OPTIMIZATION AND TROPICAL GEOMETRY: EXERCISES AND PROBLEMS 5

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Exercise 1. Determine the nondominated points of the multiobjective linear program

$$\min \begin{pmatrix} -3 & 1 & 1 & -2 \\ 2 & 1 & 1 & 0 \end{pmatrix} \cdot x \quad \text{subject to } x \in \{0,1\}^4$$

Exercise 2. Consider the monomial ideal M in $K[x_1, x_2, x_3, x_4]$ spanned by the nondominated points from Exercise 1. Check if M is Artinian; if not replace M by its Artinian closure. Compute the Alexander dual. You may use software like `Macaulay2` [4] or `Singular` [2].

Exercise 3. Let $C \subset \mathbb{TP}^{n-1}$ be a tropical cone, not necessarily polyhedral. Show that there is a unique set $R \subset C$ which is minimal with respect to inclusion such that $\operatorname{tpos}(R) = C$.

Exercise 4. In Exercise 3.1 we discussed the max-tropical polyhedron $P = \operatorname{ord}(\boldsymbol{P})$ where $\boldsymbol{P}$ is the Puiseux polyhedron given by the linear inequalities

$$\begin{aligned} \boldsymbol{x}_1 + \boldsymbol{x}_2 &\leq 2 \\ t\boldsymbol{x}_1 &\leq 1 + t^2\boldsymbol{x}_2 \\ t\boldsymbol{x}_2 &\leq 1 + t^3\boldsymbol{x}_1 \\ \boldsymbol{x}_1 &\leq t^2\boldsymbol{x}_2 \\ \boldsymbol{x}_1, \boldsymbol{x}_2 &\geq 0 \end{aligned}$$

over the ordered field $\mathbb{K}$ of *reverse* Puiseux series with real coefficients. Compute the tropical vertices of P.

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Problem 5. Study the zero-sum matrix games with multi-dimensional payoffs introduced by Hamel and Löhne [5, §3] in terms of tropical convexity. See also [3].

Problem 6 ([7, Question 25])**.** Give an interpretation of the planar resolution algorithm from [8, §3.5] and the hull resolution from [8, §4.4] in terms of tropical convexity.

The tropical upper bound theorem bounds the number k of extremal generators of a (monomial) tropical cone given as the intersection of m tropical halfspaces; cf. [1, Theorem 1] and [7, Theorem 17]. Equivalently, the number k yields the number of scalarizations required for an n-criteria optimization problem with m nondominated points. It is known that that the bound in [1, Theorem 1] is not tight for all parameters.

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Problem 7 ([7, Question 26]). Determine the exact upper bound for m as a function of n and d. See also work of Hoşten and Morris [6] and [8, Theorem 6.33].

Problem 8 ([7, Question 27]). Which bipartite graphs occur as the vertex-facet incidence graphs of monomial tropical cones?

Problem 9 ([7, Question 28]). To what extent does our approach generalize to multicriteria optimization problems which are not discrete? For instance, look into more general semigroup rings; cf. [8, Chapter 7].

References

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