CONVEX GEOMETRY RELATED TO HAMILTONIAN GROUP ACTIONS

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Abstract. Exercises and descriptions of student projects for a BMS/IMPAN block course.

1. First Week Exercises

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1.1. TJ: General Strategy and Simple Examples.

Exercise 1.1.1. Review the concept of duality for convex sets in vector spaces. Prove that a dual of a simplicial convex polytope is a simple convex polytope.

Exercise 1.1.2. Prove that simplicial convex polytopes constitute dense open set in the space of polytopes (this requires discussion of topology on polytopes).

Exercise 1.1.3. Define "*combinatorial type*" of a convex polytope. How does the combinatorial type of a simplicial polytope X relate to the structure of a simplicial complex on the boundary of X?

Exercise 1.1.4. Review the concept of barycentric subdivision for convex polytopes and for simplicial complexes (and perhaps more general convex-cell complexes). Show that the combinatorial types of dual convex polytopes are related by barycentric subdivision duality: dual cell of a vertex is the star in the barycentric subdivision.

Exercise 1.1.5 (Manifolds with corners)**.** Take a manifold with boundary, and a triangulation of the boundary. Show how barycentric duality can be used to associate with this data a structure of a manifold with corners. Construct an interesting manifold with corners which does not arise this way.

Exercise 1.1.6. Review the definition of the cubical cone over a simplicial complex X. Prove that the cubical complex has a simplicial subdivision simplicially isomorphic to the cone over the barycentric subdivision of X.

Exercise 1.1.7. Discuss actions of $\mathbb{Z}_2$ on $\mathbb{R}$, $\mathbb{T}^1$ on $\mathbb{C}$, $\mathbb{Z}_2^n$ on $\mathbb{R}^n$, $\mathbb{T}^n$ on $\mathbb{C}^n$: quotients, point stabilizers, orbits. Similarly for $\mathbb{Z}_2^n$ on $\mathbb{S}^{n-1}$, $\mathbb{Z}_2^{n-1}$ on $\mathbb{RP}^{n-1}$, $\mathbb{T}^n$ on $\mathbb{S}^{2n-1}$, $\mathbb{T}^n$ on $\mathbb{CP}^{n-1}$.

Exercise 1.1.8 (*)**.** On the Grassmann manifold of 2 planes in $\mathbb{C}^4$ the group of diagonal matrices in $\mathrm{GL}(4, \mathbb{C})$ acts. Discuss this action: point stabilizers, orbits, orbit closures, quotient.

1.2. **MJ: Affine Toric Varieties.**

Exercise 1.2.1. Sketch the *real locus* $V(f) \cap \mathbb{R}^2$ of the plane algebraic curve $V(f) \subset \mathbb{C}^2$, where f is one of the following bivariate (real) polynomials:

(1) $x^3 - y^2$,
(2) $x^3 + 3x^2 - y^2$,
(3) $x(x+1)(x-3)(x+2)(x-2) - y^2$,
(4) $(x^2+y^2)^2 + 18(x^2+y^2) - 27$ and
(5) $(x+y^2-1)^2 - 4((x-1)^2+y^2)$.

Let $\sigma \subset \mathbb{R}^n$ be a rational polyhedral cone, and let $\mathbb{C}[\sigma \cap \mathbb{Z}^n]$ be the induced semigroup algebra.

Exercise 1.2.2. Give a proof of *Gordan's Lemma*, which says that $\mathbb{C}[\sigma \cap \mathbb{Z}^n]$ is finitely generated.

Exercise 1.2.3. Give an example which shows that the rationality assumption in Gordan's Lemma is essential.

Exercise 1.2.4. Show that

(1) the dual $\sigma^\vee$ is again a rational polyhedral cone,
(2) we have $(\sigma^\vee)^\vee = \sigma$.

Under which conditions on σ is $\sigma^\vee$ strongly convex?

In the sequel t^m is short for the non-zero complex number $t_1^{m_1} \cdots t_n^{m_n}$, where $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$.

Exercise 1.2.5. Let $m_1, \ldots, m_\ell$ be generators of $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$. Show that U_σ, which is defined as the Zariski closure of the image of the map $\phi : (\mathbb{C}^*)^n \to \mathbb{C}^\ell$, $t \mapsto (t^{m_1}, \ldots, t^{m_\ell})$, is a toric variety.

Exercise 1.2.6.

