon), i.e.$ $N_{S_4}(d) \leq c(\varepsilon) d^\varepsilon$ for some constant $c(\varepsilon) > 0$.

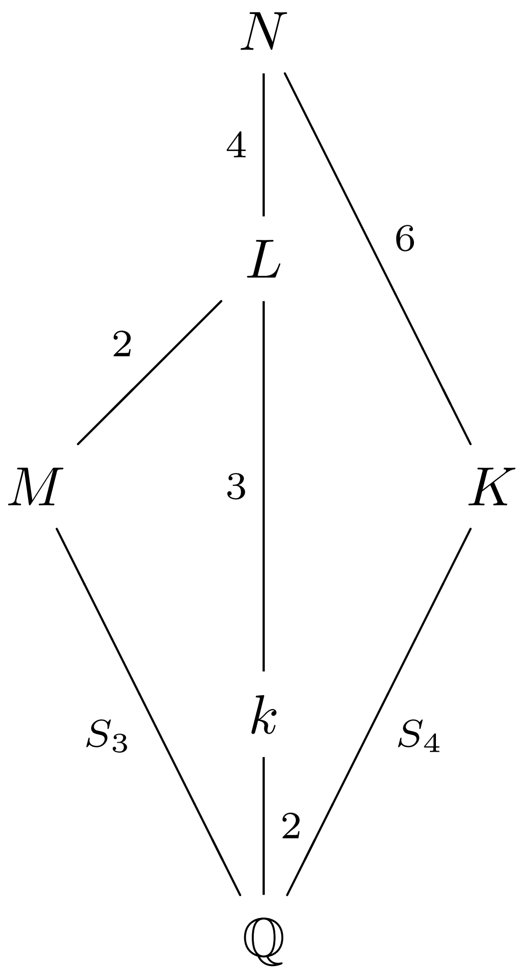
Theorem: $\forall \varepsilon > 0 : N_{S_4}(d) = O_\varepsilon(d^{1/2+\varepsilon})$.

This replaces the exponent 4/5 of Michel and Venkatesh.

In average we have:

Theorem: (Bhargava) $\lim_{x \to \infty} \frac{1}{x} \sum_{d \leq x} N_{S_4}(d) = c(S_4) > 0$.

A critical case



Suppose L/k and N/L are unramified. Then $d_k = d_M = d_K$.

$$N_{S_4}(d_K) = \frac{3^{\mathrm{rk}_3(\mathrm{Cl}_k)}-1}{3-1} 2^{\mathrm{rk}_2(\mathrm{Cl}_M)}$$

$$= O(d_k^{1/2} \log(d_k) d_M^{1/2} \log(d_M)^2)$$

$$= O(d_K \log(d_K)^3).$$

Problem: Large 2– and 3–classgroups of k and M, resp.

A theorem of Gerth III

Theorem (Gerth III): Let $M/\mathbb{Q}$ be a non-cyclic cubic extension and denote by L the normal closure of M and by k the unique quadratic subfield of L. Then the following holds.

1. If L/k is unramified, then $\mathrm{rk}_3(\mathrm{Cl}_M) = \mathrm{rk}_3(\mathrm{Cl}_k) - 1$.
2. $\mathrm{rk}_3(\mathrm{Cl}_M) = \mathrm{rk}_3(\mathrm{Cl}_k) + t - 1 - z - y$, where $y \leq t - 1$ and t is the number of prime ideals of $\mathcal{O}_k$ which ramify in L. Furthermore we have $0 \leq z \leq u$ where u is the number of primes which are totally ramified in M but split in k.
3. $\mathrm{rk}_3(\mathrm{Cl}_M) \geq \mathrm{rk}_3(\mathrm{Cl}_k) - u$

If $\mathrm{rk}_3(\mathrm{Cl}_M)$ is large, then $\mathrm{rk}_2(\mathrm{Cl}_M)$ must be small!

Parametrizing S_4–extensions

Definitions:

1. $\mathrm{Rad}(n) := \prod_{p|n} p$.
2. $\mathcal{K}$ set of quartic S_4–extensions up to isomorphy.

$$\Psi : \mathcal{K} \to \mathbb{N}^3, K \mapsto (\mathrm{Rad}(d_k), \mathrm{Rad}(\mathcal{N}(d_{L/k})), \mathrm{Rad}(\mathcal{N}(d_{N/L}))).$$

We need to solve two problems:

1. What is the discriminant of a field associated to a triple (a, b, c)?
2. How many fields are associated to a given triple (upper bounds)?

E.g. k is one of the following quadratic fields:
$\mathbb{Q}(\sqrt{a}), \mathbb{Q}(\sqrt{-a}), \mathbb{Q}(\sqrt{2a}), \mathbb{Q}(\sqrt{-2a})$.

