OPTIMIZATION AND TROPICAL GEOMETRY: EXERCISES AND PROBLEMS 4

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Exercise 1. Consider a product-mix auction for $n = 1$ indivisibile good and $m \geq 1$ agents. Suppose that the utility function $u^j : A^j \to \mathbb{R}$ of all agents are constant. Give an algorithm to decide whether or not there exists a competitive equilibrium. What is its complexity?

Exercise 2. Find two finite point sets A^1 and A^2 in $\mathbb{Z}^2$ such that there does not exist any pair of utility functions $u^1 : A^1 \to \mathbb{R}$ and $u^2 : A^2 \to \mathbb{R}$ such that the product-mix auction (for $n = 2$ indivisibile goods and $m = 2$ agents) with (u^1, u^2) does has any competitive equilibrium.

Exercise 3. Phrase the existence of competitive equilibrium in a product-mix auction (for general $m, n \geq 1$) as an integer linear program.

Exercise 4. Consider the running example (with $m = n = 2$) from the lecture defined by the tropical polynomials

$$\begin{aligned} F_1(X,Y) &= \max(0, 3+Y, 5+2Y, 9+X+2Y) \\ F_2(X,Y) &= \max(0, 1+X, 1+Y) \ , \end{aligned}$$

which combine the bundle sets A^1 and A^2 with the utility functions $u^1 : A^1 \to \mathbb{R}$ and $u^2 : A^2 \to \mathbb{R}$ of the two agents.

(1) Determine the full demand types D^1 and D^2 of u^1 and u^2, respectively.
(2) Is the union $D^1 \cup D^2$ unimodular?

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To see how the following topics concerning lattice polytopes and toric varieties are related to product-mix auctions we refer to [5, §6]. A notoriously open problem in this area is Oda's question "Is every smooth projective toric variety projectively normal?"; cf. [3] and [5, Conjecture 6.10].

But first things first. A *lattice polytope*, P, is a convex polytope all of whose vertices have integer coordinates. It has the *integer decomposition property (IDP)* if

$$(k \cdot P) \cap \mathbb{Z}^n \ = \ \underbrace{(P \cap \mathbb{Z}^n) + \cdots + (P \cap \mathbb{Z}^n)}_{k \text{ times}}$$

for all $k \geq 1$. The lattice polytope P is *smooth* if for each vertex v of P the edges through v form a lattice basis (of the integer lattice $\mathbb{Z}^n$): a smooth polytope is necessarily simple. A triangulation T of P is *unimodular* if all simplices in T have normalized volume one. The existence if a unimodular triangulation implies IDP.

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Problem 5 ([3, §1.5.4])**.** Lattice zonotopes are known to be IDP. Do they have unimodular triangulations?

A regular unimodular triangulation is *quadratic* if its minimal non-faces are edges, i.e., it is “flag”. A lattice polytope is *reflexive* if its polar dual is a lattice polytope, too.

Problem 6 ([3, §1.5.1])**.** Do all smooth reflexive polytopes have a quadratic triangulation? Specifically this question is open for 18 out of the 80 892 types in dimension 7. This probably calls for `polymake` [2] and the database `polyDB` [4].

Problem 7 ([3, §1.5.2])**.** Do polytopes whose facet normals are contained in the root systems of types C_n, D_n, E_6 or E_7 admit a unimodular triangulation? What about quadratic ones? A standard reference on root systems is [1].

References

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