(1) What are good definitions of "*homomorphism*" and "*isomorphism*" between affine toric varieties?
(2) Let $\sigma \subset \mathbb{R}^n$ and $\tau \subset \mathbb{R}^m$ both are rational polyhedral cones. Show that $U_{\sigma \times \tau} \cong U_\sigma \times U_\tau$, where "$\cong$" refers to the notion of "isomorphic" developed before.

1.3. **MJ: Projective Toric Varieties.** Let Σ be a fan of strictly convex rational polyhedral cones in $\mathbb{R}^n$.

Exercise 1.3.1. Show that for $\sigma \in \Sigma$ and $\tau \leq \sigma$ the affine toric variety U_τ is a Zariski open subset of U_σ.

Exercise 1.3.2. Show that for $\sigma, \tau \in \Sigma$ the embedding of $(\mathbb{C}^*)^n$ into $U_{\sigma \cap \tau}$ is compatible with its embeddings into U_σ and U_τ.

Exercise 1.3.3. Let Σ be the fan in $\mathbb{R}^n$ whose cones are generated by the proper subsets of $\{e_1, e_2, \ldots, e_n, -e_1 - e_2 - \cdots - e_n\}$. Show that $X_\Sigma = \mathbb{P}^n$.

Exercise 1.3.4. Let $\Sigma = \Sigma_{[0,1]^n}$ be the normal fan of the n-dimensional unit cube. Show that $X_\Sigma = (\mathbb{P}^1)^n$.

Exercise 1.3.5. Write down smooth and non-smooth examples of 2- and 3-dimensional projective toric varieties.

Exercise 1.3.6. For fixed $n \geq 2$ consider the *permutahedron* P, which is defined as the convex hull in $\mathbb{R}$ of all permutations of length n (written as vectors of length n). Show that P is a simple polytope of dimension $n-1$. Describe the combinatorics of P for $n = 2, 3, 4$. What can you say about the projective toric variety X_{Σ_P}? Is it smooth?

1.4. **TJ: Right-Angled Coxeter Groups.**

Exercise 1.4.1. Is the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on $\mathbb{RP}^2$ a reflection? What about the action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{S}^2$?

Exercise 1.4.2. Prove that the group

$$W = \langle s, t, r \mid s^2 = t^2 = r^2 = (st)^3 = (sr)^3 = (tr)^3 = 1 \rangle$$

is infinite.

Exercise 1.4.3. Draw Cayley graphs of dihedral groups, of S_4 and of the group W from Excercise 1.4.2. Draw the Davis–Vinberg and Coxeter complexes of the free products of two, three and four copies of $\mathbb{Z}_2$.

Exercise 1.4.4. Show that if a square complex has links of vertices which are flag, then the standard geodesic metric is locally CAT(0). Show that if links of vertices in a simplicial complex have girth six, then the standard geodesic metric is locally CAT(0).

Exercise 1.4.5. Show that the barycentric subdivision of any simplicial complex is flag.

Exercise 1.4.6. Compute the Euler characteristics of the (pqr) triangle groups. Prove Hurwitz theorem, which says that the automorphism group of a surface of genus g has at most $84(g-1)$ elements.

Exercise 1.4.7. Prove that any 2-dimensional simplicial complex admits a flag-no square triangulation.

1.5. **TJ: Topological Toric Manifolds.**

Exercise 1.5.1. Compute the Euler characteristics of real and complex toric spaces over a P, $\mathrm{cc}(L)$. (in particular note they are independent of a characteristic function).

Exercise 1.5.2. Toric spaces have two descriptions as identification spaces of $(P \times \mathbb{T}^n)/\sim$, where relation is given by characteristic function and as quotients $[P(\times \mathbb{T}^v/)\sim]/\mathbb{T}^{v-n}$, where $\mathbb{T}^{v-n}$ is an appropriately encoded subtorus.

Show how to pass from one description to the other.

Exercise 1.5.3. Similarly small covers come from an appropriate homomorphism $W_P \to \mathbb{Z}_2^n$. Show how to pass from this homomorphism to characteristic function and back.

Exercise 1.5.4. Describe the polytope and the characteristic function of a projective toric variety given by a fan.

Exercise 1.5.5. Describe the characteristic function of $M_{\mathbb{R}}$ in terms of the characteristic function of $M_{\mathbb{C}}$.

Exercise 1.5.6.

(1) Discuss/classify smooth toric surfaces over a square, pentagon, hexagon, ...
(2) Discuss/classify toric spaces over the triangle.