Upper bounds for the number of fields associated to (a, b, c)

Lemma 1: All fields M such that L/K is only ramified in primes dividing b are contained in the ray class field of $\mathfrak{a} := 3b\mathcal{O}_k$. The number of those extensions can be bounded by

$$\frac{3^r - 1}{3 - 1}, \text{ where } r = \mathrm{rk}_3(\mathrm{Cl}_k) + \omega(b) + 2.$$

Lemma 2: The number of S_4-extensions $N \supset M$ such that $\mathcal{N}(d_{N/L})$ is only divisible by primes dividing c is bounded by

$$2^r - 1, \text{ where } r = \mathrm{rk}_2(\mathrm{Cl}_M) + 3\omega(c) + 6.$$

Upper bounds II

The number of elements of the fibre $\Psi^{-1}(a,b,c)$ is bounded by

$$3(\frac{3^{r_1}-1}{3-1})(2^{r_2}-1) \leq 3/2 \cdot 9 \cdot 2^6 3^{\mathrm{rk}_3(\mathrm{Cl}_k)} 2^{\mathrm{rk}_3(\mathrm{Cl}_M)} 3^{\omega(b)} 8^{\omega(c)}.$$

Corollary from theorem of Gerth III

There exists a constant $C > 0$ such that

$$3^{\mathrm{rk}_3(\mathrm{Cl}_k)} 2^{\mathrm{rk}_2(\mathrm{Cl}_M)} \leq C a^{1/2} b \log(ab^2)^2 3^{\omega(b)}.$$

Theorem:

The number of elements of the fibre $\Psi^{-1}(a,b,c)$ is bounded by

$$3^3 2^5 C a^{1/2} b \log(ab^2)^2 9^{\omega(b)} 8^{\omega(c)}.$$

Discriminants

Definition: $S = \{2, 3\}, a \in \mathbb{N}$. Then we define a^S to be the largest number dividing a which is coprime to S.

Lemma: Let $\Psi(K) = (a, b, c)$. Then $d_K^S = (ab^2c^2)^S$.

Write $d = 2^{e_2} 3^{e_3} d_1 d_2^2 d_3^3$ with $6 d_1 d_2 d_3$ squarefree. Then
$a^S = d_1 d_3, d_3 \mid c^S, (bc)^S = d_2 d_3$.

Theorem: $\forall \varepsilon > 0 : N_{S_4}(d) = O_\varepsilon(d^{1/2+\varepsilon})$.

Theorem: The number of degree 4 fields of given discriminant d is bounded by $O_\varepsilon(d^{1/2+\varepsilon})$.

Remark: d squarefree. Then $N_{S_4}(d) = O(d^{1/2} \log(d)^2)$.

Connections to modular forms of given conductor

	D_p	I_p	$v_p(N)$	$v_p(d)$		$p \mid$
$\mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3$	C_2	C_2	1	1		a
$\mathfrak{p}_1^2\mathfrak{p}_2$	$C_2 \times C_2$	C_2	2	1		a
$\mathfrak{p}_1^2$	$C_2 \times C_2$ or C_4	C_2	2 or 1	2		c
$\mathfrak{p}_1^2\mathfrak{p}_2^2$	$C_2 \times C_2$ or C_2	C_2	2 or 1	2		c
$\mathfrak{p}_1^4$	D_4	C_4	2	3	$p \equiv 3 \bmod 4$	a, c
$\mathfrak{p}_1^4$	C_4	C_4	1	3	$p \equiv 1 \bmod 4$	a, c
$\mathfrak{p}_1^3\mathfrak{p}_2$	C_3	C_3	1	2	$p \equiv 1 \bmod 3$	b
$\mathfrak{p}_1^3\mathfrak{p}_2$	D_3	C_3	2	2	$p \equiv 2 \bmod 3$	b

Connections to modular forms of given conductor II

Theorem: Let $N = 2^{n_2} 3_3^n N_{1,1} N_{1,2} N_2^2$ such that $6N_{1,1}N_{1,2}N_2$ is squarefree. Assume that $p \mid N_{1,i}$ if and only if $p \equiv i \bmod 3$ $(i = 1, 2)$. Then the number of S_4–fields of given conductor N is bounded by

$$C 54^{\omega(N)} N_{1,1} N_{1,2}^{1/2} N_2 \log(N)^2$$

for a suitable $C > 0$.

Corollary: Let p be a prime. Then the dimension of the space of octahedral modular forms of weight 1 and conductor p or p^2 is bounded above by $O(p^{1/2} \log(p)^2)$.

Corollary: Assume $p \mid N \Rightarrow p \equiv 2 \bmod 3$. Then the dimension of the space of octahedral forms of weight 1 and conductor N is bounded above by $O_\varepsilon(N^{1/2+\varepsilon})$ for all $\varepsilon > 0$.