OPTIMIZATION AND TROPICAL GEOMETRY: EXERCISES AND PROBLEMS 3

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Exercise 1. Consider the polyhedron $\boldsymbol{\mathcal{P}} \subset \mathbb{K}^2$, defined over the ordered field $\mathbb{K}$ of *reverse* Puiseux series with real coefficients, given by the linear inequalities:

$$
\begin{aligned}
\boldsymbol{x}_1 + \boldsymbol{x}_2 &\leq 2 \\
t\boldsymbol{x}_1 &\leq 1 + t^2\boldsymbol{x}_2 \\
t\boldsymbol{x}_2 &\leq 1 + t^3\boldsymbol{x}_1 \\
\boldsymbol{x}_1 &\leq t^2\boldsymbol{x}_2 \\
\boldsymbol{x}_1, \boldsymbol{x}_2 &\geq 0 \ .
\end{aligned}
\tag{1}
$$

Sketch $\boldsymbol{\mathcal{P}}$ and its tropicalization (with respect to max).

Exercise 2. Let $S = S_{A,B}$ be the Shapley operator associated with a pair of matrices $A, B \in \mathbb{T}_{\max}^{m \times n}$. Show that

(1) S is order-preserving, i.e., $x \leq y$ implies $S(x) \leq S(y)$ for all $x, y \in \mathbb{T}_{\max}^n$;
(2) S is additively homogeneous, i.e., $S(\lambda\mathbf{1} + x) = \lambda\mathbf{1} + S(x)$ for all $\lambda \in \mathbb{T}_{\max}$ and $x \in \mathbb{T}_{\max}^n$;
(3) S is continuous. In order to render this true: What is the suitable topology on the set $\mathbb{T}_{\max} = \mathbb{R} \cup \{-\infty\}$? Assume that, whatever topology we choose for $\mathbb{T}_{\max}$, we take the induced product topology on $\mathbb{T}_{\max}^n$.

Exercise 3. Let $A, B \in \mathbb{T}_{\max}^{m \times n}$. We study the mean-payoff game on the bipartite digraph $D(A, B)$. Suppose that player Min chooses a fixed *positional* strategy, i.e., the choice of Min's next move only depends on the current position of the token, not on the entire history of the game so far.

(1) What is then the optimal strategy for player Max? Do you recognize the classical combinatorial optimization problem which this corresponds to?
(2) What is the complexity of finding such an optimal strategy for Max?

Exercise 4. Determine the value of the mean-payoff game associated with the tropicalization of the linear inequalities (1).

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Problem 5. Describe `polymake`'s mechanism for rule based computation [5] in terms of AND/OR scheduling networks [7] and tropical linear programs.

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Problem 6. Like the classical simplex method also the tropical simplex method visits a portion of a finite graph (i.e., by travelling along improving edges toward an optimal vertex). In the tropical setting that graph formally depends on the given inequality description of the tropical polyhedron formed by the feasible solutions, cf. [2, Proposition-Definition 3.10].

Do there exist matrices $A, B, A', B' \in \mathbb{T}^{m\times n}$ with $A \odot x \leq B \odot x \iff A' \odot x \leq B' \odot x$ such that the graphs induced by (A, B) and (A', B') are distinct?

Problem 7. The article [3] contains a construction of a family of (nondegenerate) linear programs $\mathbf{LW}_r^\epsilon(t)$ over convergent real Puiseux series such that, for $t \gg 0$, the associated ordinary linear programs over the reals have a central path (with respect to the log-barrier interior point method) whose total curvature is exponential in r. That parameter r simultaneously controls the dimension $2r$ and the number of constraints $3r + 1$.

Does there exist another family of linear programs with similar properties but with arbitrarily many constraints in fixed dimension? What about the running time of interior point algorithms on such a family of linear programs?

Problem 8. Develop quantifier elimination over the tropical semiring $\mathbb{T}$, and study the algorithmic complexity; cf. [1] and [6]. Pay attention to the special case of quadratic polynomials and the connection to tropical semidefinite programming [4].

References

1. Daniele Alessandrini, *Logarithmic limit sets of real semi-algebraic sets*, Adv. Geom. **13** (2013), no. 1, 155–190. MR 3011539
2. Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig, *Tropicalizing the simplex algorithm*, SIAM J. Discrete Math. **29** (2015), no. 2, 751–795.
3. ———, *Log-barrier interior point methods are not strongly polynomial*, SIAM J. Appl. Algebra Geom. **2** (2018), no. 1, 140–178.
4. Xavier Allamigeon, Stéphane Gaubert, and Mateusz Skomra, *Solving generic nonarchimedean semidefinite programs using stochastic game algorithms*, J. Symbolic Comput. **85** (2018), 25–54. MR 3707850
5. Ewgenij Gawrilow and Michael Joswig, *`polymake`: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR MR1785292 (2001f:52033)
6. Philipp Jell, Claus Scheiderer, and Josephine Yu, *Real tropicalization and analytification of semialgebraic sets*, 2018, Preprint `arXiv:1810.05132`.
7. Rolf H. Möhring, Martin Skutella, and Frederik Stork, *Scheduling with AND/OR precedence constraints*, SIAM J. Comput. **33** (2004), no. 2, 393–415. MR 2048448

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