Exercise 1.5.7. Discuss the topology of a link of a vertex in a toric space in case P is not a simple polytope.

Exercise 1.5.8. Do not use the Four Color Theorem, but still prove: For any simple closed convex polytope P there is a smooth toric space over P.

Show an example of a polytope in dimension 4 which does not have this propoerty.

1.6. MJ: Even Simple Polytopes.

Exercise 1.6.1. Let Δ and Δ' be finite simplicial complexes with facets σ_0 and σ_0', respectively. Show that $\Pi(\Delta * \Delta', \sigma_0 \cup \sigma_1) = \Pi(\Delta, \sigma_0) \times \Pi(\Delta', \sigma_0')$. Here $\Delta * \Delta'$ is the join of Δ and Δ'.

Exercise 1.6.2. What is the minimal dimension of a simply-connected combinatorial manifold Δ (with or without boundary) for which $\Pi(\Delta)$ is not trivial?

Exercise 1.6.3. Generalize the theorem which describes a system of generators for the group of projectivities of a combinatorial manifold Δ to the situation where Δ has a non-trivial boundary.

Exercise 1.6.4. Determine the groups of projectivities of all regular polytopes which are simple. Which Coxeter groups do these correspond to? Does the 24-cell admit a group of projectivities?

A *full truncation* of a polytope Q is a polytope obtained from Q by first truncating all the vertices, then truncating all the edges and further all faces with increasing dimension. The boundary of a full truncation is dual to the barycentric subdivision of the boundary of $Q^\vee$. The combinatorial type is unique, which is why we also speak of *the* full truncation.

Exercise 1.6.5. Show that the full truncation of *any* polytope is an even simple polytope.

Exercise 1.6.6. Construct even simple polytopes which are neither full truncations nor zonotopes.

Exercise 1.6.7. For any even simple polytope construct a *canonical characteristic function*.

Exercise 1.6.8. Can you construct a subdivision scheme for simplicial complexes which gets arbitrarily fine and which preserves the group of projectivities?

1.7. **MJ: A Colorful Lebesgue Theorem.** Let X be a topological space. Its *covering dimension* is defined to be the minimal d, such that every open cover of X has an open refinement which covers each point at most $d+1$ times.

Exercise 1.7.1. Classify the topological spaces of covering dimension zero.

Exercise 1.7.2. Show that $\mathbb{R}^d$ has covering dimension d.

1.8. **TJ: Right-Angled Buildings.**

2. Second Week Exercises

Bedlewo, March 19–22, 2018.

2.1. **MJ: h-Vectors of Polytopes and Spheres.**

Exercise 2.1.1. Determine the h-vectors of the cubes/cross polytopes in all dimensions.

Exercise 2.1.2. Determine the h-vectors of the permutahedra in all dimensions.

Exercise 2.1.3. What is the dual of an abstract objective function?

Exercise 2.1.4. Let P be a simple d-polytope with graph $\Gamma = \Gamma(P)$. Show that the facets of P bijectively correspond to the connected and $(d-1)$-regular subgraphs of Γ which are initial with respect to some abstract objective function.

Exercise 2.1.5. Let Δ be a simplicial $(d-1)$-sphere. Show that $h_d(\Delta) = 1$.

Exercise 2.1.6. Let $\ell, i \geq 1$ be integers. Show that there is a unique $j \in \mathbb{N}$ and a unique sequence of integers

$$n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$$

such that

$$\ell \ = \ \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j} \ .$$

2.2. **MJ: The g-Theorem.**

Exercise 2.2.1. Compute the h-vector of the simplicial complex Δ on four vertices with facets $\{1,2\}$ and $\{2,3,4\}$. Describe its Stanley-Reisner ring $\mathbb{R}[\Delta]$.

Exercise 2.2.2. Let P and Q be simple polytopes. Show that the product $P \times Q$ is a simple polytope again. What is the h-vector of $P \times Q$ expressed in the h-vectors of P and Q?

Exercise 2.2.3. Find out what *stacked* and *k-stacked polytopes* are (with or without the help of `Wikipedia`). What can you say about their f- and h-vectors?

Exercise 2.2.4. For $C = [0,1]^d$ and $i \in \{0, 1, \ldots, \lfloor d/2 \rfloor\}$ determine the Lefschetz isomorphism

$$\omega^{d-2i} : H^{2i}(X_C; \mathbb{Q}) \to H^{d-2i}(X_C; \mathbb{Q}) \ .$$

Start out with describing bases of the vector spaces $H^{2i}(X_C; \mathbb{Q})$.

2.3. **MJ: Computing Face Lattices and f-Vectors.** Let us reconsider Exercise 2.1.2 in more detail. Let Π_n be the convex hull of all permutations in the symmetric group S_n, written in one-line representation. We have $d := \dim \Pi_n = n-1$. The polytope Π_{d+1} is the d-dimensional *permutahedron*.

Exercise 2.3.1. What are the facets of Π_n? How many are there? What are the lower-dimensional faces and how many are there?

Let $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ with $a_1 + a_2 + \cdots + a_k = n$. This is a *partition* of n into k parts.

Exercise 2.3.2. Determine how many of such partitions of n exist. How many partitions of n into *distinct* parts are there? How many of them have at most k parts?

Exercise 2.3.3. Show that Π_n is a secondary polytope for the prism over the $(n-1)$-simplex.

3. Projects

3.1. **Counting Lattice Points.** For P a lattice polytope in $\mathbb{R}^n$ and $k \in \mathbb{N}$ let

$$\eta_P(k) \; := \; \#\{m \mid m \in k \cdot P \cap \mathbb{Z}^n\}$$

be the number of lattice points in the kth dilate of P.

Theorem. *The function $\eta_P(k)$ is a polynomial in k.*

- Prove the theorem; you might want to borrow one or two lemmas from suitable literature.
- Motivate those lemmas that you are citing by examples.
- What is known about these polynomials?
- What is not known?

3.2. **Polygons With a Given Number of Interior Lattice Points.**

Theorem. *For every fixed integer $g \geq 1$, there are only finitely many lattice polygons with exactly g interior lattice points, up to integer affine isomorphisms in $\mathbb{Z}^2$.*

- Prove the theorem.
- What is wrong with $g = 0$?
- How many distinct classes of polygons exist for $g = 1, 2, 3$?
- How can we enumerate them in general?

3.3. **In the Art Gallery.** In computational geometry language a *simple polygon* is the bounded region described by a polygonal Jordan curve in $\mathbb{R}^2$. A simple polygon in the sense of this definition does not need to be convex.

Theorem. *Every simple polygon admits a balanced triangulation.*

- Prove the theorem.
- Describe examples and non-examples.
- What is the "Art Gallery Theorem", and how does it follow from the theorem?
- What is known about generalizations?

References

1. Peter Abramenko, *Twin buildings and applications to S-arithmetic groups*, Lecture Notes in Mathematics, vol. 1641, Springer-Verlag, Berlin, 1996. MR 1624276
2. Peter Abramenko and Kenneth S. Brown, *Buildings*, Graduate Texts in Mathematics, vol. 248, Springer, New York, 2008, Theory and applications. MR 2439729
3. Dorde Baralić and Rade Živaljević, *Colorful versions of the Lebesgue, KKM, and Hex theorem*, J. Combin. Theory Ser. A **146** (2017), 295–311. MR 3574233
4. Victor M. Buchstaber and Taras E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, vol. 24, American Mathematical Society, Providence, RI, 2002. MR 1897064
5. ———, *Toric topology*, Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015. MR 3363157
6. David Cox, *What is a toric variety?*, Topics in algebraic geometry and geometric modeling, Contemp. Math., vol. 334, Amer. Math. Soc., Providence, RI, 2003, pp. 203–223. MR 2039974
7. David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322
8. Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR 2360474
9. Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451. MR 1104531
10. Albrecht Dold, *Lectures on algebraic topology*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1972 edition. MR 1335915
11. William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037
12. James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR 1066460
13. Ivan Izmestiev and Michael Joswig, *Branched coverings, triangulations, and 3-manifolds*, Adv. Geom. **3** (2003), no. 2, 191–225. MR MR1967999 (2004b:57004)
14. Michael Joswig, *Projectivities in simplicial complexes and colorings of simple polytopes*, Math. Z. **240** (2002), no. 2, 243–259. MR 1900311
15. Michael Joswig and Thorsten Theobald, *Polyhedral and algebraic methods in computational geometry*, Universitext, Springer, London, 2013, Revised and updated translation of the 2008 German original. MR 2905853
16. Roman N. Karasev, *Covering dimension using toric varieties*, Topology Appl. **177** (2014), 59–65. MR 3258182
17. Richard P. Stanley, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1453579
18. Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York-Berlin, 1983, Corrected reprint of the 1971 edition. MR 722297
19. Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR 1311028

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